

Coinductive types in Coq

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In Coq nontermination has to be avoided

General recursion will make Coq inconsistent

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Fixpoint bad (u : unit) : P := bad u.
```

Fixpoint definition has its “guard condition” (recursive calls has to be done on structurally smaller terms) and it reduces only when aki (the argument one does recursion on) starts with a constructor:

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(Fix f a1 ... aki) -> ti a1 ... aki
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Fix and Case reductions are called together iota reduction

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Fix and Case reductions are called together iota reduction

Lazy lists — LList

```
CoInductive LList (A:Set) :Set :=  
  LNil : LList A  
| LCons : A -> LList A -> LList A
```

- terms built from constructors
- LList is the greatest set of terms built from LNil | LCons containing infinite and finite terms
- induction principle does not hold
- constructors are injective and distinct (one may use tactics injection and discriminate)

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  LLeaf : LTree A  
| LBin : A -> LTree A -> LTree A -> LTree A
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- finite and infinite trees
- some branches can be infinite

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Streams — Stream

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CoInductive Stream (A:Set) :Set :=  
  Cons : A -> Stream A -> Stream A
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Pattern-matching

```
Definition isEmpty (A:Type) (l:LList A) : Prop :=  
  match l with  
  | LNil => True  
  | LCons a l' => False  
end.
```

```
Definition LHead (A:Type) (l:LList A) : option A :=  
  match l with  
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Pattern-matching

```

Fixpoint LNth (A:Type) (n:nat) (l:LList A) {struct n} :
option A :=
  match l with
  | LNil => None
  | LCons a l' => match n with
                  | 0 => Some a
                  | S p => LNth p l'
                  end
  end.

```

```

Eval compute in (LNth 2 (LCons 4 (LCons 3 (LCons 90 LNil)))).
= Some 90 : option nat

```

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Building infinite objects

Goal: to represent infinite objects in a finite way. Failed attempt:

```
Fixpoint from (n:nat) {struct n} : LList nat :=  
  Lcons n (from (S n)).
```

Reason: recursive call `from` is not applied to structurally smaller argument. Successful attempt:

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Corecursive functions — introduction

- all computations in Coq are finite,
- recursive function consumes values of an inductive type,
- corecursive function produces values in a coinductive type,
- result may be infinite, but its every finite approximation should be computable in finite time,
- corecursive functions have its “guard conditions”.

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Corecursive functions — guard condition

Definition by `cofixpoint` is correct if every (co)recursive call is one of the arguments of some constructor of a coinductive type.

- similarity: in lazy programming languages constructors do not evaluate its arguments
- if coinductive values are matched against patterns, then guard condition ensures that every recursive call of a corecursive function produces in a finite time its head-constructor
- recursive function reduces when it is applied to a value with constructor in head position; corecursive function reduces when it is an argument to pattern-matching

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Examples

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Eval simpl in (from 3).
= from 3 : LList nat
```

```
Eval simpl in (LHead (LTail (from 3))).
= Some 4 : option nat
```

```
CoFixpoint forever (A:Type)(a:A):LList A:=LCons a (forever a).
```

```
CoFixpoint LAppend (A:Type) (u v:LList A) : LList A :=
  match u with
  | LNil => v
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  end.
```

```
Eval compute in (LNth 123 (LAppend (forever 33) Nats)).
= Some 33 : option nat
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```
Eval compute in
  (LNth 123 (LAppend (LCons 0 (LCons 1 (LCons 2 LNil))) Nats)).
= Some 120 : option nat
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Incorrect definitions by cofixpoint

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CoFixpoint filter (A:Set) (p: A->bool) (l:LList A) : LList A
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LHead (filter (fun p:nat =>
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would cause an infinite computation

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Decomposition lemmas

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Definition LList_decompose (A:Type) (l:LList A) : LList A :=
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Eval simpl in (LList_decompose (forever 33)).
= LCons 33 (forever 33) : LList nat

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```

Lemma LList_decomposition : forall (A:Type) (l:LList A), l =
LList_decompose l.

