The 23rd Austrian–Polish Mathematics Competition

Baranów Sandomierski, June 28 - June 30, 2000

1. Determine all polynomials $P(x)$ with real coefficients possessing the following property: there exists a positive integer n such that the equality

$$\sum_{k=1}^{2n+1} (-1)^k \left[\frac{k}{2}\right] P(x+k) = 0$$

holds for infinitely many real numbers x.

2. In a cube with the edge of length equal to 1, $ABCD$ is a face of the cube and CG is an edge perpendicular to $ABCD$. O_1 is the incircle of the square $ABCD$ and O_2 is the circumcircle of the triangle BDG. Find $\min\{|XY| : X \in O_1, Y \in O_2\}$.

3. For each positive integer $n \geq 3$ solve in real numbers the following system of equations:

$$\begin{cases} x_1^3 &= x_2 + x_3 + 1 \\ & \dots \\ x_k^3 &= x_{k+1} + x_{k+2} + 1 \\ & \dots \\ x_{n-1}^3 &= x_n + x_1 + 1 \\ x_n^3 &= x_1 + x_2 + 1\,. \end{cases}$$

4. Find all positive integers N possessing only 2 and 5 as prime divisors, such that the number $N + 25$ is the square of an integer.

5. For which integers $n \geq 5$ is it possible to colour the vertices of the regular n-gon using at most 6 colours in such a way that each 5 consecutive vertices have different colours?

6. Consider the following solid:

$$Q = Q_0 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5 \cup Q_6\ ,$$

where Q_i ($i = 0, 1, 2, ..., 6$) are unit cubes and Q_0 has a common face with each Q_i for $i = 1, 2, ..., 6$. Prove or disprove that the space can be filled up with such solids in such a way that each two of them do not have common interior points.

7. In the plane the triangle $A_0B_0C_0$ is given. Consider all triangles ABC satisfying:
(C1) the straight lines AB, BC, CA pass through the points C_0, A_0, B_0, respectively,
(C2) the triangles ABC and $A_0B_0C_0$ are similar (i.e. there exists a similarity transforming A onto A_0, B onto B_0 and C onto C_0).
Find the set containing all circumcenters of the triangles ABC.

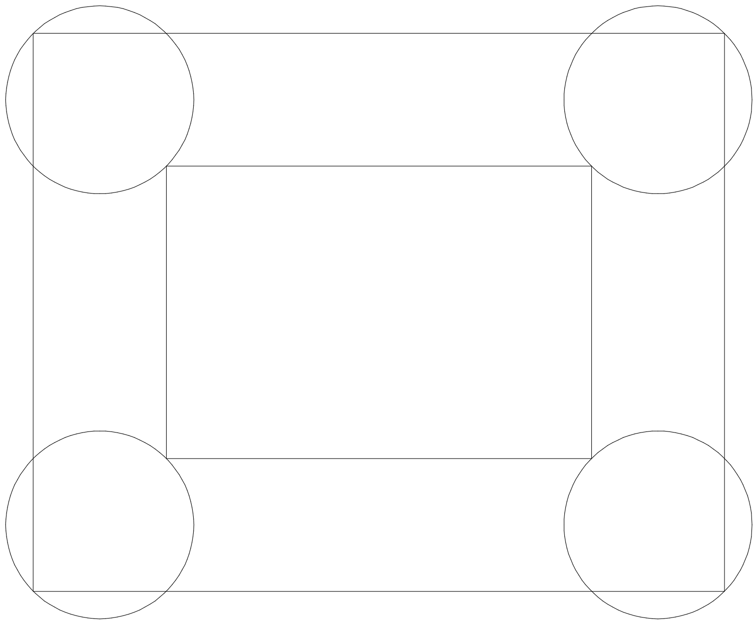
8. In the plane are given 27 points from which no three are collinear. 4 points from this set are vertices of the unit square. The other 23 points lie inside this square. Prove that there exist 3 points in this set forming a triangle with area not greater than $\frac{1}{48}$.

9. For each nonnegative real numbers a, b, c satisfying $a+b+c = 1$ prove the following inequalities

$$2 \leq \left(1-a^2\right)^2 + \left(1-b^2\right)^2 + \left(1-c^2\right)^2 \leq (1+a)(1+b)(1+c)\ .$$

For each inequality determine all a, b, c such that the inequality becomes the equality.

10. The plan of the castle in Baranów Sandomierski can be presented as the following graph with 16 vertices:



A night guard plans a closed round along the edges of this graph.
a) How many rounds (directions are not taking into account) passing through each vertex exactly once are there?
b) How many rounds (taking directions into account) containing each edge of the graph exactly once and not having self-cross points are there?