On Gromov's Macroscopic Dimension

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Outline

Definition of Sc

- The scalar curvature Sc_v of a Riemannian manifold V^n at a point $v \in V^n$ is the number defined by

$$\frac{Vol\ B_V(\epsilon, v)}{Vol\ B_{\mathbb{R}^n}(\epsilon, 0)} = 1 - \frac{Sc_v}{6n}\epsilon^2 + o(\epsilon^2)$$

where $B_V(\epsilon, v)$ is the ϵ-ball centered at $v \in V^n$.

- $Sc_v =$ the sum of sectional curvatures over all 2-planes $e_i \wedge e_j$ in the tangent space to v, where $e_1, \dots, e_n$ is the orthonormal basis.

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Properties

- There is the product formula:

$$Sc_{(v_1,v_2)} = Sc_{v_1} + Sc_{v_2}$$

  for $(V_1 \times V_2, \mathcal{G}_1 \oplus \mathcal{G}_2)$.
- Thus, for every closed manifold M, the product $M \times S^2$ admits a metric with $Sc > 0$
- Take the S^2 factor to be ϵ-small !
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Why $Sc > 0$?

- Every manifold of dim ≥ 3 admits a metric with $Sc < 0$ [J.L. Kazdan and F. Warner]. So, no restriction on topology.
- If $Sc(M^n) \geq 0$, $n \geq 3$, then M^n admits a metric with $Sc > 0$ (with some exceptions) [J.L. Kazdan and F. Warner].
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- GROMOV CONJECTURE (intuitive). *If a closed n-manifold M has $Sc(M) > 0$, then the universal cover $\widetilde{M}$ is at most $(n-2)$-dimensional on large scales.*
- EXAMPLE: $M^{n-2} \times S^2$ admits a metric with $Sc > 0$. The universal cover $\widetilde{M^{n-2} \times S^2} = \widetilde{M^{n-2}} \times S^2$ looks at most $n-2$-dimensional on a large scale.

GROMOV CONJECTURE (formal).

$dim_{mc}\widetilde{M}^n \leq n-2$ for every closed n-manifold with $Sc(M) > 0$.

- The conjecture is from Gelfand-80 book [1996] but first time it appeared in Gromov's "filling" paper [1983] in a different language.

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Macroscopic Dimension

- Here dim_{mc} is the *macroscopic dimension*.
- For a metric space X,

$$dim_{mc} X \leq k$$

  iff there is a uniformly cobounded map $f : X \to N^k$ to a k-dimensional simplicial complex.
- A map $f : X \to N$ is *uniformly cobounded* if there is $b > 0$ such that $diam(f^{-1}(y)) \leq b$ for all $y \in N$.
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Examples

- $dim_{mc}\mathbb{R}^n = n$ and generally, $dim_{mc}V^n = n$ for every uniformly contractible manifold with proper metric.
  Proof. Assume that $dimV^n \leq n-1$. Let $f : V^n \to K^{n-1}$ be a uniformly cobounded map. There is a map $s : K^{n-1} \to V^n$ such that $d(s \circ f, id) < D$. Hence $f^*s^* = 1$:

  $$H_c^n(V^n) \xrightarrow{s^*} H_c^n(K^{n-1}) \xrightarrow{f^*} H_c^n(V^n)$$

  Contradiction, since $s^* : H_c^n(V^n) \to H_c^n(K^{n-1})$ is zero and $H_c^n(V^n) \neq 0$.
- V is uniformly contractible if $\exists\, \rho : \mathbb{R}_+ \to \mathbb{R}_+$ s.t. $B(t,x)$ contracts to a point in $B(\rho(t),x)$ for all $x \in V$ and $t \in \mathbb{R}_+$.

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- Gromov's Conjecture implies the Gromov-Lawson:

Gromov-Lawson Conjecture

A closed aspherical manifold M cannot carry a metric with $Sc > 0$.

- *Proof.* The result follows from the facts that $\widetilde{M}$ is uniformly contractible and $dim_{mc} V^n = n$ for all uniformly contractible manifolds.
- The Gromov-Lawson is a Novikov type conjecture.

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Main Result

Let $ko = KO\langle 0 \rangle$ be the connective cover of the real K-theory.

Theorem (Bolotov-Dr.)

Suppose that a discrete group π has the following properties:

- The Strong Novikov Conjecture holds for π.
- The natural map $per : ko_n(B\pi) \to KO_n(B\pi)$ is injective.

Then the Gromov Macroscopic Dimension Conjecture holds true for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$.

