On the Model of Bodner-Partom with Nonhomogeneous Boundary Data

Krzysztof Chełmiński *

Technical University of Darmstadt

and

Piotr Gwiazda †

Warsaw University

Abstract

In the present paper we study existence and the uniqueness of global in time, strong, large solutions to the inelastic model of Bodner–Partom with nonhomogeneous boundary data, and with the perturbation term $A(\frac{y-y_2}{y_1})^r$ in the equation for the isotropic hardening function y. Moreover we consider the limit case $A \longrightarrow 0^+$ and prove the convergence result in a suitable topology to the unperturbed problem.

AMS classification: 35Q72, 73F99

1 Introduction

In this article we study an example of inelastic models of material behaviour of metals. The model was proposed by S.R. Bodner and Y. Partom in [3]. The system of equations, which we consider consists of linear partial differential equations (the balance of momentum) and of nonlinear ordinary differential equations for internal variables (constitutive equations describing the dependence of material on the deformation history). The set of internal variables proposed in the model consists of nine components of the plastic strain tensor ε_p, and of one positive variable y describing isotropic hardening of the metal. We use also the assumption of small deformations, which allows the balance of momentum to be linearized. Moreover we assume that the stress tensor T depends linearly on the elastic part $\varepsilon - \varepsilon_p$ of the strain tensor ε. Thus resulting from these assumptions the initial-boundary value problem, which we consider can be written in the form:
Let $\Omega \subset \mathbb{R}^3$ be an open, connected set with smooth boundary $\partial\Omega$, find the displacement

*Supported by Sonderforschungsbereich 298.
†Supported by DAAD.

vector $u:\Omega\times\mathbb{R}_+\longrightarrow\mathbb{R}^3$, the stress tensor $T:\Omega\times\mathbb{R}_+\longrightarrow\mathbb{R}^{3\times3}_{\mathrm{sym}}$ and the internal variables $(\varepsilon_p,y):\Omega\times\mathbb{R}_+\longrightarrow\mathbb{R}^{3\times3}_{\mathrm{sym}}\times\mathbb{R}_+$ which satisfy the following system of equations

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) &= \mathrm{div}_x T(x,t)+F(x,t)\\
T(x,t) &= \mathcal{D}(\varepsilon(\nabla_x u(x,t))-\varepsilon_p(x,t))\\
\varepsilon(\nabla_x u(x,t)) &= \frac{1}{2}(\nabla_x u(x,t)+(\nabla_x u(x,t))^T)\\
\frac{\partial}{\partial t}\varepsilon_p(x,t) &= \mathcal{G}(\frac{|\sigma(x,t)|}{y(x,t)})\,\frac{\sigma(x,t)}{|\sigma(x,t)|}\\
\frac{\partial}{\partial t}y(x,t) &= \gamma(y(x,t))\mathcal{G}(\frac{|\sigma(x,t)|}{y(x,t)})|\sigma(x,t)|-A\delta(y(x,t))
\end{aligned}\tag{BP}$$

(where $F:\Omega\times\mathbb{R}_+\longrightarrow\mathbb{R}^3$ is a given external force, $\sigma=T-\frac{1}{3}(\mathrm{tr}\,T)\cdot I$ is the stress deviator, $\mathcal{G}:\mathbb{R}_+\to\mathbb{R}_+$, $\gamma:D(\gamma)\subset\mathbb{R}_+\to\mathbb{R}_+$, $\delta:D(\delta)\subset\mathbb{R}_+\to\mathbb{R}_+$ are given functions and $A\geq 0$ is a given material constant)
with the Dirichlet boundary condition

$$u(x,t)_{|\partial\Omega}=b(x,t)$$

and with the initial conditions

$$\begin{aligned}
u(x,0) &= u^0(x)\,,\\
\tfrac{\partial}{\partial t}u(x,0) &= u^1(x)\,,\\
\varepsilon_p(x,0) &= \varepsilon_p^{(0)}(x)\,,\\
y(x,0) &= y^{(0)}(x)\,.
\end{aligned}$$

where $b:\partial\Omega\times\mathbb{R}_+\longrightarrow\mathbb{R}^3$, $u^0:\Omega\longrightarrow\mathbb{R}^3$, $u^1:\Omega\longrightarrow\mathbb{R}^3$, $\varepsilon_p^{(0)}:\Omega\longrightarrow\mathbb{R}^{3\times3}_{\mathrm{sym}}$, $y^{(0)}:\Omega\longrightarrow\mathbb{R}_+$ are given functions.
We see now that the initial boundary-value problem depends on properties of the functions $\mathcal{G},\gamma,\delta$. For example the functions yield all nonlinearities to the system. In [3] Bodner and Partom have proposed functions $\mathcal{G},\gamma,\delta$ in the form

$$\begin{aligned}
\mathcal{G}_{BP}(s) &= d\,\exp\left(-\alpha(\frac{1}{s})^n\right)\quad \alpha=\frac{n/2+1}{n}\\
\gamma_{BP}(y) &= m(y_1-y)\\
\delta_{BP}(y) &= Ay_1(\frac{y-y_2}{y_1})^r
\end{aligned}$$

where $n,r>1$, $d,m>0$, $y_1>y_2>0$ are constants depending on the material considered. For mathematical methods used in this paper we generalize the (BP) equations and require only that the functions $\mathcal{G},\gamma,\delta$ satisfy:

Assumptions

1. $\mathcal{G}(p),\frac{\mathcal{G}(p)}{p}$, are $\mathbb{C}^\infty(\mathbb{R}_+,\mathbb{R}_+)$ functions, $\quad (A1)$
2. $\exists_{g>0}\,\forall_{p\in\mathbb{R}_+}\,\mathcal{G}(p)<g$, ($\mathcal{G}$ is bounded) $\quad (A2)$

3. $\forall_{p\in\mathbb{R}_+}\ \mathcal{G}'(p) \geq 0$ ($\mathcal{G}$ is nondecreasing) $\quad (A3)$

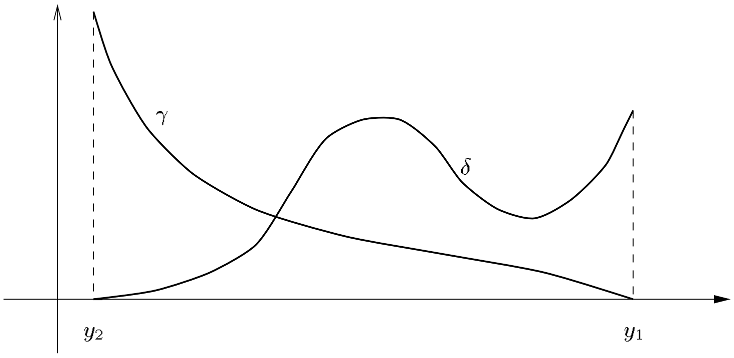
4. $\exists_{g'>0}\ \forall_{p\in\mathbb{R}_+}\ \mathcal{G}'(p)p < g'$. $\quad (A4)$

5. $D(\gamma) = D(\delta) = [y_2, y_1]$ where $0 < y_2 < y_1$ and $\gamma(p), \delta(p)$ are $\mathbb{C}^\infty(D(\gamma), \mathbb{R}_+)$ functions, $\quad (A5)$

6. $\forall_{p\in[y_2,y_1]}\ \gamma'(p) \leq 0$ (γ is nonincreasing) $\quad (A6)$

7. $\gamma(y_2) > \gamma(y_1) = 0$, $\delta(y_2) = 0$ and $\forall_{p\in(y_2,y_1]}\ \delta(p) > 0$ $\quad (A7)$

An example of functions γ, δ satisfying the assumptions $(A5) - (A7)$ is presented in the picture below.



In section 3 we study uniqueness of solutions for the problem (BP) under a stronger assumption than $(A4)$:

4'. $\exists_{g'>0}\ \forall_{p\in\mathbb{R}_+}\ \mathcal{G}'(p)p^2 < g'$. $\quad (A4')$

It is easy to see that the functions $\mathcal{G}_{BP}, \gamma_{BP}, \delta_{BP}$ proposed by Bodner and Partom satisfy all assumptions $(A1) - (A7) + (A4')$. Here we want to point out that the function δ_{BP} is monotone increasing and we don't need the property in our work. This article yields, under the assumptions $(A1)-(A4)$, in the case $A \to 0^+$ existence of strong, global in time, large solutions for the unperturbed problem ($A = 0$). This result is stronger than the existence theorem proved in [5], which requires $(A1)-(A3)+(A4')$ and the homogeneous boundary data. Thus we show that the term $A\delta(y)$ for $A \to 0^+$ yields in all bounded time intervals only small perturbations of the solution for the case ($A = 0$). We prove also that the difference between the solution for the perturbed system and for the unperturbed one in the $\mathbb{L}^2$ topology is proportional to A and depends on the length of the time interval. In the last section of our work we obtain that a similar result does not hold in infinite time intervals. We show namely that for $t \to \infty$ the solutions y_A to a initial data y^0 from the interval (y_2, y_1) tend to y_2 and it is easy to see that the solution y for the case $A = 0$ is a nondecreasing function in time. Moreover we prove the uniqueness theorem for all initial data, for which we have the existence result. In the literature this problem was studied in [6] for the quasistatic case and in [7] for the dynamical system but in the both articles the uniqueness question was only partially answered .
In our article we denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the standard $\mathbb{L}^2$ norm and scalar product respectively. Moreover by $\|\cdot\|_s$ we denote the norm in the Sobolev space $\mathbb{H}^s$.