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Proof.

```

  intros A l; destruct l; trivial.

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Qed.

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Proofs using decomposition

```
Ltac LList_unfold term := apply trans_equal with
  (1 := LList_decomposition term).
```

```
Lemma LAppend_LNil : forall (A:Type) (v:LList A),
  LAppend LNil v = v.
```

Proof.

```
  intros A v.
```

```
  LList_unfold (LAppend LNil v).
```

```
  simpl; destruct v; reflexivity.
```

Qed.

Inductive predicates on coinductive types

```

Inductive Finite (A:Type) : LList A -> Prop :=
  | Finite_LNil : Finite LNil
  | Finite_LCons : forall (a:A) (l:LList A), Finite l -> Finite
    (LCons a l).

```

```

Remark one_two_three : Finite (LCons 1 (LCons 2 (LCons 3 LNil)))

```

Proof.

```

  repeat constructor.

```

Qed.

```

Theorem Finite_of_LCons :

```

```

  forall (A:Type) (a:A) (l:LList A),
    Finite (LCons a l) -> Finite l.

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  intros A a l H; inversion H; assumption.

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Qed.

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Coinductive predicates

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    forall (a:A) (l:LList A), Infinite l -> Infinite (LCons a l).
```

We want to prove that `forall n:nat, Infinite (from n)`.

We need an auxiliary decomposition lemma for `from`:

```
Lemma from_unfold : forall n:nat, from n = LCons n (from (S
n)).
```

Proof.

```
  intro n.
  LList_unfold (from n).
  simpl; trivial.
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Qed.

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Proof of forall n:nat, Infinite (from n)

The proof will be a corecursive function — the greatest fixpoint of `F_from`:

Definition `F_from` :

```
(forall n:nat, Infinite (from n)) -> forall n:nat, Infinite
(from n).
```

```
intros H n; rewrite (from_unfold n).
constructor; auto.
```

Defined.

```
Theorem from_Infinite_V0 : forall n:nat, Infinite (from n).
Proof (cofix H : forall n:nat, Infinite (from n) := F_from H).
```

```
Lemma from_Infinite : forall n:nat, Infinite (from n).
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Proof.

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constructor; apply H.
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Qed.

Wrong proof of forall n:nat, Infinite (from n)

Lemma from_Infinite_buggy : forall n:nat, Infinite (from n).

Proof.

cofix H.

assumption.

Qed.

Error: Recursive definition of "H" is ill-formed.

In environment

H: $\forall n:nat, Infinite (from n)$

ungarded recursive call in H

Note: you may use command Guarded, to check that "guard condition" is still satisfied

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Elimination of coinductive assumptions

Tactics `case` and `inversion` work for coinductive types:

```
Lemma LNil_not_Infinite : forall A:Type, ~ Infinite (@LNil A).  
Proof.  
  intros A H; inversion H.  
Qed.
```

Equality of coinductive objects

Equality `eq` is adequate if finite number of simplification results in identical terms. There are examples when it does not hold:

```
Lemma Lappend_of_Infinite_0 :  
  forall (A:Type) (u:LList A), Infinite u -> forall v:LList A,  
  u = LAppend u v.
```

Equality `eq` is too strong, one needs a weaker predicate.

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```

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Bisimilarity

```
CoInductive bisimilar (A:Type) : LList A -> LList A -> Prop
:=
| bisim0 : bisimilar LNil LNil
| bisim1 :
  forall (a:A) (l l':LList A),
    bisimilar l l' -> bisimilar (LCons a l) (LCons a l').
```

Bisimulation

```

Definition bisimulation (A:Type) (R:LList A -> LList A -> Prop)
:=
  forall l1 l2:LList A,
    R l1 l2 ->
      match l1 with
      | LNil => l2 = LNil
      | LCons a l'1 =>
          match l2 with
          | LNil => False
          | LCons b l'2 => a = b  $\wedge$  R l'1 l'2
          end
      end
end.

```


Park principle

Bisimilarity is the greatest relation containing the pair `LNil`, `LNil` and closed under application of `LCons`.

Bisimulation is any relation satisfying these closure properties. Hence:

Theorem `park_principle` :

```
forall (A:Type) (R:LList A -> LList A -> Prop),
  bisimulation R -> forall l1 l2:LList A, R l1 l2 ->
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Coinductive operational semantics for while-programs (example from CPDT)

Nonterminating (and terminating) programs will be modeled using coinductive types.