Corollary 1.

The Gromov conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M)$ equal the product of free groups $F_1 \times \cdots \times F_n$. In particular, it holds for free abelian groups.

Proof. The formula for homology with coefficients in a spectrum **E**:

$$H_i(X \times S^1; \mathbf{E}) \cong H_i(X; \mathbf{E}) \oplus H_{i-1}(X; \mathbf{E})$$

implies that if $ko_*(X) \to KO_*(X)$ is monomorphism, then $ko_*(X \times S^1) \to KO_*(X \times S^1)$ is a monomorphism. By induction on m using the Mayer-Vietoris sequence one can show that $ko_*(X \times (\bigvee_m S^1)) \to KO_*(X \times (\bigvee_m S^1))$ is a monomorphism.

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Corollary 2.

The Gromov Conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$ having $cd\pi \leq n+3$ and satisfying the Strong Novikov Conjecture.

Proof. Let $\mathbf{F} \to ko \to KO$ be the fibration of spectra induced by the morphism $ko \to KO$. Then $\pi_k(\mathbf{F}) = 0$ for $k \geq 0$ and $\pi_k(\mathbf{F}) = \pi_k(KO) = KO_k(pt) = 0$ if $k = -1, -2, -3$ mod 8. AHSS for the F-homology of $B\pi$ implies that $H_n(B\pi; \mathbf{F}) = 0$ since all entries on the n-diagonal in the E^2-term are 0. Then the coefficient exact sequence for homology

$$H_n(B\pi; \mathbf{F}) \to ko_n(B\pi) \to KO_n(B\pi) \to \dots$$

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Finite index subgroups

- PROPOSITION. *Let* $\pi' \subset \pi$ *be a subgroup of finite index,* $[\pi' : \pi] < \infty$. *If Gromov's conjecture holds for manifolds with fundamental group* π', *then it holds for manifolds with the fundamental group* π.
- Proof. Let $\pi_1(M) = \pi$ and let M have a PSC metric. Then M' corresponding to π' has a PSC metric. Then $dim_{mc}\tilde{M}' \leq n - 2$ by Gromov's Conjecture for π'. Note that $\tilde{M}' = \tilde{M}$.

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Outline

- There is a *real analytic assembly map*

$$\alpha : KO_*(B\pi) \to KO_*(C_r^*(\pi))$$

defined as the "slant product" with the class $[\nu_{B\pi}] \in KO^0(B\pi; C_r^*(\pi))$ generated by the "$C_r^*(\pi)$-line bundle" $E\pi \times_\pi C_r^*(\pi) \to B\pi$.

- $C_r^*(\pi)$ is the completion of $\mathbb{R}\pi$ in the operator norm where $\mathbb{R}\pi$ acts on $\ell^2(\pi)$ by multiplication on the left.

Strong Novikov Conjecture

The real analytic assembly map is a monomorphism for torsion free groups π.

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Let $f : M \to B\pi$ be the classifying map for the universal cover of M.

Rosenberg's Theorem

Let M be a closed connected spin manifold with $Sc > 0$. Then $\alpha \circ f_*([M]_{KO}) = 0$.

This result led to the Gromov-Lawson-Rosenberg Conjecture which is an extension of the Gromov-Lawson conjecture to general manifolds. (disproved by T. Schick)

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A fact from real K-theory

We use the following fact: The induced homomorphism

$$h_* : KO_n(S^n) \to KO_n(S^{n-1})$$

is nontrivial ($\mathbb{Z} \to \mathbb{Z}_2$) where $h : S^n \to S^{n-1}$ is the iterated suspension of the Hopf bundle.

Outline

Definition of the BS-class

- A "characteristic class" arising from the universal Ganea fibrations over the classifying space $B\pi$ is called the Berstein-Schwarz class $\beta_\pi \in H^1(\pi; I(\pi))$ of π where $I(\pi)$ the augmentation ideal of the group ring $\mathbb{Z}(\pi)$
- Formally, β_π is the image of the generator under connecting homomorphism $H^0(\pi; \mathbb{Z}) \to H^1(\pi; I(\pi))$ in the long exact sequence generated by the short exact sequence of coefficients

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Universality of the BS-class

The "cup product" $\alpha \cup \beta \in H^{p+q}(X; A \otimes B)$ is defined for $\alpha \in H^p(X; A)$ and $\beta \in H^q(X; B)$ for any π-modules A and B where $\pi = \pi_1(X)$.