2 Existence theorem for all $A > 0$

We start with an existence theory for the perturbed case $A > 0$. The main idea is to construct an approximate sequence of solutions using the Galerkin method and to prove suitable energy estimates for this sequence.

2.1 Transformation to a homogeneous boundary-value problem

We want to begin this subsection with an apriori $\mathbb{L}^\infty$ estimate for the isotropic hardening function y. This result is very important for the further energy estimates. In the articles [4, 5, 6, 7, 8] such estimates were proved with an additional assumption that the initial data y^0 is uniformly bounded from below by a positive constant. In the perturbed (BP) problem this constant is determined by the special form of the equation (BP5).

Lemma 2.1
If we assume that the function σ is continuous then for all $\mathbb{C}^1$-solutions $y(x,t)$ of the equation $(BP5)$ with the initial data satisfying $y_2 \le y^{(0)}(x) \le y_1$ we have the estimation $y_2 \le y(x,t) \le y_1$.
Proof. Let us assume $|\sigma(x,t)|$ be a given function. Thus for each $x \in \Omega$, (BP5) represents an ordinary differential equation for the function $y(x,\cdot)$. For the right hand side of the equation we have

$$\gamma(y_2)\mathcal{G}\left(\frac{|\sigma(x,t)|}{y_2}\right)|\sigma(x,t)| - A\delta(y_2) \ge 0$$

and

$$\gamma(y_1)\mathcal{G}\left(\frac{|\sigma(x,t)|}{y_1}\right)|\sigma(x,t)| - A\delta(y_1) < 0.$$

Hence the solution $y(x,t)$ cannot leave out the interval $[y_2, y_1]$.

We are now going to prove existence of solutions for the problem (BP) with the energy method. Thus first we should transform our problem to a homogeneous boundary-value problem. Let us define $\tilde{u}(x,t), \tilde{T}(x,t)$ as the unique solution of the following linear system of equations

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\tilde{u}(x,t) &= \operatorname{div}_x \tilde{T}(x,t) + F(x,t) \\ \tilde{T}(x,t) &= \mathcal{D}(\varepsilon(\nabla_x \tilde{u}(x,t))) \\ \varepsilon(\nabla_x \tilde{u}(x,t)) &= \frac{1}{2}(\nabla_x \tilde{u}(x,t) + (\nabla_x \tilde{u}(x,t))^T) \end{aligned} \tag{LP}$$

with the Dirichlet boundary condition

$$\tilde{u}(x,t)_{|\partial\Omega} = b(x,t)$$

and with smooth initial conditions

$$\tilde{u}(x,0) = \tilde{u}^0(x)\,, \qquad \tilde{u}_t(x,0) = \tilde{u}^1(x)$$

compatible with the boundary data

$$\tilde{u}^0(x)_{|\partial\Omega} = b(x,0)\,, \qquad \tilde{u}^1(x)_{|\partial\Omega} = b_t(x,0)\,.$$

Lemma 2.2
Let the function F belong to the space $\mathbb{W}^1((0,T);\mathbb{L}^2(\Omega))$, b belong to the space $\mathbb{W}^{1,\infty}((0,T);\mathbb{H}^{\frac{3}{2}}(\partial\Omega))$, and the second time derivative b_{tt} belong to $\mathbb{W}^{1,\infty}((0,T);\mathbb{L}^2(\partial\Omega))$ for all $T>0$. Then for the initial data $(\tilde{u}^0, \tilde{u}^1)$ from the space $\mathbb{H}^2(\Omega)\times\mathbb{H}^1(\Omega)$ compatible with the boundary data the solution $\tilde{u}$ of the problem (LP) belongs to $\mathbb{L}^\infty((0,T);\mathbb{H}^2(\Omega))$ and the second time derivative $\tilde{u}_{tt}$ belongs to $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega))$. Moreover the norms are bounded by the given functions F, b, $\tilde{u}^0$, $\tilde{u}^1$:

$$
\begin{aligned}
\|\tilde{u}(t)\|_2^2 + \|\tilde{u}_{tt}(t)\|^2 \;\leq\; & C(T)\Big(\|F(t)\|^2 + \int_0^t (\|F_t(\tau)\|^2)d\tau \\
& + \|B(t)\|_2^2 + \|B_{tt}(t)\|^2 + \int_0^t (\|B_t(\tau)\|_2^2 + \|B_{ttt}(\tau)\|^2)d\tau \\
& + \|F(0)\|^2 + \|B(0)\|_2^2 + \|B_t(0)\|_1^2 + \|B_{tt}(0)\|^2 \\
& + \|\tilde{u}^0\|_2^2 + \|\tilde{u}^1\|_1^2\Big)
\end{aligned}
$$

where $B(x,t)$ is an extension of $b(x,t)$ into Ω compatibile up to the third time derivative.

The proof of the lemma is based on the same energy estimates, which we are going to prove for the nonlinear problem (BP). Therefore we shall omit the proof.
If we assume that u, T are solutions for the problem (BP) then the differences $u-\tilde{u} =: \hat{u}$, $T-\tilde{T} =: \hat{T}$ satisfy

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\hat{u}(x,t) &= \operatorname{div}_x \hat{T}(x,t) \\
\hat{T}(x,t) &= \mathcal{D}(\varepsilon(\nabla_x \hat{u}(x,t)) - \varepsilon_p(x,t)) \\
\varepsilon(\nabla_x\hat{u}(x,t)) &= \frac{1}{2}(\nabla_x\hat{u}(x,t) + (\nabla_x\hat{u}(x,t))^T) \\
\frac{\partial}{\partial t}\varepsilon_p(x,t) &= \mathcal{G}\left(\frac{|\sigma(x,t)|}{y(x,t)}\right)\frac{\sigma(x,t)}{|\sigma(x,t)|} \\
\frac{\partial}{\partial t}y(x,t) &= \gamma(y(x,t))\mathcal{G}\left(\frac{|\sigma(x,t)|}{y(x,t)}\right)|\sigma(x,t)| - A\delta(y(x,t))
\end{aligned}
\tag{HBP}
$$

(here $\sigma = \tilde{T}+\hat{T}-\frac{1}{3}\mathrm{tr}\,(\tilde{T}+\hat{T})I)$ with the homogeneous Dirichlet boundary condition, and with the initial conditions

$$
\begin{aligned}
\hat{u}(x,0) &= u^0(x) - \tilde{u}^0(x) =: \hat{u}^0(x), \\
\tfrac{\partial}{\partial t}\hat{u}(x,0) &= u^1(x) - \tilde{u}^1(x) =: \hat{u}^1(x), \\
\varepsilon_p(x,0) &= \varepsilon_p^{(0)}(x), \\
y(x,0) &= y^{(0)}(x).
\end{aligned}
$$

We are now going to use the energy method for the problem (HBP). Thus we want to approximate a solution of the problem by the Galerkin sequence of smooth functions. Therefore first we regularize the function $\mathcal{G}$. This procedure is similar to that used in [4, 5, 8]. We define for all $k\in\mathbb{N}$ the function $\mathcal{G}_k : [0,\infty)\to[0,\infty)$ by $\mathcal{G}_k(p) = \chi(kp)\mathcal{G}(p)$

where $\chi \in \mathbb{C}^\infty(\mathbb{R})$ is a cut-off function with $\chi(p) = 0$ for $p < \frac{1}{2}$, $\chi(p) = 1$ for $p > 1$, $0 \leq \chi \leq 1$, $\chi' \geq 0$ and $p \cdot \chi'(p) \leq \chi_0$ for all $p \in \mathbb{R}_+$, (χ_0 is a positive constant). Note that for all k the function $\mathcal{G}_k$ satisfies the assumptions $(A1) - (A4) + (A4')$ where the constants g_k and g'_k for $k \to \infty$ converge to g and g' respectively (see [4, 5]).
Moreover we assume that the given functions F, b and the initial data $\varepsilon_p^{(0)}$ and $y^{(0)}$ are $\mathbb{C}^\infty$ functions. For the function $\varepsilon_p^{(0)}$ we additionally require to be symmetric and to satisfy for all $x \in \Omega$ the equality $\mathrm{tr}\,\varepsilon_p^{(0)} = 0$. This yields immediately that all solutions ε^p of the equation (HBP4) have the same properties. Now by the standard Galerkin procedure we define a sequence of smooth approximate solutions $(\hat{u}_k, \varepsilon_k^p, y_k)$ of the problem (HBP) where in the k-th step the function $\mathcal{G}$ is replaced by $\mathcal{G}_k$ (see [4, 5, 8]). Let us denote by $[0, T_k^{max})$ the time interval on which the functions $(\hat{u}_k, \varepsilon_k^p, y_k)$ are defined. The energy estimate proved below shows, that $T_k^{max} = \infty$.

