

# Infinite Automata 2025/26

## Exercise Sheet 8

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**Exercise 8.1** Show that, from any initial configuration  $(p, \mathbf{u})$ , the reachability set of a given integer  $d$ -VASS ( $\mathbb{Z}$ -VASS) intersected with the positive quadrant  $\mathbb{N}^d$  is semilinear.

**Exercise 8.2.** Show that, from any initial configuration  $(p, \mathbf{u})$ , the reachability set of a given bidirected VASS is semilinear.

*Hint 1. Show that if there is a run from  $(p, \mathbf{u})$  to  $(q_1, \mathbf{v})$ , and there is a run from  $(p, \mathbf{u})$  to  $(q_2, \mathbf{u} + \mathbf{x})$  then there is a run from  $(p, \mathbf{u})$  to  $(q_2, \mathbf{v} + \mathbf{x})$ .*

*Hint 2. Show that if there is a run from  $(p, \mathbf{u})$  to  $(p, \mathbf{u} + \mathbf{r})$ , then  $\mathbf{r} = \mathbf{x}_1 + \dots + \mathbf{x}_k$  such that there is a run from  $(p, \mathbf{u})$  to  $(p, \mathbf{u} + \mathbf{x}_i)$  and  $\mathbf{x}_i$  is minimal (i.e. there does not exist  $\mathbf{y} \leq \mathbf{x}$  such that  $(p, \mathbf{u})$  can reach  $(p, \mathbf{u} + \mathbf{y})$ ).*

*Hint 3. Let  $B$  be the set of configurations  $(q, \mathbf{v})$  that are reachable from  $(p, \mathbf{u})$  and such that there does not exist a configuration  $(q, \mathbf{w})$  that is also reachable from  $(p, \mathbf{u})$  such that  $\mathbf{w} \geq \mathbf{v}$ . In other words  $B$  is the minimal set of reachable configurations from  $(p, \mathbf{u})$ . Let  $P$  be the set of vectors  $\mathbf{x} \in \mathbb{N}^d$  in such that there is a run from  $(p, \mathbf{u})$  to  $(p, \mathbf{u} + \mathbf{x})$  and there does not exist a vector  $\mathbf{y} \leq \mathbf{x}$  such that  $\mathbf{y} \neq \mathbf{0}$  and  $(p, \mathbf{u} + \mathbf{y})$  is reachable from  $(p, \mathbf{u})$ . Then,  $\text{Reach}((p, \mathbf{u}))$  is equal to  $\{(q, \mathbf{v} + \mathbf{x}_1 + \dots + \mathbf{x}_k) : (q, \mathbf{v}) \in B \text{ and } \mathbf{x}_i \in P\}$ .*

**Exercise 8.3.** Show the reachability problem for bidirected VASS is decidable.

*Hint 1. Design two semi-procedures. One using runs and the other using semilinear separators.*

*Hint 2. If there is no run from  $(p, \mathbf{u})$  to  $(q, \mathbf{u})$ , then we can guess the sets  $B$  and  $P$  and verify that  $\text{Reach}((p, \mathbf{u})) = B + P^*$  (as per Hint 3 of Exercise 8.2) is an inductive invariant for reachability.*

**Exercise 8.4.** Show that, if we have an exponential bound for the size (in the one-norm) of minimal solutions to a homogeneous systems of linear equations, then we also have an exponential bound for the size of minimal solutions of arbitrary (not necessarily homogeneous) systems of linear equations.