Definition $\text{var} := \mathbf{nat}$.

Definition $\text{vars} := \text{var} \rightarrow \mathbf{nat}$.

Definition set $(vs : \text{vars}) (v : \text{var}) (n : \mathbf{nat}) : \text{vars} :=$
 $\text{fun } v' \Rightarrow \text{if beq_nat } v \ v' \text{ then } n \text{ else } vs \ v'$.

Expressions

```
Inductive exp : Set :=  
| Const : nat → exp  
| Var : var → exp  
| Plus : exp → exp → exp.
```

```
Fixpoint evalExp (vs : vars) (e : exp) : nat :=  
  match e with  
  | Const n ⇒ n  
  | Var v ⇒ vs v  
  | Plus e1 e2 ⇒ evalExp vs e1 + evalExp vs e2  
  end.
```

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```

```
Fixpoint evalExp (vs : vars) (e : exp) : nat :=
```

```
  match e with
```

```
  | Const n ⇒ n
```

```
  | Var v ⇒ vs v
```

```
  | Plus e1 e2 ⇒ evalExp vs e1 + evalExp vs e2
```

```
end.
```

Instructions

```
Inductive cmd : Set :=  
| Assign : var → exp → cmd  
| Seq : cmd → cmd → cmd  
| While : exp → cmd → cmd.
```

Operational semantics

A program that does not terminate in a particular initial state is related to any final state.

```

CoInductive evalCmd : vars → cmd → vars → Prop :=
| EvalAssign : ∀ vs v e, evalCmd vs (Assign v e) (set vs v (evalExp vs e))
| EvalSeq : ∀ vs1 vs2 vs3 c1 c2, evalCmd vs1 c1 vs2
  → evalCmd vs2 c2 vs3
  → evalCmd vs1 (Seq c1 c2) vs3
| EvalWhileFalse : ∀ vs e c, evalExp vs e = 0
  → evalCmd vs (While e c) vs
| EvalWhileTrue : ∀ vs1 vs2 vs3 e c, evalExp vs1 e ≠ 0
  → evalCmd vs1 c vs2
  → evalCmd vs2 (While e c) vs3
  → evalCmd vs1 (While e c) vs3.

```

Operational semantics

A program that does not terminate in a particular initial state is related to any final state.

```

CoInductive evalCmd : vars → cmd → vars → Prop :=
| EvalAssign : ∀ vs v e, evalCmd vs (Assign v e) (set vs v (evalExp vs e))
| EvalSeq : ∀ vs1 vs2 vs3 c1 c2, evalCmd vs1 c1 vs2
  → evalCmd vs2 c2 vs3
  → evalCmd vs1 (Seq c1 c2) vs3
| EvalWhileFalse : ∀ vs e c, evalExp vs e = 0
  → evalCmd vs (While e c) vs
| EvalWhileTrue : ∀ vs1 vs2 vs3 e c, evalExp vs1 e ≠ 0
  → evalCmd vs1 c vs2
  → evalCmd vs2 (While e c) vs3
  → evalCmd vs1 (While e c) vs3.
  
```


Bisimulation for evalCmd

Section evalCmd_coind.

Variable $R : \text{vars} \rightarrow \mathbf{cmd} \rightarrow \text{vars} \rightarrow \text{Prop}$.

Hypothesis *AssignCase* : $\forall vs1\ vs2\ v\ e, R\ vs1\ (\text{Assign}\ v\ e)\ vs2$
 $\rightarrow vs2 = \text{set}\ vs1\ v\ (\text{evalExp}\ vs1\ e)$.

Hypothesis *SeqCase* : $\forall vs1\ vs3\ c1\ c2, R\ vs1\ (\text{Seq}\ c1\ c2)\ vs3$
 $\rightarrow \exists vs2, R\ vs1\ c1\ vs2 \wedge R\ vs2\ c2\ vs3$.