2.2 Energy estimate

For the sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ we define the energy function :

$$\mathcal{E}(\hat{u}_k, \varepsilon_k^p, y_k)(t) = \frac{1}{2}(\rho\partial_t\hat{u}_k(t), \partial_t\hat{u}_k(t)) + \frac{1}{2}(\hat{T}_k(t), \hat{\varepsilon}_k(t) - \varepsilon_k^p(t)).$$

The first term in the above sum is the kinetic energy and the second one describes the potential energy. Using the properties of the elastic tensor $\mathcal{D}$ we obtain that there exists a positive constant d such that

$$\mathcal{E}(\hat{u}_k, \varepsilon_k^p, y_k)(t) \geq \frac{\rho}{2}\|\partial_t\hat{u}_k(t)\|^2 + \frac{d}{2}\|\hat{\varepsilon}_k(t) - \varepsilon_k^p(t))\|^2.$$

We will prove now the boundedness in time of the energy function. For the homogeneous and unperturbed problem (see [4, 5]) the energy was a nondecreasing function. In the nonhomogeneous case the energy will be controlled additionally by the given external force and by the boundary data.

Lemma 2.3
The energy function $\mathcal{E}(t)$ defined for the sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ is bounded in time and moreover for all $t < T_k^{max}$ the following inequality holds

$$\begin{aligned}\mathcal{E}(\hat{u}_k, \varepsilon_k^p, y_k)(t) \quad &+ \quad \frac{1}{2}\int_0^t\int_\Omega \frac{\mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}_k|}{y_k}\right)}{|\hat{\sigma}_k + \tilde{\sigma}_k|}|\hat{\sigma}_k|^2 dx d\tau \\ &\leq \quad \mathcal{E}(\hat{u}_k, \varepsilon_k^p, y_k)(0) + \frac{1}{2}\left(\sup_{p \geq 0}\frac{\mathcal{G}_k(p)}{p}\right)\int_0^t\int_\Omega |\tilde{\sigma}_k|^2 dx d\tau\end{aligned} \tag{EI}$$

where $\tilde{\sigma}_k = \tilde{T}_k - \frac{1}{3}\mathrm{tr}\,(\tilde{T}_k)I$ and $\hat{\sigma}_k = \hat{T}_k - \frac{1}{3}\mathrm{tr}\,(\hat{T}_k)I$.

Proof. We note first that the sequence of initial energies $\mathcal{E}(\hat{u}_k, \varepsilon_k^p, y_k)(0)$ tends to $\mathcal{E}(\hat{u}^0, \varepsilon_p^{(0)}, y^{(0)})$ where $\hat{T}(0) = \mathcal{D}(\varepsilon(\nabla_x\hat{u}^0) - \varepsilon_p^{(0)})$ (see [4]) and thus is bounded. Calculating the time derivative of $\mathcal{E}(t)$ we obtain similarly as in [4, 5, 8]

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\hat{u}_k, \varepsilon^p_k, y_k)(t) &= (\rho\partial^2_t\hat{u}_k(t), \partial_t\hat{u}_k(t)) + (\hat{T}_k(t), \partial_t\hat{\varepsilon}_k(t)) - (\hat{T}_k(t), \partial_t\varepsilon^p_k(t))\\ &= (\rho\partial^2_t\hat{u}_k(t), \partial_t\hat{u}_k(t)) + (\hat{T}_k(t), \nabla\partial_t\hat{u}_k(t)) - (\hat{\sigma}_k(t), \partial_t\varepsilon^p_k(t))\\ &= -(\hat{\sigma}_k(t), \partial_t\varepsilon^p_k(t)).\end{aligned}$$

Using the equation (HBP4) for the function $\partial_t\varepsilon^p$ we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\hat{u}_k, \varepsilon^p_k, y_k)(t) &= -\int_\Omega \mathcal{G}_k\left(\frac{|\hat{\sigma}_k+\tilde{\sigma}|}{y_k}\right)\frac{\hat{\sigma}_k+\tilde{\sigma}}{|\hat{\sigma}_k+\tilde{\sigma}|}\cdot\hat{\sigma}_k dx\\ &= -\int_\Omega \frac{\mathcal{G}_k\left(\frac{|\hat{\sigma}_k+\tilde{\sigma}|}{y_k}\right)}{|\hat{\sigma}_k+\tilde{\sigma}|}|\hat{\sigma}_k|^2dx - \int_\Omega \frac{\mathcal{G}_k\left(\frac{|\hat{\sigma}_k+\tilde{\sigma}|}{y_k}\right)}{|\hat{\sigma}_k+\tilde{\sigma}|}\hat{\sigma}_k\cdot\tilde{\sigma}dx\\ &\le -\frac{1}{2}\int_\Omega \frac{\mathcal{G}_k\left(\frac{|\hat{\sigma}_k+\tilde{\sigma}|}{y_k}\right)}{|\hat{\sigma}_k+\tilde{\sigma}|}|\hat{\sigma}_k|^2dx + \frac{1}{2y_2}\left(\sup_{p\ge 0}\frac{\mathcal{G}_k(p)}{p}\right)\int_\Omega|\tilde{\sigma}|^2dx.\end{aligned}$$

Integrating the last inequality over the time interval $(0,t)$ we complete the proof of the inequality (EI). By the assumptions $(A1)$ and $(A2)$ the functions sequence $\frac{\mathcal{G}_k(p)}{p}$ is uniformly bounded with respect to k and from lemma 2.2 $\tilde{\sigma}$ belongs to $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega))$ for all $T>0$. Hence we have immediately boundedness of the energy function and this denotes global in time definiteness of the approximate sequence (see [4, 5, 8]).

2.3 Estimates of $t-$derivatives

In this subsection we are going to prove a similar result for the time derivatives $(\partial_t\hat{u}_k, \partial_t\varepsilon^p_k, \partial_t y_k)$. This is equivalent to the $\mathbb{L}^2$ boundedness of the left hand sides of the differential equations from the system (HBP). We want to point out that we prove the lemma below with the assumption $(A4)$ instead $(A4')$, used in [4, 5, 8]. The energy function for the time derivatives is given by the formula:

$$\mathcal{E}(\partial_t\hat{u}_k, \partial_t\varepsilon^p_k, \partial_t y_k)(t) = \frac{1}{2}(\rho\partial^2_t\hat{u}_k(t), \partial^2_t\hat{u}_k(t)) + \frac{1}{2}(\partial_t\hat{T}_k(t), \partial_t\hat{\varepsilon}_k(t) - \partial_t\varepsilon^p_k(t)).$$

This is the same function as in the subsection above and only the arguments are changed. Therefore we use the same name "energy function", even the function does not describes a physical energy.

Lemma 2.4
There exists a positive constant $C_1(T)$ defined for all positive T, such that for all k it holds that:

$$\begin{aligned}\mathcal{E}(\partial_t\hat{u}_k, \partial_t\varepsilon^p_k, \partial_t y_k)(t) &\le C_1(T)\,(\mathcal{E}(\partial_t\hat{u}_k, \partial_t\varepsilon^p_k, \partial_t y_k)(0)\\ &\quad + \int_0^t\|\hat{\sigma}_k\|^2d\tau + \int_0^t\|\partial_t\tilde{\sigma}\|^2d\tau + \int_0^t\|\tilde{\sigma}\|^2d\tau)\end{aligned}$$

Moreover $C_1(T)$ does not depend on k.

Proof.

