An “almost” full embedding of the category of graphs into the category of groups

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$$F : \mathcal{G}raphs \to \mathcal{G}roups$$

- full and faithful:

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(FX, FY)$$

- “almost” full:

$$\mathrm{Hom}_{\mathcal{G}raphs}(X, Y) \cup \{*\} \xrightarrow{\cong} \mathrm{Rep}(FX, FY)$$

where $\mathrm{Rep}(FX, FY) = \mathrm{Hom}(FX, FY)/FY$

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Choice of categories: target

- $\mathcal{G}roups$ - is interesting in itself
- $\mathcal{G}roups \xrightarrow{B} Ho$ (unpointed homotopy category) yields, up to constant maps, a full embedding

$$BF : \mathcal{G}raphs \longrightarrow Ho$$

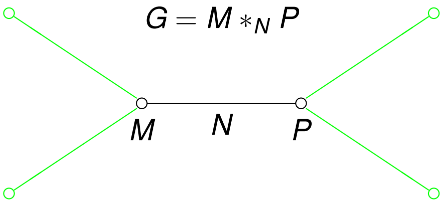
Choice of categories: source

$\mathcal{G}$*raphs* is very comprehensive and well researched. Many "non-homotopy" categories are contained in $\mathcal{G}$*raphs* as full subcategories:

- category of groups
- category of fields
- category of R-modules
- category of Hilbert spaces
- category of partially ordered sets
- category of simplicial sets
- category of metrizable spaces and continuous maps
- category of CW-complexes and continuous maps
- category of models of some first order theory
- many more

Tool: Adámek, Rosický *Locally presentable and accessible categories*, Theorem 2.65.

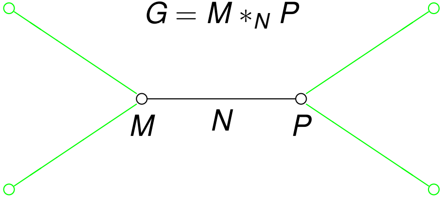
Bass-Serre theory on groups acting on trees



- If $A \subseteq G$ is finite then it stabilizes a vertex of the tree hence is conjugated to a subgroup of M or P.
- Take $A = M$ finite, s.t. $\mathrm{Hom}(M, P) = *$ and $M \to M$ is either trivial or an inner automorphism.
- Let $N_M(N) = N$, $N \subseteq M$ does not extend to $P \to M$, $N \subseteq P$ extends uniquely to $P \to P$ then $M \subseteq G$ uniquely extends to $G \to G$.

Bass-Serre theory on groups acting on trees

$G = M *_N P$



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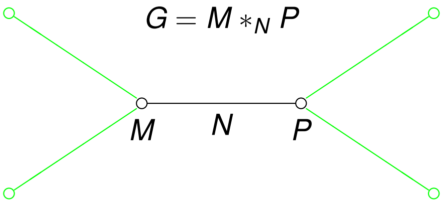

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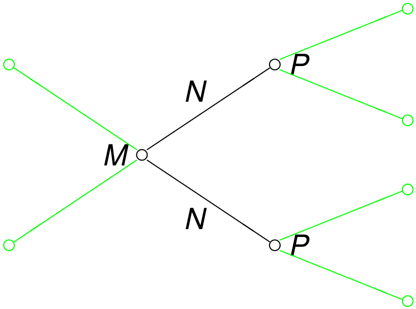
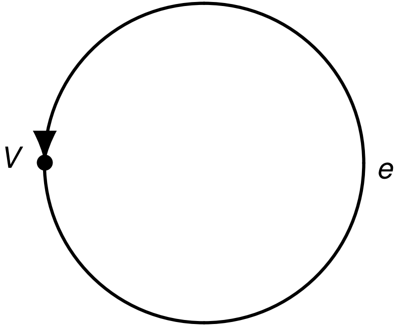
$$G = (M *_N P) *_N P$$