Hypothesis *WhileCase* : $\forall vs1\ vs3\ e\ c, R\ vs1\ (\text{While}\ e\ c)\ vs3$
 $\rightarrow (\text{evalExp}\ vs1\ e = 0 \wedge vs3 = vs1)$

$\vee \exists vs2, \text{evalExp}\ vs1\ e \neq 0 \wedge R\ vs1\ c\ vs2 \wedge R\ vs2\ (\text{While}\ e\ c)$
 $vs3$.

Bisimulation for evalCmd cont.

Theorem evalCmd_coind : $\forall vs1\ c\ vs2, R\ vs1\ c\ vs2 \rightarrow \mathbf{evalCmd}\ vs1\ c\ vs2$.

cofix; intros; destruct c.

rewrite (*AssignCase H*); constructor.

destruct (*SeqCase H*) as [? [? ?]]; econstructor; eauto.

destruct (*WhileCase H*) as [[? ?] | [? [? [? ?]]]]; subst;

econstructor; eauto.

Qed.

End evalCmd_coind.

Optimization

```
Fixpoint optExp (e : exp) : exp :=  
  match e with  
  | Plus (Const 0) e  $\Rightarrow$  optExp e  
  | Plus e1 e2  $\Rightarrow$  Plus (optExp e1) (optExp e2)  
  | _  $\Rightarrow$  e  
end.
```

```
Fixpoint optCmd (c : cmd) : cmd :=  
  match c with  
  | Assign v e  $\Rightarrow$  Assign v (optExp e)  
  | Seq c1 c2  $\Rightarrow$  Seq (optCmd c1) (optCmd c2)  
  | While e c  $\Rightarrow$  While (optExp e) (optCmd c)  
end.
```

Optimization

```

Fixpoint optExp (e : exp) : exp :=
  match e with
  | Plus (Const 0) e ⇒ optExp e
  | Plus e1 e2 ⇒ Plus (optExp e1) (optExp e2)
  | _ ⇒ e
  end.

```

```

Fixpoint optCmd (c : cmd) : cmd :=
  match c with
  | Assign v e ⇒ Assign v (optExp e)
  | Seq c1 c2 ⇒ Seq (optCmd c1) (optCmd c2)
  | While e c ⇒ While (optExp e) (optCmd c)
  end.

```

Optimization correctness for expressions

Lemma `optExp_correct` : \forall vs e, `evalExp vs (optExp e) = evalExp vs e`.

Optimization correctness for instructions

Lemma `optCmd_correct1` : $\forall vs1\ c\ vs2, \mathbf{evalCmd}\ vs1\ c\ vs2$
 $\rightarrow \mathbf{evalCmd}\ vs1\ (\mathbf{optCmd}\ c)\ vs2.$

Lemma `optCmd_correct2` : $\forall vs1\ c\ vs2, \mathbf{evalCmd}\ vs1\ (\mathbf{optCmd}\ c)\ vs2$
 $\rightarrow \mathbf{evalCmd}\ vs1\ c\ vs2.$

`intros; apply (evalCmd_coind (fun vs1 c vs2 => evalCmd vs1
 (optCmd c) vs2));`
`crush; finisher.`

Qed.

Optimization correctness for instructions

Lemma `optCmd_correct1` : $\forall vs1\ c\ vs2, \mathbf{evalCmd}\ vs1\ c\ vs2$
 $\rightarrow \mathbf{evalCmd}\ vs1\ (\mathbf{optCmd}\ c)\ vs2.$

Lemma `optCmd_correct2` : $\forall vs1\ c\ vs2, \mathbf{evalCmd}\ vs1\ (\mathbf{optCmd}\ c)\ vs2$
 $\rightarrow \mathbf{evalCmd}\ vs1\ c\ vs2.$

`intros; apply (evalCmd_coind (fun vs1 c vs2 => evalCmd vs1
 (optCmd c) vs2));`
`crush; finisher.`

Qed.

Optimization correctness for instructions, cont.

Theorem `optCmd_correct` : $\forall vs1\ c\ vs2, \mathbf{evalCmd}\ vs1\ (\mathbf{optCmd}\ c)\ vs2$

$\leftrightarrow \mathbf{evalCmd}\ vs1\ c\ vs2.$

`split; apply optCmd_correct1 || apply optCmd_correct2;`

`assumption.`

`Qed.`