A similar calculation to that used in the lemma above yields:

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\partial_t \hat{u}_k, \partial_t \varepsilon^p_k, \partial_t y_k)(t)\\
&= (\rho\partial_t^3 \hat{u}_k(t), \partial_t^2 \hat{u}_k(t)) + (\partial_t \hat{T}_k(t), \partial_t^2 \hat{\varepsilon}_k(t)) - (\partial_t \hat{T}_k(t), \partial_t^2 \varepsilon^p_k(t))\\
&= (\rho\partial_t^3 \hat{u}_k(t), \partial_t^2 \hat{u}_k(t)) + (\partial_t \hat{T}_k(t), \partial_t^2 \nabla \hat{u}_k(t)) - (\partial_t \hat{\sigma}_k(t), \partial_t^2 \varepsilon^p_k(t))\\
&= -(\partial_t \hat{\sigma}_k(t), \partial_t^2 \varepsilon^p_k(t))
\end{aligned}
\tag{2.1}
$$

By equation (HBP5) using properties of the function $\mathcal{G}_k$ we obtain:

$$
\begin{aligned}
\partial_t^2 \varepsilon^p \quad &= \quad \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\left(\frac{\partial_t|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\frac{\hat{\sigma}_k + \tilde{\sigma}}{|\hat{\sigma}_k + \tilde{\sigma}|} - \frac{\partial_t y_k(\hat{\sigma}_k + \tilde{\sigma})}{y^2}\right)\\
&\quad + \mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\left(\frac{\partial_t(\hat{\sigma}_k + \tilde{\sigma})}{|\hat{\sigma}_k + \tilde{\sigma}|} - \frac{(\hat{\sigma}_k + \tilde{\sigma})\partial_t|\hat{\sigma}_k + \tilde{\sigma}|}{|\hat{\sigma}_k + \tilde{\sigma}|^2}\right).
\end{aligned}
$$

Inserting the last inequality into (2.1) and using equation (HBP5) for the isotropic hardening function we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\partial_t \hat{u}_k, \partial_t \varepsilon^p_k, \partial_t y_k)(t)\\
&\quad -\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{(\hat{\sigma}_k + \tilde{\sigma})\cdot(\partial_t \hat{\sigma}_k + \partial_t \tilde{\sigma})(\hat{\sigma}_k + \tilde{\sigma})\cdot \partial_t \hat{\sigma}_k}{y_k|\hat{\sigma}_k + \tilde{\sigma}|^2}dx\\
&\quad +\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{\partial_t y_k(\hat{\sigma}_k + \tilde{\sigma})}{y_k^2}\cdot \partial_t \hat{\sigma}_k dx\\
&\quad -\int_\Omega \mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\left(\frac{\partial_t(\hat{\sigma}_k + \tilde{\sigma})}{|\hat{\sigma}_k + \tilde{\sigma}|} - \frac{(\hat{\sigma}_k + \tilde{\sigma})\partial_t|\hat{\sigma}_k + \tilde{\sigma}|}{|\hat{\sigma}_k + \tilde{\sigma}|^2}\right)\cdot \partial_t \hat{\sigma}_k dx\\
\leq &\quad -\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{[(\hat{\sigma}_k + \tilde{\sigma})\cdot \partial_t \hat{\sigma}_k]^2}{y_k|\hat{\sigma}_k + \tilde{\sigma}|^2}dx\\
&\quad -\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{(\hat{\sigma}_k + \tilde{\sigma})\cdot \partial_t \tilde{\sigma}(\hat{\sigma}_k + \tilde{\sigma})\cdot \partial_t \hat{\sigma}_k}{y_k|\hat{\sigma}_k + \tilde{\sigma}|^2}dx\\
&\quad +\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\gamma(y_k)\mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{|\hat{\sigma}_k + \tilde{\sigma}|^2}{y_k^2}|\partial_t \hat{\sigma}_k|dx\\
&\quad +\int_\Omega \mathcal{G}'_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k^2}A\delta(y)|\partial_t \hat{\sigma}_k|dx\\
&\quad +2\int_\Omega \mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{|\partial_t \hat{\sigma}_k||\partial_t \tilde{\sigma}|}{|\hat{\sigma}_k + \tilde{\sigma}|}dx + \int_\Omega \mathcal{G}_k\left(\frac{|\hat{\sigma}_k + \tilde{\sigma}|}{y_k}\right)\frac{|\partial_t \hat{\sigma}_k|^2}{|\hat{\sigma}_k + \tilde{\sigma}|}dx
\end{aligned}
$$

The first term in the right hand side of the last inequality is nonpositive. In the second one we estimate the product $((\hat\sigma_k+\tilde\sigma)\cdot\partial_t\tilde\sigma)((\hat\sigma_k+\tilde\sigma)\cdot\partial_t\hat\sigma_k)$ by $\frac{1}{2}|\hat\sigma_k+\tilde\sigma|^2|\partial_t\tilde\sigma|^2+\frac{1}{2}|\hat\sigma_k+\tilde\sigma|^2|\partial_t\hat\sigma_k|^2$ and the part containing $|\partial_t\hat\sigma_k|^2$ can be absorbed by the first nonpositive term. In the third and fourth integrals we use the following simple inequality $|\hat\sigma_k+\tilde\sigma||\partial_t\hat\sigma_k|\le(\frac{1}{2}|\hat\sigma_k|^2+\frac{1}{2}|\tilde\sigma|^2+|\partial_t\hat\sigma_k|^2)$. Thus together we obtain

$$\frac{d}{dt}\mathcal{E}(\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k)(t)\le\Big(\sup_{p\ge0}\frac{\mathcal{G}_k'(p)}{2y_2}+\sup_{p\ge0}\frac{\mathcal{G}_k(p)}{p}\Big)\int_\Omega|\partial_t\tilde\sigma|^2dx$$
$$+\Big(\sup_{p\ge0}\mathcal{G}_k'(p)p\mathcal{G}_k(p)\frac{(\gamma(y_2)}{y_2}+A\sup_{p\ge0}\mathcal{G}_k'(p)\sup_{y\in[y_2,y_1]}\frac{\delta(y)}{y^2}+2\sup_{p\ge0}\frac{\mathcal{G}_k(p)}{p}\Big)\int_\Omega|\partial_t\hat\sigma_k|^2dx$$
$$+\Big(\sup_{p\ge0}\mathcal{G}_k'(p)p\mathcal{G}_k(p)\frac{(\gamma(y_2)}{y_2}+A\sup_{p\ge0}\mathcal{G}_k'(p)\sup_{y\in[y_2,y_1]}\frac{\delta(y)}{y^2}\Big)\Big(\int_\Omega|\hat\sigma_k|^2dx+\int_\Omega|\tilde\sigma|^2dx\Big)$$

Hence there exists a positive constant C such that

$$\frac{d}{dt}\mathcal{E}(\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k)(t)\le C\left(\mathcal{E}(\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k)(t)+\|\partial_t\tilde\sigma(t)\|^2+\|\hat\sigma_k(t)\|^2+\|\tilde\sigma(t)\|^2\right).$$

Gronwall's lemma completes the proof.

The sequence $\mathcal{E}(\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k)(0)$ converges to $\mathcal{E}(\hat u^1,\varepsilon_p^{(1)},y^{(1)})$ where $\hat u^1$ is the initial velocity and $\varepsilon_p^{(1)}$, $y^{(1)}$ are calculated from the differential equations (HBP4)+(HBP5) with the initial data $\hat u^0,\varepsilon_p^{(0)},y^{(0)}$ (see [4]). The function $\tilde\sigma$ belongs to $\mathbb{W}^{1,\infty}((0,T);\mathbb{L}^2(\Omega))$ (lemma 2.2) and by lemma 2.3 the functions sequence $\hat\sigma_k$ is bounded in $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega))$. Thus lemma 2.4 yields the boundedness in time uniformly with respect to k of the energy function for the time derivatives $\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k$.

2.4 Estimates of $x-$derivatives

Lemma 2.4 gives us $\mathbb{L}^2$ bounds of the left hand side of the partial differential equation (HBP1). Thus in the manner of [5, 8] we want to prove estimates of $x-$derivatives of the functions $(\hat u_k,\varepsilon^p_k,y_k)$ using ellipticity of the operator $\operatorname{div}\mathcal{D}(\frac{1}{2}(\nabla+\nabla^T))$. The main problem (compare with [5]) is to obtain $\mathbb{L}^2$ estimates for the gradient of the stress tensor T. These are derived in the following theorem.

Theorem 2.1 *For the constructed approximate sequence $(\hat u_k,\varepsilon^p_k,y_k)$ there exists a positive, undepended of k constant $C_2(T)$ defined for all positive T, such that for all $t\in[0,T)$ it holds that:*

$$\|\nabla\hat T_k(t)\|^2\le\ C_2(T)\ \Big(\|\nabla\hat T(0)\|^2+\mathcal{E}(\partial_t\hat u_k,\partial_t\varepsilon^p_k,\partial_t y_k)(t)+\|\nabla y^{(0)}\|^2+2\|\nabla\varepsilon_p^{(0)}\|^2$$
$$+\ \int_0^t\|\nabla\tilde\sigma\|^2d\tau\Big)$$

Proof.
For the clarity of the proof we shall divide it into four steps.
Step 1: First we are going to estimate $x-$derivatives of the isotropic hardening function

y_k.

