52nd Mathematical Olympiad in Poland

Problems of the first round, September – December 2000

1. Solve in integers the equation
$$x^{2000} + 2000^{1999} = x^{1999} + 2000^{2000}\,.$$

2. The points D and E lie on the sides BC and AC of the triangle ABC, respectively. The lines AD and BC meet in the point P. The points K and L lie on the sides BC and AC, respectively and are chosen so that $CLPK$ is a parallelogram. Prove that
$$\frac{AE}{EL} = \frac{BD}{DK}\,.$$

3. Find all positive integers $n \geq 2$, such that the inequality
$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n \leq \frac{n-1}{n}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)$$
is satisfied for all positive real numbers $x_1, x_2, \ldots, x_n$.

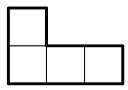
4. Prove or disprove: into a cube box with edge 4, one can put 65 balls of diameter 1.

5. Prove that for all integers $n \geq 2$ and for all prime numbers p the number
$$n^{p^p} + p^p$$
is composite.

6. The integers a, b, x, y satisfy the equality
$$a + b\sqrt{2001} = \left(x + y\sqrt{2001}\right)^{2000}\,.$$
Prove that $a \geq 44b$.

7. ABC is an isosceles triangle with the angle $\sphericalangle A = 90°$. The points D and E lie on the side BC and $\sphericalangle DAE = 45°$. The circumcircle of the triangle ADE meets the sides AB and AC in the points P and Q, respectively. Prove that $BP + CQ = PQ$.

8. For which positive integers m, n, can the rectangle of dimensions $m \times n$ be cut into the pieces congruent to the one at the figure. Each of the little squares at the figure has side 1.



9. Prove that among any 12 consecutive integers there exists an integer which is not equal to the fourth power of an integer.

10. Prove that each triangle ABC contains an interior point P possessing the following property: each line passing through P divides the perimeter and the area of the triangle ABC in the same ratio.

11. The set $\{c_1, c_2, \ldots, c_n\}$ of positive integers is called admissible if each integer $k \in \{1, 2, \ldots, 2(c_1 + c_2 + \ldots + c_n)\}$ can be represented in the form
$$k = \sum_{i=1}^{n} a_i c_i \quad \text{with} \quad a_i \in \{-2, -1, 0, 1, 2\}\,.$$
For each n find
$$\max\left\{\sum_{i=1}^{n} c_i \;:\; \{c_1, c_2, \ldots, c_n\} \text{ is admissible}\right\}.$$

12. Consider the sequences $x_0, x_1, \ldots, x_{2000}$ of integers satisfying
$$x_0 = 0 \quad \text{and} \quad |x_n| = |x_{n-1} + 1| \qquad \text{for } n = 1, 2, \ldots, 2000\,.$$
Find the minimum value of the expression $|x_1 + x_2 + \ldots + x_{2000}|$.