Optimizing Tree Decompositions in MSO*†

Mikołaj Bojańczyk[1] and Michał Pilipczuk[2]

1 **Institute of Informatics, University of Warsaw, Warsaw, Poland**
`bojan@mimuw.edu.pl`
2 **Institute of Informatics, University of Warsaw, Warsaw, Poland**
`michal.pilipczuk@mimuw.edu.pl`

Abstract

The classic algorithm of Bodlaender and Kloks [J. Algorithms, 1996] solves the following problem in linear fixed-parameter time: given a tree decomposition of a graph of (possibly suboptimal) width k, compute an optimum-width tree decomposition of the graph. In this work, we prove that this problem can also be solved in MSO in the following sense: for every positive integer k, there is an MSO transduction from tree decompositions of width k to tree decompositions of optimum width. Together with our recent results [LICS 2016], this implies that for every k there exists an MSO transduction which inputs a graph of treewidth k, and nondeterministically outputs its tree decomposition of optimum width.

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1 Introduction

Consider the following problem: given a tree decomposition of a graph of some width k, possibly suboptimal, we would like to compute an optimum-width tree decomposition of the graph. A classic algorithm of Bodlaender and Kloks [4] solves this problem in linear fixed-parameter time complexity, where the input width k is the parameter.

▶ **Theorem 1** (Bodlaender and Kloks, [4]). *There exists an algorithm that, given a graph G on n vertices and its tree decomposition of width k, runs in time $2^{\mathcal{O}(k^3)} \cdot n$ and returns a tree decomposition of G of optimum width.*

The algorithm of Bodlaender and Kloks proceeds by a bottom-up dynamic programming procedure on the input decomposition. For every subtree, a set of partial optimum-width decompositions is computed. The crucial ingredient is a combinatorial analysis of partial decompositions which shows that only some small subset of them, of size bounded only by a function of k, needs to be remembered for future computation.

The algorithm of Bodlaender and Kloks is a key subroutine in the linear-time algorithm for computing the treewidth of a graph, due to Bodlaender [2]. The fact that the algorithm is essentially governed by a run of a finite-state automaton on the input tree decomposition was also used in the recent approximation algorithm of Bodlaender et al. [3]. The notion of

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typical sequences, which is the main component of the analysis of partial decompositions, has found applications in algorithms for computing other width measures, like branchwidth [5], cutwidth [16, 17], or pathwidth of matroids [12]. The concept of typical sequences originates in the previous work of Courcelle and Lagergren [8] and of Lagergren and Arnborg [15].

Our results. The main result of this paper (Theorem 2) is that the problem of Bodlaender and Kloks can be solved by an MSO *transduction*, which is a way of describing nondeterministic transformations of relational structures using monadic second-order logic. More precisely, we show that for every $k \in \{0, 1, 2, \ldots\}$ there is an MSO transduction that inputs a tree decomposition of width k of a graph G, and outputs nondeterministically a tree decomposition of G of optimum width.

As a corollary of our main result, we show (Corollary 3) that an MSO transduction can compute an optimum-width tree decomposition, even if the input is only the graph and not a (possibly suboptimal) tree decomposition. This application is obtained by combining the main result of this paper with Theorem 2.4 from [6], which says that for every $k \in \{0, 1, 2, \ldots\}$ there is an MSO transduction which inputs a graph of treewidth k and outputs nondeterministically one of its tree decompositions of possibly suboptimal width at most $f(k)$, for some function f. In particular, we thus strengthen Theorem 2.4 of [6] by making the output a decomposition of exactly the optimum width, instead of only bounded by a function of the optimum.

Our proof is divided into a few steps. First, we prove a result called the *Dealternation Lemma*, which shows that there always exists an optimum-width tree decomposition that has bounded "alternation" with respect to the input suboptimal decomposition. Intuitively, small alternation is the key property allowing an optimum-width tree decomposition to be captured by an MSO transduction or by a dynamic programming algorithm that works on the input suboptimal decomposition. This part of the proof essentially corresponds to the machinery of typical sequences of Bodlaender and Kloks. However, we find the approach via alternation more intuitive and combinatorially less complicated, and we hope that it will find applications for computing other width measures. In fact, a similar approach has very recently been used by Giannopoulou et al. [10] in a much simpler setting of cutwidth to give a new fixed-parameter algorithm for this graph parameter.

Next, we derive a corollary of the Dealternation Lemma called the *Conflict Lemma*, which directly prepares us to construct the MSO transduction for the Bodlaender-Kloks problem. The Conflict Lemma is stated in purely combinatorial terms, but intuitively it shows that some optimum-width tree decomposition of the graph can be interpreted in the given suboptimum-width tree decomposition using subtrees that cross each other in a restricted fashion, guessable in MSO. Finally, we formalize the intuition given by the Conflict Lemma in MSO, thus constructing the MSO transduction promised in our main result.

2 Preliminaries and statement of the main result

Trees, forests and tree decompositions. Throughout this paper all graphs are undirected, unless explicitly stated. A *forest* (which is sometimes called a *rooted forest* in other contexts) is defined to be an acyclic graph, where every connected component has one designated node called the *root*. This naturally imposes parent–child and ancestor–descendant relations in a (rooted) forest. We use the usual tree terminology: root, leaf, child, parent, descendant and ancestor. We assume that every node is its own descendant, to exclude staying in the same node we use the name *strict descendant*. Likewise for ancestors. For forests we often use the name *node* instead of vertex. A tree is the special case of a forest that is connected and thus

has one root. Two nodes in a forest are called siblings if they have a common parent, or if they are both roots. Note that there is no order on siblings, unlike some models of unranked trees and forests where siblings are ordered from left to right.