Claim 1 *For the approximate sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ we have the following estimation*

$$\|\partial_{x_i} y_k(t)\|^2 \le \|\partial_{x_i} y^{(0)}\|^2 + C_3 \int\limits_0^t \left(\|\partial_{x_i}\sigma_k\|^2 + \|\partial_{x_i} y_k\|^2\right) d\tau$$

for $i = 1,2,3$ where the positive constant C_3 does not depend on k and $\sigma_k = \hat{\sigma}_k + \tilde{\sigma}$.

Proof of Claim 1 We calculate the x_i-derivative of equation $(P6)$:

$$\begin{aligned}\partial_t\partial_{x_i} y_k &= \partial_{x_i} y_k \gamma'(y_k)\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right)|\sigma_k| + \gamma(y_k)\mathcal{G}_k'\left(\frac{|\sigma_k|}{y_k}\right)|\sigma_k|\frac{\partial_{x_i}|\sigma_k| y_k - \partial_{x_i} y_k|\sigma_k|}{y_k^2}\\ &\quad + \partial_{x_i}|\sigma_k|\gamma(y_k)\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right) - A\partial_{x_i} y_k\delta'(y_k).\end{aligned}$$

Multiplication by $\partial_{x_i} y_k$, and integration over Ω yields:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_{x_i} y_k\|^2 &= \int\limits_\Omega (\partial_{x_i} y_k)^2\gamma'(y_k)\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right)|\sigma_k| dx\\ &- \int\limits_\Omega (\partial_{x_i} y_k)^2\gamma(y_k)\mathcal{G}_k'\left(\frac{|\sigma_k|}{y_k}\right)\frac{|\sigma_k|^2}{y_k^2}dx - A\int\limits_\Omega (\partial_{x_i} y_k)^2\delta'(y_k)dx\\ &+ \int\limits_\Omega \partial_{x_i} y_k\partial_{x_i}|\sigma_k|\gamma(y_k)\mathcal{G}_k'\left(\frac{|\sigma_k|}{y_k}\right)\frac{|\sigma_k|}{y_k}dx + \int\limits_\Omega \partial_{x_i} y_k\partial_{x_i}|\sigma_k|\gamma(y_k)\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right)dx\end{aligned}$$

The first two integrals by the assumptions $(A6)$ and $(A3)$ are nonpositive (!). Hence using Young's inequality we obtain

$$\begin{aligned}\frac{d}{dt}\|\partial_{x_i} y_k\|^2 \;&\le\; \gamma(y_2)\Big(\sup_{p\ge 0}\mathcal{G}_k'(p)p + \sup_{p\ge 0}\mathcal{G}_k(p)\Big)\left(\|\partial_{x_i} y_k\|^2 + \|\partial_{x_i}\sigma_k\|^2\right)\\ &\quad + \sup_{y\in[y_2,y_1]}|\delta'(y)|\;\|\partial_{x_i} y_k\|^2 .\end{aligned}$$

Now by the assumptions $(A4)$ and $(A2)$ and by Gronwall's lemma we end the proof of Claim 1.

Step 2: Next we want to obtain a similar estimation to that proved above for the inelastic stress tensor ε_k^p.

Claim 2 *The approximate sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ for the problem (HBP) for $i = 1,2,3$ satisfies the inequality*

$$\|\partial_{x_i}\varepsilon_k^p(t)\|^2 \le 2\|\partial_{x_i}\varepsilon_p^{(0)}\|^2 + tC_4\int\limits_0^t \left(\|\partial_{x_i}\sigma_k\|^2 + \|\partial_{x_i} y_k\|^2\right) d\tau$$

where the positive constant C_4 does not depend on k and $\sigma_k = \hat{\sigma}_k + \tilde{\sigma}$.
Proof of Claim 2 For $i = 1, 2, 3$ by (HBP5) we obtain:

$$\begin{aligned}
\|\partial_{x_i}\varepsilon_k^p\|^2 &\leq \int_\Omega \Big| \int_0^t \partial_{x_i}\left(\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right)\frac{\sigma_k}{|\sigma_k|}\right) d\tau + \partial_{x_i}\varepsilon_p^{(0)}\Big|^2 dx \\
&\leq 2t \int_0^t \left\|\partial_{x_i}\left(\mathcal{G}_k\left(\frac{|\sigma_k|}{y_k}\right)\frac{\sigma_k}{|\sigma_k|}\right)\right\|^2 d\tau + 2\left\|\partial_{x_i}\varepsilon_p^{(0)}\right\|^2 .
\end{aligned}$$

Thus in the manner of [5] we have

$$\begin{aligned}
\|\partial_{x_i}\varepsilon_k^p\|^2 \leq\; & 2\|\partial_{x_i}\varepsilon_p^{(0)}\|^2 \\
+\; & 8t\int_0^t \Big[\Big((\sup_{p\geq 0}\frac{\mathcal{G}_k'(p)}{y_2})^2 + (\sup_{p\geq 0}\frac{\mathcal{G}_k(p)}{p})^2\Big)\|\partial_{x_i}\sigma_k\|^2 + (\sup_{p\geq 0}\frac{\mathcal{G}_k'(p)p}{(y_2)^2})^2\|\partial_{x_i}y_k\|^2\Big] d\tau
\end{aligned}$$

The assumptions $(A1)$, $(A2)$ and $(A4)$ end the proof of Claim 2.

Step 3: If we assume that u_{tt} be a given $\mathbb{L}^2$ function and ε_k^p a given $\mathbb{H}^1$ function than for solutions u of the equation (HBP1) we have the usual $\mathbb{H}^2$ regularity. Unfortunately our sequence $\hat{u}_k$ satisfies (HBP1) only in an approximate sense (see [4, 5, 8]). Therefore next we shall prove $\mathbb{H}^2$ regularity for the sequence $\hat{u}_k$.
Claim 3 *If for all k the acceleration vector $\partial_t^2\hat{u}_k$ belongs to $\mathbb{L}^\infty((0,T);\mathbb{L}(\Omega))$ and the inelastic strain tensor ε_k^p belongs to $\mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega))$ then $\hat{u}_k \in \mathbb{L}^\infty((0,T);\mathbb{H}^2(\Omega))$ and the following estimate holds*

$$\|\hat{u}_k(t)\|_2^2 \leq C_5\left(\|\nabla\varepsilon_k^p(t)\|^2 + \|\partial_t^2\hat{u}_k(t)\|^2\right)$$

with the positive constant C_5, independed of k.
The claim is proved in [5].

Step 4: Adding the inequalities derived in the claims 1 and 2 we have

$$\|\partial_{x_i}y_k(t)\|^2 + \|\partial_{x_i}\varepsilon_k^p(t)\|^2 \leq \|\partial_{x_i}y^{(0)}\|^2 + 2\|\partial_{x_i}\varepsilon_p^{(0)}\|^2 + (C_3 + tC_4)\int_0^t (\|\partial_{x_i}\sigma_k\|^2 + \|\partial_{x_i}y_k\|^2) d\tau$$

and using Gronwall's inequality we obtain $\mathbb{L}^2$ bounds for the inelastic strain tensor ε_k^p

$$\begin{aligned}
&\|\partial_{x_i}\varepsilon_k^p(t)\|^2 \leq \|\partial_{x_i}y_k(t)\|^2 + \|\partial_{x_i}\varepsilon_k^p(t)\|^2 \\
&\quad\leq e^{\int_0^t (C_3+\tau C_4)d\tau}\left(\|\partial_{x_i}y^{(0)}\|^2 + 2\|\partial_{x_i}\varepsilon_p^{(0)}\|^2 + (C_3 + tC_4)\int_0^t \|\partial_{x_i}\sigma_k\|^2 d\tau\right).
\end{aligned}$$

Finally differentiation with respect to x of the equation (HBP2) yields that there exists a positive constant D such that

$$\|\nabla\hat{T}_k(t)\|^2 \;\leq\; D(\|\hat{u}_k(t)\|_2^2 + \|\nabla\varepsilon_k^p(t)\|^2).$$

The last inequality, Claim 3 and the inequality (2.2) give

$$\|\nabla\hat{T}_k(t)\|^2 \leq D(t)\Big(\|\partial_t^2\hat{u}(t)\|^2 + \|\nabla y^{(0)}\|^2 + 2\|\nabla\varepsilon_p^{(0)}\|^2\int_0^t \|\nabla\tilde{\sigma}\|^2 d\tau + \int_0^t \|\nabla\hat{T}_k(t)\|^2 d\tau\Big)$$

where the constant $D(t)$ depends on D, C_3, C_4, C_5 and does not depend on k. Gronwall's inequality used again completes the proof of the theorem.

Theorem 2.1 ends the energy estimates. The constructed approximate sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ with all first derivatives is bounded in the space $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega))$ where $T > 0$ is arbitrary. Hence we are ready to prove a general existence theorem for all $A > 0$.