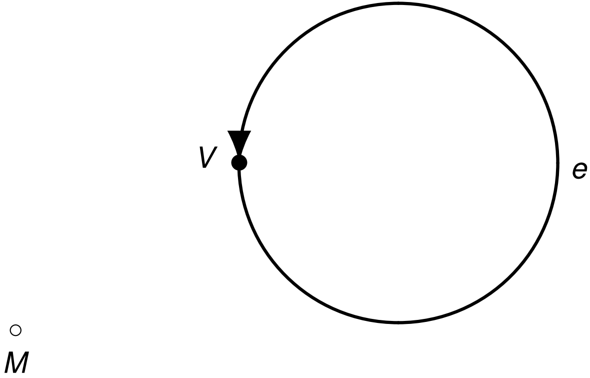


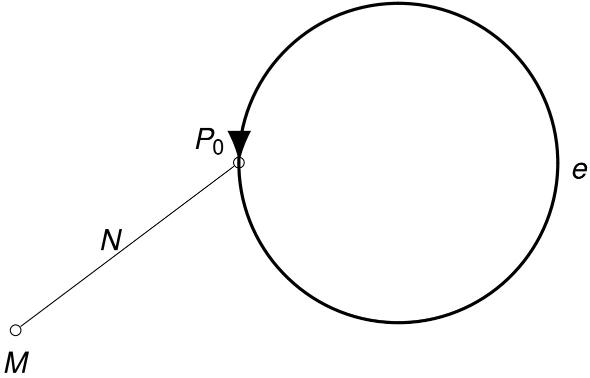
$$\mathrm{Rep}(G, G) = \{*\} \cup \mathrm{Hom}(\underline{2}, \underline{2})$$

Start with a graph Γ

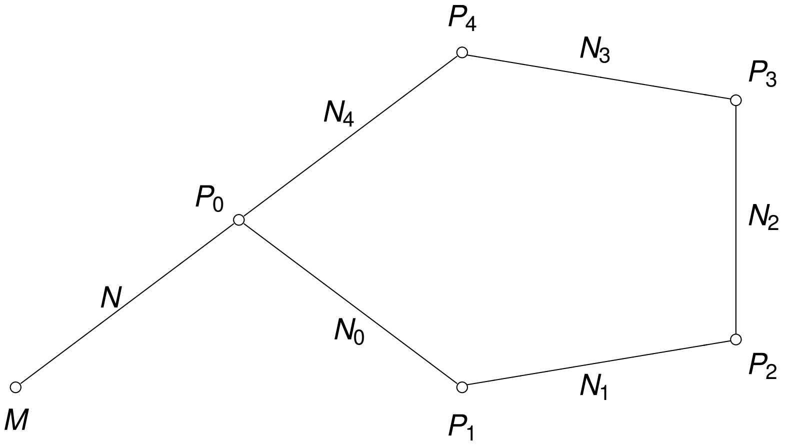


$\emptyset$

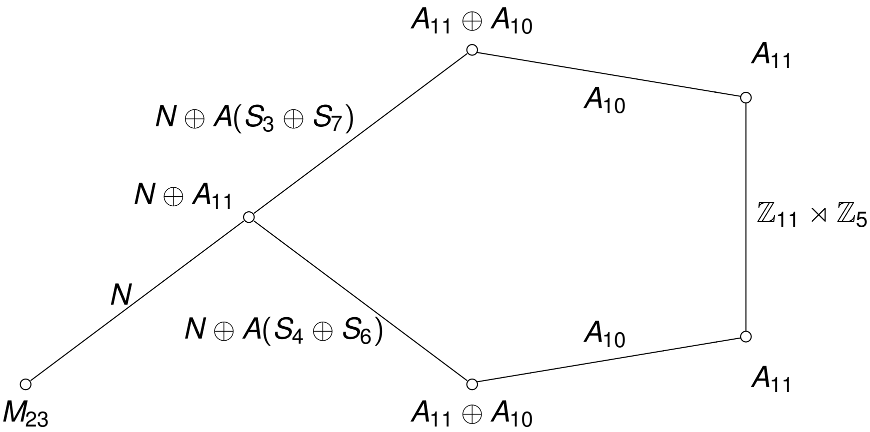

V
e
M


P_0
e
N
M

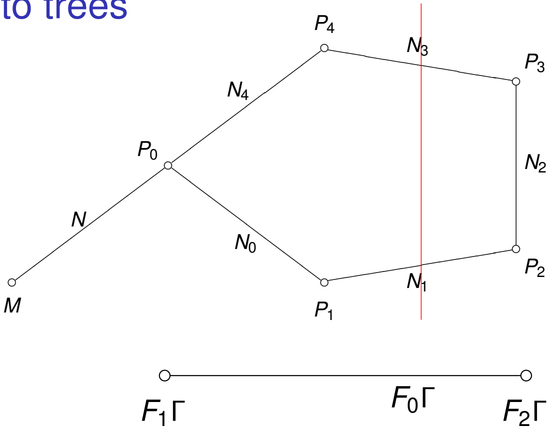
Obtain graph of groups $G\Gamma$, define $F\Gamma = \mathrm{colim}\, G\Gamma$



Example


$A_{11} \oplus A_{10}$
A_{11}
A_{10}
$N \oplus A(S_3 \oplus S_7)$
$N \oplus A_{11}$
$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$
N
$N \oplus A(S_4 \oplus S_6)$
A_{10}
M_{23}
$A_{11} \oplus A_{10}$
A_{11}