Universality Theorem (Schwarz, Dr.-Rudyak)

For every π-module L, every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\beta_\pi)^k$ under a suitable coefficients homomorphism $\psi : I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L$.

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Proof of UT

There is a projective resolution

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $C_k = \mathbb{Z}\pi \otimes I(\pi)^k$ and with $\partial_k = \alpha_{k-1} \circ \beta_k$:

$$\mathbb{Z}\pi \otimes I(\pi)^k \xrightarrow{\beta_k} \mathbb{Z} \otimes I(\pi)^k \xrightarrow{\alpha_{k-1}} \mathbb{Z}\pi \otimes I(\pi)^{k-1}.$$

where α_k,β_k are from

$$0 \to I(\pi)^{k+1} \xrightarrow{\alpha_k} \mathbb{Z}\pi \otimes I(\pi)^k \xrightarrow{\beta_k} I(\pi)^k \to 0$$

which is obtained by taking tensor product with $I(\pi)^k$ from

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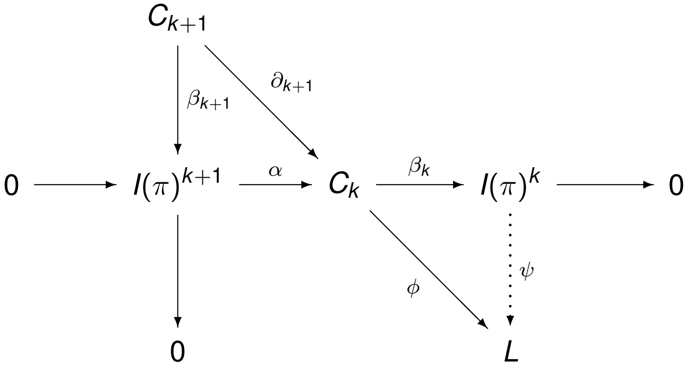
Proof of UT

Given $[\phi]$ we define ψ that takes $[\beta_k]$ to $[\phi]$. Then the theorem follows from the fact $\beta_\pi^k = [\beta_k]$.



Essential Manifolds

- An n-manifold M is called *essential* if the classifying map $f : M \to B\pi$ of its universal covering $\widetilde{M}$ cannot be deformed to the $(n-1)$-skeleton.
- Otherwise it is called *inessential*.

Theorem

TFAE

- An n-manifold M is inessential
- $f^*(\beta_\pi^n) = 0$
- The LS-category of $M < n$.

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- The LS-category of $M < n$.

Sketch of Proof 1 $\Leftrightarrow$ 2

- (M^n is inessential $\Rightarrow f^*(\beta_\pi^n) = 0$).
  Let $f \sim g$, $g : M^n \to B\pi^{(n-1)}$. Then $f^*(\beta_\pi^n) = g^*(\beta_\pi^n) = 0$ by a dimensional reason.
- (M^n is inessential $\Leftarrow f^*(\beta_\pi^n) = 0$).
  Let $c_r \in H^n(B\pi; \pi_{n-1}(B\pi^{(n-1)}))$ be the first obstruction to retraction of $B\pi$ to the $(n-1)$-skeleton. By UT $\beta_\pi^n \to c_r$ for some coefficient homomorphism. Since $f^*(\beta_\pi^n) = 0$, $f^*(c_r) = 0$. Thus, the obstruction to deform $f : M^n \to B\pi$ to the $(n-1)$-skeleton is zero.

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Lusternik-Schnirelmann category

- The *Lusternik-Schnirelmann category*, $cat_{LS}X \leq n$ if there is an open cover $U_0, \dots, U_n$ by contractible in X subset.
- $cat_{LS}X \leq dimX$.
- $\forall X$ there are Ganea's fibrations $p_n : G_n(X) \to X$ such that $cat_{LS}X \leq n$ if and only if there is a section $s : X \to G_n$.
- If $dimX = n$ the only obstruction to a section of p_{n-1} is $f^*(\beta_\pi^n)$ where $f : X \to B\pi$ induces iso of π_1.
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Inessential Manifolds

Theorem

If a closed n-manifold M is inessential then not only $f^*(\beta_\pi^n) = 0$ but also $f^*(\beta_\pi^{n-1}) = 0$.

Proof. Let $\beta_M = f^*(\beta_\pi)$. Assume that $\beta_M^{n-1} \neq 0$. By the Poincare Duality $a = \beta_M^{n-1} \cap [M] \neq 0$. Hence, there is $\alpha \in H^1(M; L)$ for some L such that $\langle \alpha, a \rangle \neq 0$. Therefore, $\beta_M^{n-1} \cup \alpha \neq 0$. Then the twisted cup-length of M equals n. Contradiction with $cat_{LS} M < n$.