Theorem 2.2
Assume that the initial data satisfy

$$\hat{u}^0 \in \mathbb{H}^2(\Omega)\cap\mathbb{H}_0^1(\Omega), \quad \hat{u}^1 \in \mathbb{H}_0^1(\Omega), \quad \varepsilon_p^{(0)} \in \mathbb{H}^1(\Omega), \quad y^{(0)} \in \mathbb{H}^1(\Omega)$$

with $\varepsilon_p^{(0)}$ symmetric and with $\mathrm{tr}\,(\varepsilon_p^{(0)}(x)) = 0$ for almost all $x \in \Omega$. For the initial data $y^{(0)}$ we require additionally that for almost all $x \in \Omega$ $y_2 \leq y^{(0)}(x) \leq y_1$. Moreover suppose that the given functions F and b have the regularity required in lemma 2.2 and the assumptions $(A1) - (A7)$ hold. Then there exists a global in time, strong solution $(u, \varepsilon_p, y) \in \mathbb{H}^2((0,T)\times\Omega) \times \mathbb{H}^1((0,T)\times\Omega) \times \mathbb{H}^1((0,T)\times\Omega)$ of the problem (HBP).

Proof. First we regularize the given data $(\varepsilon_p^{(0)}, y^{(0)}, F, b)$ by a smooth sequence $(\varepsilon_p^{(0),j}, y^{(0),j}, F^j, b^j)$, which converges to $(\varepsilon_p^{(0)}, y^{(0)}, F, b)$ in the topology required above. For the sequence $(\varepsilon_p^{(0),j}, y^{(0),j}, F^j, b^j)$ we construct the Galerkin sequence $(\hat{u}_{jk}, \varepsilon_{jk}^p, y_{jk})$ of approximate smooth solutions of the problem (HBP). The energy estimates proved in lemma 2.3, lemma 2.4 and in theorem 2.1 yield that for all $T > 0$ the twice indexed sequence $(\hat{u}_{jk}, \varepsilon_{jk}^p, y_{jk})$ with all first derivatives is bounded in the space $\mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega))$. Hence by the compactness method we can select a subsequence, which converges strong to (u, ε^p, y) in the space $\mathbb{L}^2((0,T)\times\Omega)$. Thus in the manner of [5, 8] we obtain that (u, ε^p, y) is a strong solution of the problem (HBP).
Finally we shall remark that the constructed solution (u, ε^p, y) has the following regularity:

$$u \in \mathbb{L}^\infty((0,T);\mathbb{H}^2(\Omega)), \quad u_t \in \mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega)), \quad u_{tt} \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega)),$$

$$\varepsilon^p \in \mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega)), \quad \varepsilon_t^p \in \mathbb{L}^\infty((0,T)\times\Omega),$$

$$y \in \mathbb{L}^\infty((0,T);\mathbb{H}^1(\Omega)), \quad y_t \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega)).$$

Moreover the isotropic hardening function y belongs to $\mathbb{L}^\infty((0,\infty)\times\Omega)$ and for almost all $x \in \Omega$ satisfies $y_2 \leq y(x) \leq y_1$.

Remark At the end of the section we want to remark that for F, b having regularity from lemma 2.2 and for initial data $u^0 \in \mathbb{H}^2(\Omega)$, $u^1 \in \mathbb{H}^1(\Omega)$, $\varepsilon_p^{(0)} \in \mathbb{H}^1(\Omega)$ and $y^{(0)} \in \mathbb{H}^1(\Omega)$ compatibile with the boundary data $u^0_{|\partial\Omega} = b(0)$, $u^1_{|\partial\Omega} = b_t(0)$, theorem 2.2 with lemma 2.2 yield the existence of global in time strong solutions of the problem (BP). Note that the initial data $\hat{u}^0 = u^0 - \tilde{u}^0$, $\hat{u}^1 = u^1 - \tilde{u}^1$ are compatible with the homogeneous (equal to zero) boundary data.

3 Uniqueness theorem and the limit case $A \longrightarrow 0$

In this section we are going to prove the uniqueness result for solutions of the problem (BP) constructed in the section above. Moreover we shall study the limit case $A \longrightarrow 0$. Hence let us define $(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta) = (u^1, \varepsilon^{p,1}, y^1) - (u^2, \varepsilon^{p,2}, y^2)$, where $(u^i, \varepsilon^{p,i}, y^i)$ $(i = 1, 2)$ are solutions of the problem (BP), with constants $A = A_i$ to the same initial and boundary conditions. The differences $(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)$ satisfy the following system of equations

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2} u^\Delta(x,t) &= \operatorname{div}_x T^\Delta(x,t) \\
T^\Delta(x,t) &= \mathcal{D}(\varepsilon(\nabla_x u^\Delta(x,t)) - \varepsilon^{p,\Delta}(x,t)) \\
\varepsilon(\nabla_x u^\Delta(x,t)) &= \frac{1}{2}(\nabla_x u^\Delta(x,t) + (\nabla_x u^\Delta(x,t))^T) \\
\frac{\partial}{\partial t} \varepsilon^{p,\Delta}(x,t) &= \mathcal{G}\left(\frac{|\sigma^1(x,t)|}{y^1(x,t)}\right) \frac{\sigma^1(x,t)}{|\sigma^1(x,t)|} \\
&\quad - \mathcal{G}\left(\frac{|\sigma^2(x,t)|}{y^2(x,t)}\right) \frac{\sigma^2(x,t)}{|\sigma^2(x,t)|} \\
\frac{\partial}{\partial t} y^\Delta(x,t) &= \gamma(y^1(x,t))\mathcal{G}\left(\frac{|\sigma^1(x,t)|}{y^1(x,t)}\right) |\sigma^1(x,t)| - A_1 \delta(y^1(x,t)) \\
&\quad - \gamma(y^2(x,t))\mathcal{G}\left(\frac{|\sigma^2(x,t)|}{y^2(x,t)}\right) |\sigma^2(x,t)| + A_2 \delta(y^2(x,t))
\end{aligned}
$$

(where $\sigma^i = T^i - \frac{1}{3}\mathrm{tr}\,(T^i)I$ and $T^i = \mathcal{D}(\varepsilon(\nabla_x u^i) - \varepsilon^{p,i})$ for $i = 1, 2$) with the homogeneous (equal to zero) initial and Dirichlet boundary conditions. If we now define in the standard manner the energy function for the differences $(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)$

$$
\mathcal{E}(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)(t) = \frac{1}{2}(\rho \partial_t u^\Delta(t), \partial_t u^\Delta(t)) + \frac{1}{2}(T^\Delta(t), \varepsilon^\Delta(t) - \varepsilon_p^\Delta(t))
$$

then we can try to prove energy estimates similar to those obtained in lemma 2.3.

Theorem 3.1
Let us suppose that $A_1, A_2 \leq A^*$. For the differences $(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)$ of two solutions $(u^1, \varepsilon_p^1, y^1)$, $(u^2, \varepsilon_p^2, y^2)$ for the problem (BP) to the same given data, satisfying for almost all $(x,t) \in \Omega \times (0, \infty)$ the inequality $y_2 \leq y^i(x) \leq y_1$, $(i = 1, 2)$, the following estimation holds:

$$
\mathcal{E}(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)(t) + \|y^\Delta(t)\|^2 \leq |A_1 - A_2|^2 \exp(ct) .
$$

The positive constant c depends affine on A^* and moreover if δ is a nondecreasing function then c does not depend on A^*.

Proof. With the method used in lemma 2.3 and lemma 2.4 we immediately have

$$
\frac{d}{dt} \mathcal{E}(u^\Delta, \varepsilon^{p,\Delta}, y^\Delta)(t) = -(\sigma^\Delta(t), \partial_t \varepsilon^{p,\Delta}(t))
$$

where $\sigma^{\Delta} = \sigma^1 - \sigma^2$. Thus inserting the right hand side of the ordinary differential equation for the difference $\varepsilon^{p,\Delta}$ we obtain

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{E}^{\Delta}(t) &= -\left(\sigma^1(t) - \sigma^2(t), \mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right)\frac{\sigma^1(t)}{|\sigma^1(t)|} - \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right)\frac{\sigma^2(t)}{|\sigma^2(t)|}\right) \\
&= -\int_\Omega \mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right)|\sigma^1(t)|\,dx - \int_\Omega \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right)|\sigma^2(t)|\,dx + \\
&\quad + \int_\Omega \mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right)\frac{\sigma^1(t)\cdot\sigma^2(t)}{|\sigma^1(t)|}\,dx + \int_\Omega \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right)\frac{\sigma^1(t)\cdot\sigma^2(t)}{|\sigma^2(t)|}\,dx \\
&\le -\int_\Omega \left\{\mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right) - \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right)\right\}(|\sigma^1(t)| - |\sigma^2(t)|)\,.dx
\end{aligned}$$