Reduction to trees



$$F\Gamma = F_1\Gamma *_{F_0\Gamma} F_2\Gamma$$

Two definitions of a localization $L : \mathcal{C} \to \mathcal{C}$

1. L is a left adjoint of an inclusion $\mathcal{D} \subseteq \mathcal{C}$ of some subcategory.
2. L is a functor with coaugmentation $\eta : Id \to L$ such that $\eta_{LX} = L\eta_X : LX \to LLX$ is an isomorphism

Localizations may be viewed as projections
onto the class of local objects $\mathcal{D}$
along the class of L-equivalences $\mathcal{E} = \{f \mid Lf \text{ is an equivalence}\}$
For every

$f : A \to B$ in $\mathcal{E}$, an L-equivalence and
Z in $\mathcal{D}$, an L-local object

we have:

$$\mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

$$\mathrm{map}(B, Z) \xrightarrow{\simeq} \mathrm{map}(A, Z)$$

Orthogonality classes

If for $f : A \to B$ and Z we have

$$\mathrm{Hom}(B, Z) \xrightarrow{\cong} \mathrm{Hom}(A, Z)$$

$$\mathrm{map}(B, Z) \xrightarrow{\simeq} \mathrm{map}(A, Z)$$

then we say that f is *orthogonal* to Z and write $f \perp Z$.
A pair $(\mathcal{E}, \mathcal{D})$ is orthogonal if $\mathcal{E} = \mathcal{D}^\perp$ and $\mathcal{D} = \mathcal{E}^\perp$.
A localization always yields an orthogonal pair.

Whether every orthogonal pair yields a localization depends on set theory

in $\mathcal{G}$raphs:
NO is consistent with ZFC
weak Vopěnka's principle is equivalent to YES

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More properties of $F : \mathcal{G}raphs \to \mathcal{G}roups$

1. $\mathrm{Hom}_{\mathcal{G}raphs}(X, Y) \cup \{*\} \xrightarrow{\cong} \mathrm{Rep}(FX, FY)$
2. $f \perp Z$ if and only if $Ff \perp FZ$
3. F preserves directed colimits
4. F preserves intersections and countably co-directed limits
5. $\Delta \subseteq \Gamma$ implies $F\Delta \subseteq F\Gamma$
6. for every $g \in F\Gamma$ there exists a finite subgraph $\Delta \subseteq \Gamma$ s.t. $g \in F\Delta$
7. F does not preserve products.

Large localizations of finite groups

Theorem

There exist localizations $L : \mathcal{G}roups \to \mathcal{G}roups$ whose values LM on a finite group M have arbitrarily large cardinalities.

Proof.

Vopěnka (1965): there exist arbitrarily large graphs Γ s.t. $\mathrm{Hom}(\Gamma, \Gamma) = \{id\}$.
The inclusion $i : \emptyset \subseteq \Gamma$ is orthogonal to Γ.
$Fi \perp F\Gamma$
If $L = L_{Fi}$ then $LM = F\Gamma$. □

Proved by Shelah and Göbel (2002) on 28 pages.

Theorem

The following are equivalent:

1. *Every orthogonal pair* $(\mathcal{E}, \mathcal{D})$ *in* $\mathcal{G}$*roups is associated with a localization.*
2. *Every orthogonal pair* $(\mathcal{E}, \mathcal{D})$ *in* $\mathcal{G}$*raphs is associated with a localization (weak Vopěnka's principle).*

$2 \implies 1$ was proved by Adámek and Rosický (1994)

$1 \implies 2$ follows from properties of F

Theorem

The following are equivalent:

1. *For every orthogonal pair* $(\mathcal{E}, \mathcal{D})$ *in* $\mathcal{G}$*roups there exists a homomorphism f such that* $\mathcal{D} = \{f\}^\perp$*.*
2. *For every orthogonal pair* $(\mathcal{E}, \mathcal{D})$ *in* $\mathcal{G}$*raphs there exists a map f such that* $\mathcal{D} = \{f\}^\perp$ *(Vopěnka's principle).*
3. *For every orthogonal pair* $(\mathcal{E}, \mathcal{D})$ *in Ho there exists a map f such that* $\mathcal{D} = \{f\}^\perp$*.*

$2 \implies 3$ was proved by Casacuberta, Scevenels, Smith (2005)
$1 \implies 2$ and $3 \implies 1$ follow from properties of F

Summary

An almost full embedding $F : \mathcal{G}raphs \to \mathcal{G}roups$

$$\mathrm{Hom}_{\mathcal{G}raphs}(X, Y) \cup \{*\} \xrightarrow{\cong} \mathrm{Rep}(FX, FY)$$

is a "black box" tool translating some categorical constructions from many point-set categories to the category of groups (or to the homotopy category).

- Question
  Is there an embedding $F : \mathcal{G}raphs \to \mathcal{A}b - Groups$ such that $f \perp Z$ if and only if $Ff \perp FZ$.

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thank you