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Essentiality and dim_{mc}

- If M^n is inessential, then $dim_{mc}\widetilde{M} \leq n-1$.
- Converse is not true: $\mathbb{R}P^n$
- If $f : M^n \to B\pi$ can be deformed to the $(n-2)$-skeleton, then $dim_{mc}\widetilde{M} \leq n-2$.

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Outline

First Step of the proof

- The proof of the Main Theorem uses the Obstruction Theory to get a deformation of $f : M^n \to B\pi$ to $B\pi^{(n-2)}$.
- Assume that M has one top dimensional cell.
- 1st obstruction to deformation of f to $B\pi^{(n-1)}$ is zero: It lives in $H^n(M; \pi_n(B\pi, B\pi^{(n-1)}))$. By PD with twisted coefficients the later equals $\pi_n(B\pi/B\pi^{(n-1)})$.
- Moreover, the obstruction is the class of the induced map $\bar{f} : M/M^{(n-1)} = S^n \to B\pi/B\pi^{(n-1)}$. It's null-homotopic, since otherwise it induces a nonzero ko_*-homomorphism which contradicts to the assumptions and Rosenberg's theorem.
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The diagram

$$\begin{array}{ccccccc}
ko_n(M) & \xrightarrow{f_*} & ko_n(B\pi) & \xrightarrow{per} & KO_n(B\pi) & \xrightarrow{\alpha} & KO \\
\downarrow & & \downarrow & & & & \\
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Second step of the proof

- Deform $f : M \to B\pi^{(n-1)}$ to a map with the property $f(M^{(n-1)}) \subset B\pi^{(n-2)}$. Here we use the properties of the Berstein-Schwarz class of an inessential manifold.
- Show that the 1st obstruction for deforming this f to $B\pi^{(n-2)}$ is zero.
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The converse

- *Does the converse hold true: If $\dim_{mc}\widetilde{M}^n \leq n-2$ them M admits a metric with $Sc(M) > o$?*
- The answer is 'No' even in simply connected case.

Stolz' Theorem

A closed simply connected spin manifold M admits a metric of positive scalar curvature if and only if $c_*([M]_{KO}) = 0$ where $c : M \to pt$ is the constant map.

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- A VERSION of CHARACTERIZATION *A manifold M admits a metric with* $Sc(M) > 0$ *if and only if* $dim_{mc}\widetilde{M} \leq n-2$ *and* $\alpha \circ f_*([M]_{KO}) = 0$ *in* $KO_*(C_r^*(\pi))$.
- Schick's counterexample to the GLR has $dim_{mc}\tilde{M} \geq n-1$, so it is not a counterexample to this conjecture.
- But the followup counterexample by Joachim and Schick has $dim_{mc}\tilde{M} \leq n-2$.
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Torsion free case

- An n-manifold M is called *k-essential* if the classifying map $f : M \to B\pi$ cannot be deformed to $B\pi^{(k)}$. Otherwise it is called k-inessential.
- Thus, n-inessential means inessential.
- Clearly, $dim_{mc}\widetilde{M} < k$ for k-inessential n-manifolds
- A version of Gromov's Conjecture. *If a closed n-manifold M with torsion free fundamental group admits a metric with $Sc(M) > 0$, then M is $(n-1)$-inessential.*
- REMARK. The proof of our main theorem implies that spin PSC manifold is $(n-1)$-inessential (under our conditions on π).

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- THEOREM. *The characterization Conjecture holds true under the condition of the Main Theorem*
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Rosenberg-Stolz Theorem

Suppose that a discrete group π has the following properties:

- The Strong Novikov Conjecture holds for π.
- The natural map $per : ko_n(B\pi) \to KO_n(B\pi)$ is injective.

Then the Gromov-Lawson Conjecture holds true for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$.

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- 'Yes' if $\pi_1(M)$ is amenable.

Other problems on dim_{mc}

- PROBLEM 2. Does the macroscopic dimension coincide with the cohomological macroscopic dimension ?
- REMARK. Gromov's conjecture can be proven for cohomological macroscopic dimension (at least rational).
- PROBLEM 3. Does the formula

$$dim_{mc}(X \times \mathbb{R}) = dim_{mc}X + 1$$

holds for all metric spaces X? for manifolds?
- Clearly, 'yes' for P2 implies 'yes' for P3.

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For Further Reading I

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