Up to now our proof is similar to that presented in [6]. We see that the isotropic hardening function violates the monotonicity of the function $\mathcal{G}$. The main idea of the following considerations is a "good" estimation of the difference $|\mathcal{G}(p_1) - \mathcal{G}(p_2)|$ where $p_1,\, p_2 > 0$ are arbitrary. In the article [6] the difference is estimated by $L|p_1 - p_2|$ where L is the global Lipschitz constant of the function $\mathcal{G}$. But we observe that by the assumptions $(A1) - (A3)$ the difference $|\mathcal{G}(p_1) - \mathcal{G}(p_2)|$ is small provided that the arguments are sufficiently large. Unfortunatly the property disappears immediately if we use only the Lipschitz bounds. Therefore we should estimate more carefully. By the assumptions $(A1)$ and $(A4)$ there exists a positive constant C such that for all $p \ge 0$, $\mathcal{G}'(p) \le \frac{C}{1+p}$ and this allows us to write the following inequality

$$|\mathcal{G}(p_1) - \mathcal{G}(p_2)| \le \frac{C}{1 + \min\{p_1, p_2\}}|p_1 - p_2|\,. \tag{GE}$$

Hence using (GE) we obtain

$$\begin{aligned}
\left|\mathcal{G}\left(\frac{|\sigma^1|}{y^1}\right) - \mathcal{G}\left(\frac{|\sigma^2|}{y^2}\right)\right| &\le \frac{C}{1 + \min\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\}}\left|\frac{|\sigma^1|}{y^1} - \frac{|\sigma^2|}{y^2}\right| \\
&\le \frac{C}{1 + \min\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\}}\left|\max\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\} - \min\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\}\right| \\
&\le \frac{C}{1 + \min\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\}}\max\left\{\frac{1}{y^1}, \frac{1}{y^2}\right\}\big||\sigma^1| - |\sigma^2|\big| \\
&\quad + \frac{C}{1 + \min\left\{\frac{|\sigma^1|}{y^1}, \frac{|\sigma^2|}{y^2}\right\}}\min\left\{|\sigma^1|, |\sigma^2|\right\}\left|\frac{1}{y^1} - \frac{1}{y^2}\right| \\
&\le \frac{C}{y_2}\left(|\sigma^1 - \sigma^2| + \frac{y_1}{y_2}|y^1 - y^2|\right).
\end{aligned}$$

Finally for the time derivative of the function $\mathcal{E}(u^{\Delta}, \varepsilon^{p,\Delta}, y^{\Delta})$ we have

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{E}(u^{\Delta}, \varepsilon^{p,\Delta}, y^{\Delta})(t) &\le \left(\frac{C}{y_2} + \frac{Cy_1}{(y_2)^2}\right)\int_\Omega |\sigma^1(t) - \sigma^2(t)|^2 dx \\
&\quad + \frac{Cy_1}{(y_2)^2}\int_\Omega |y^1(t) - y^2(t)|^2 dx\,.
\end{aligned} \tag{3.1}$$

The same approach will be used to estimate the norm $\|y^\Delta\|$. Multiplying the equation for y^Δ by $y^1 - y^2$ and integrating over Ω we have

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|y^\Delta(t)\|^2_{\mathbb{L}^2} &= \int_\Omega \left\{ \gamma\left(y^1(t)\right) \mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right) |\sigma^1(t)| - \gamma\left(y^2(t)\right) \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right) |\sigma^2(t)| \right\} \\
&\quad \left(y^1(t) - y^2(t)\right) dx \quad - \int_\Omega \left(A_1 \delta\left(y^1(t)\right) - A_2 \delta\left(y^2(t)\right)\right) \left(y^1(t) - y^2(t)\right) dx \\
&\le \int_\Omega \left[\gamma\left(y^1(t)\right) - \gamma\left(y^2(t)\right)\right] \left(y^1(t) - y^2(t)\right) \mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right) |\sigma^1(t)| \qquad (3.2) \\
&\quad + \gamma(y_1) \int_\Omega \left(\mathcal{G}\left(\frac{|\sigma^1(t)|}{y^1(t)}\right) |\sigma^1(t)| - \mathcal{G}\left(\frac{|\sigma^2(t)|}{y^2(t)}\right) |\sigma^2(t)| \right) \left(y^1(t) - y^2(t)\right) dx \\
&\quad + |A_1 - A_2| \sup_{y_2 \le y \le y_1} \delta(y) \int_\Omega |y^1(t) - y^2(t)|\, dx + A^* L_\delta \int_\Omega |y^1(t) - y^2(t)|^2\, dx
\end{aligned}
$$

where L_δ is the Lipschitz constant of the function δ. By the assumption $(A6)$ the first term in the right hand side of the last inequality is negative. Now we shall estimate the second one. To do this we need a stronger assumption than $(A4)$. Namely by $(A1)$ and $(A4')$ there exists a positive constant C' such that $\mathcal{G}'(p)p^2 + \mathcal{G}'(p) \le C'$ and we obtain the inequality

$$
\begin{aligned}
\left| \mathcal{G}\left(\frac{|\sigma^1|}{y^1}\right) |\sigma^1| - \mathcal{G}\left(\frac{|\sigma^2|}{y^2}\right) |\sigma^2| \right| &\le \max\left\{ \mathcal{G}\left(\frac{|\sigma^1|}{y^1}\right), \mathcal{G}\left(\frac{|\sigma^2|}{y^2}\right) \right\} \left| |\sigma^1| - |\sigma^2| \right| \\
&\quad + \min\left\{ |\sigma^1|, |\sigma^2| \right\} \left| \mathcal{G}\left(\frac{|\sigma^1|}{y^1}\right) - \mathcal{G}\left(\frac{|\sigma^2|}{y^2}\right) \right| \\
&\le \sup_{p \ge 0} \mathcal{G}(p) \left|\sigma^1 - \sigma^2\right| \\
&\quad + \frac{C' y_1}{(y_2)^2} \left|\sigma^1 - \sigma^2\right| + \frac{C'(y_1)^2}{(y_2)^2} \left|y^1 - y^2\right|.
\end{aligned}
$$

Inserting the last estimation into the second integral of (3.2) we obtain

$$
\frac{d}{dt} \|y^\Delta(t)\|^2 \le C^1 \left(\|y^\Delta(t)\|^2 + \|\sigma^\Delta(t)\|^2 \right) + C^2 |A_1 - A_2|^2 \qquad (3.3)
$$

where the constant C^1 depends affine on A^* and C^2 does not depend on A^*. Adding (3.1) and (3.3) and integrating over $(0, t)$ we end the proof.

Theorem 3.1 yields immediately the following two conclusions.

Corollary 3.1
For given data F, b, u^0, u^1, $\varepsilon_p^{(0)}$, $y^{(0)}$ with regularity required in theorem 2.2 and for all $A > 0$ the problem (BP) has unique global in time strong solution (u, ε^p, y) satisfying the $\mathbb{L}^\infty$ estimate for the isotropic hardening function: for almost all $(x,t) \in \Omega \times (0, \infty)$ $y_2 \le y(x,t) \le y_1$.

Proof. We set in theorem 3.1 $A_1 = A_2 = A$ and integrate the inequality obtained that.

Corollary 3.2
Let us denote by $(u_A, \varepsilon^p_A, y_A)$ the solution of the problem (BP) constructed in section 2 and

suppose that the given data F, b, u^0, u^1, $\varepsilon_p^{(0)}$, $y^{(0)}$ satisfying the assumptions of theorem 2.2. Then for $A \to 0$ the sequence $(u_A, \varepsilon^p_A, y_A)$ converges in the energy norm to a global in time, strong solution (u, ε^p, y) of the unperturbed problem ($A = 0$). Moreover if the function δ is nondecreasing then the convergence rate is proportional to A.

Proof. Use theorem 3.1 with $A_1 = 0$ and $A_2 = A$.

Remark
We shall remark that theorem 3.1 proves the uniqueness result for the unperturbed problem ($A = 0$) too. Thus the uniqueness questions answered only partially in [6, 7] are by theorem 3.1 validated.

4 $\mathbb{L}^p$–stability of the hardening function

In the last section of our work we are going to prove that the convergence result for $A \to 0$ holds only on finite time intervals. We show that for all $A > 0$ and $p \geq 1$ the isotropic hardening function converges strongly in $\mathbb{L}^p(\Omega)$ for $t \to \infty$ to the constant function y_2. Obviously the property is not valid for the unperturbed case ($A = 0$).

Lemma 4.1
Let (u, ε^p, y) be the solution of the problem (BP) constructed in section 2. Suppose that the function $\|\tilde{\sigma}\|$ obtained in the lemma 2.2 belongs to $\mathbb{L}^2((0,\infty);\mathbb{R}_+)$, then for all $A > 0$ it holds that:

$$\lim_{t \longrightarrow \infty} \|y(t) - y_2\|^2 \longrightarrow 0\,.$$

Proof. First let us define the nonnegative number

$$\epsilon_\delta = \inf_{y_2 \leq p \leq y_1} \frac{\delta(p) - \delta(y_2)}{p - y_2}.$$

In the proof we consider two cases:
1. $\epsilon_\delta > 0$.
Multiplying (BP6) by $(y - y_2)$ and integrating over Ω we obtain:

$$\frac{d}{dt}\frac{1}{2}\|y - y_2\|^2 = -A(\delta(y) - \delta(y_2), y - y_2) + \int_\Omega \gamma(y)\mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|(y - y_2)\,dx\,.$$

Using the assumptions $(A5) - (A7)$ and the $\mathbb{L}^\infty$ a priori estimates for y we conclude that

$$\frac{d}{dt}\frac{1}{2}\|y - y_2\|^2 \leq -A\epsilon_\delta\|y - y_2\|^2 + \gamma(y_2)(y_1 - y_2)\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|\,dx\,.$$

Now we multiply the last inequality by $e^{A\epsilon_\delta t}$ to get

$$\frac{d}{dt}\left(e^{A\epsilon_\delta t}\frac{1}{2}\|y - y_2\|^2\right) \leq Ce^{A\epsilon_\delta t}\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dx$$

and integrate over $(0, t)$, yielding

$$\|y(t) - y_2\|^2 \leq e^{-A\epsilon_\delta t}\|y(0) - y_2\|^2 + C\int_0^t e^{-A\epsilon_\delta(t-\tau)}\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau$$

(here $C=\gamma(y_2)(y_1-y_2)$). The energy estimate proved in lemma 2.3 with the assumption $\|\tilde{\sigma}\|\in\mathbb{L}^2((0,\infty);\mathbb{R}_+)$ yields

$$\int_0^\infty\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau \;=G<\infty\,.$$

Let $\epsilon>0$ then there exists a positive number N such that $\int_N^\infty\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau<\frac{\epsilon}{2}$. Hence for $t>N$ we obtain

$$\int_0^t e^{-A\epsilon_\delta(t-\tau)}\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau \;\;\le\;\; \int_0^N e^{-A\epsilon_\delta(t-\tau)}\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau+\frac{\epsilon}{2}\,.$$

Moreover for fixed N there exists a positive number $N'(N)$ such that for all $t>N'(N)$ we have $\sup\limits_{0\le\tau\le N} e^{-A\epsilon_\delta(t-\tau)}\le\frac{\epsilon}{2G}$. Thus for $t>\max\{N'(N),N\}$

$$\int_0^t e^{-A\epsilon_\delta(t-\tau)}\int_\Omega \mathcal{G}\left(\frac{|\sigma|}{y}\right)|\sigma|dxd\tau\le\epsilon$$

which is equivalent to

$$\lim_{t\longrightarrow\infty}\|y(t)-y_2\|^2\longrightarrow 0\,.$$

2. $\epsilon_\delta=0$.
For the given function y we define additionally:

$$\chi_{\frac{1}{n}}(y)=\left\{\begin{array}{ll} 0 & \text{for}\quad y_2<y<y_2+\frac{1}{n}\,,\\ \inf\limits_{y_2+\frac{1}{n}\le p\le y_1}\frac{A\delta(p)}{p-(y_2+\frac{1}{n})}(y-(y_2+\frac{1}{n})) & \text{for}\quad y_2+\frac{1}{n}<y<y_1\end{array}\right.$$

and

$$y^{(0)}_{\frac{1}{n}}(x)=\left\{\begin{array}{ll} y_2+\frac{1}{n} & \text{for}\quad y_2<y^{(0)}(x)\le y_2+\frac{1}{n}\,,\\ y^{(0)}(x) & \text{for}\quad y_2+\frac{1}{n}<y^{(0)}(x)<y_1\,.\end{array}\right.$$

Let us denote by $y_{\frac{1}{n}}(t)$ a solution of the ordinary differential equation

$$\frac{d}{dt}y_{\frac{1}{n}}(t)=-\chi_{\frac{1}{n}}(y_{\frac{1}{n}}(t))+\gamma(y(t))\mathcal{G}\left(\frac{|\sigma(t)|}{y(t)}\right)|\sigma(t)| \tag{4.1}$$

with the initial condition

$$y_{\frac{1}{n}}(t,x)=y^{(0)}_{\frac{1}{n}}(x)\,.$$

Note that in the equation (4.1) y and σ are given functions. By the special definitions of the functions $\chi_{\frac{1}{n}}(y_{\frac{1}{n}}(t))$ and $y^{(0)}_{\frac{1}{n}}(x)$ we conclude that the difference $y_{\frac{1}{n}}-y$ satisfies the equation

$$\frac{d}{dt}(y_{\frac{1}{n}}(t)-y(t))=-\chi_{\frac{1}{n}}(y_{\frac{1}{n}}(t))+A\delta(y(t))\in\mathbb{L}^2(\Omega,\mathbb{R}_+)$$

with the initial condition

$$y_{\frac{1}{n}}(0,x)-y(0,x)=y^{(0)}_{\frac{1}{n}}(x)-y^{(0)}(x)\ge 0\quad\text{for almost all } x\in\Omega\,.$$

Thus the diference $y_{\frac{1}{n}} - y$ belongs to $\mathbb{L}^2(\Omega, \mathbb{R}_+)$ for all $t \geq 0$. Hence using the method from case 1 we have

$$\lim_{t \longrightarrow \infty} \left\| y_{\frac{1}{n}}(t) - (y_2 + \frac{1}{n}) \right\|^2 \longrightarrow 0$$

for all n. Finally we obtain

$$\begin{aligned} \lim_{t \longrightarrow \infty} \|y(t) - y_2\|^2 &\leq \lim_{t \longrightarrow \infty} \|y_{\frac{1}{n}}(t) - y_2\|^2 \\ &\leq 2 \int_\Omega \frac{1}{n^2} dx + 2 \lim_{t \longrightarrow \infty} \left\| y_{\frac{1}{n}}(t) - (y_2 + \frac{1}{n}) \right\|^2 \\ &\leq 2 \left(\frac{1}{n} \right)^2 |\Omega| . \end{aligned}$$

Remark For all $p \geq 1$ lemma 4.1 with the $\mathbb{L}^\infty$ a priori estimates of the function y yields immediately

$$\lim_{t \longrightarrow \infty} \|y(t) - y_2\|_{\mathbb{L}^p(\Omega)} \longrightarrow 0 .$$

References

[1] Alber, H.D.: "Initial-Boundary Value Problems for the Inelastic Material Behaviour of Metals" – Lecture Notes in Mathematics, Springer (in preparation)

[2] Alber, H.D.: "Mathematische Theorie des inelastischen Materialverhaltens von Metallen" – Preprint Nr. 1682 FB Mathematik TH Darmstadt 1994

[3] Bodner, S.R., Partom, Y. : "Constitutive Equations for Elastic–Viscoplastic Strain–Hardening Materials" – J. of Appl Mech. p. 385-389, 1975

[4] Chełmiński, K. : "Global in time existence of solutions to the constitutive model of Bodner–Partom with isotropic hardening" – Demonstratio Mathematica v.28/3 p. 667-688, 1995

[5] Chełmiński, K. : "Energy estimates and global in time results for a problem from nonlinear viscoelasticity" – Bull. of Pol. Acad. of Sci.: Math. v. 44 p. 465-477, 1996

[6] Chełmiński, K.: "Stress L^∞– estimates and the uniqueness problem for the quasistatic equations to the model of Bodner–Partom" – accepted for publication in Math. Meth. in App. Sci.

[7] Chełmiński, K. : "Stress L^∞-Estimates and the Uniqueness Problem for the Equations to the Model of Bodner-Partom in the Two Dimensional Case" – accepted for publication in Math. Meth. in App. Sci.

[8] Gwiazda, P. : "On Singular Limits to the Bodner-Partom Model" – Preprint Nr. 1882 FB Mathematik TH Darmstadt 1996

[9] Johnson, C. : "On plasticity with hardening" – J. Math. Anal. Appl. v. 62 p. 325-336, 1978

[10] Le Tallec, P. : "Numerical analysis of viscoelastic problems" – Paris: Masson. Berlin, Heidelberg, New York: Springer 1990

Addresses:

Krzysztof Chełmiński
Fachbereich Mathematik
Technische Hochschule Darmstadt
Schloßgartenstr. 7
64289 Darmstadt, Germany
e-mail: chelminski@mathematik.th-darmstadt.de

Piotr Gwiazda
Institute of Applied Math. and Mech.
Warsaw University
Banacha 2
00–913 Warsaw, Poland
e-mail: pgwiazda@appli.mimuw.edu.pl