A *tree decomposition* of a graph G is a pair $t = (F, \beta)$, where F is a rooted forest and β is a function that associates *bags* to the nodes of F.A bag is a nonempty subset of vertices of G. We require the following two properties:

(T1) whenever uv is an edge of G, then there exists a node of F whose bag contains both u and v; and

(T2) for every vertex u of G, the set of nodes of F whose bags contain u is nonempty and induces a connected subtree in F.

The *width* of a tree decomposition is its maximum bag size minus 1, and the *treewidth* of a graph is the minimum width of its tree decomposition. An *optimum-width* tree decomposition is one whose width is equal to the treewidth of the underlying graph. Note that throughout this paper all tree decompositions will be rooted forests. This slightly diverges from the literature where usually the shape of a tree decomposition is an unrooted tree.

For a tree decomposition $t = (F, \beta)$ of a graph G, and each node x of F, we define the following vertex sets:

- The *adhesion* of x, denoted $\sigma(x)$, is equal to $\beta(x) \cap \beta(x')$, where x' is the parent of x in F. If x is a root of F, we define its adhesion to be empty.
- The *margin* of x, denoted $\mu(x)$, is equal to $\beta(x) \setminus \sigma(x)$.
- The *component* of x, denoted $\alpha(x)$, is the union of the margins of all the descendants of x (including x itself). Equivalently, it is the union of the bags of all the descendants of x, minus the adhesion of x.

Whenever the tree decomposition t is not clear from the context, we specify it in the subscript, i.e., we use operators $\beta_t(\cdot)$, $\sigma_t(\cdot)$, $\mu_t(\cdot)$, and $\alpha_t(\cdot)$.

Observe that, by property (T2) of a tree decomposition, for every vertex of G there is a unique node whose bag contains u, but the bag of its parent (if exists) does not contain u. In other words, there is a unique node whose margin contains u. Consequently, the margins of the nodes of a tree decomposition form a partition of the vertex set of the underlying graph.

Relational structures and MSO. Define a *vocabulary* to be a finite set of *relation names*, each with associated arity that is a nonnegative integer. A *relational structure* over the vocabulary Σ consists of a set called the *universe*, and for each relation name in the vocabulary, an associated relation of the same arity over the universe. To describe properties of relational structures, we use logics, mainly *monadic second-order logic* (MSO for short). This logic allows quantification both over single elements of the universe and also over subsets of the universe. For a precise definition of MSO, see [7].

We use MSO to describe properties of graphs and tree decompositions. To do this, we need to model graphs and tree decompositions as relational structures. A graph is viewed as a relational structure, where the universe is a disjoint union of the vertex set and the edge set of a graph. There is a single binary incidence relation, which selects a pair (v, e) whenever v is a vertex and e is an incident edge. The edges can be recovered as those elements of the universe which appear on the second coordinate of the incidence relation; the vertices can be recovered as the rest of the universe. For a tree decomposition of a graph G, the universe of the corresponding structure consists of the disjoint union of: the vertex set of G, the edge set of G, and the node set of the tree decomposition. There is the incidence relation between vertices and edges, as for graphs, a binary descendant relation over the nodes of the tree decomposition, and a binary bag relation which selects pairs (v, x) such that x is

a node of the tree decomposition whose bag contains vertex v of the graph. The nodes of the decomposition can be recovered as those which are their own descendants, since we assume that the descendant relation is reflexive. Note that thus, the representation of a tree decomposition as a relational structure contains the underlying graph as a substructure.

MSO transductions. Suppose that Σ and Γ are vocabularies. Define a *transduction* with input vocabulary Σ and output vocabulary Γ to be a set of pairs

(input structure over Σ, output structure over Γ)

which is invariant under isomorphism of relational structures. When talking about transductions on graphs or tree decompositions, we use the representations described in the previous paragraph. Note that a transduction is a relation and not necessarily a function, thus it can have many different possible outputs for the same input. A transduction is called *deterministic* if it is a partial function (up to isomorphism). For example, the subgraph relation is a transduction from graphs to graphs, but it is not deterministic since a graph can have many subgraphs. On the other hand, the transformation that inputs a tree decomposition and outputs its underlying graph is a deterministic transduction.

This paper uses MSO transductions, as defined in the book of Courcelle and Engelfriet [7], which are a special case of transductions that can be defined using the logic MSO. The precise definition is in Section 5, but the main idea is that an MSO transduction is a finite composition of transductions of the following types: copy the input a fixed number of times, nondeterministically color the universe of the input, and add new predicates to the vocabulary with interpretations given by MSO formulas over the input vocabulary. We refer to Courcelle and Engelfriet [7] for a broader discussion of the role of MSO transduction in the theory of formal languages for graphs.

The main result. We now state the main contribution of this paper, which is an MSO version of the algorithm of Bodlaender and Kloks.

▶ **Theorem 2.** *For every $k \in \{0, 1, 2, \ldots\}$ there is an* MSO *transduction from tree decompositions to tree decompositions such that for every input tree decomposition t:*

- *if t has width at most k, then there is at least one output; and*
- *every output is an optimum-width tree decomposition of the underlying graph of t.*

We remark that the transduction of Theorem 2 is not deterministic, i.e. it might have several outputs on the same input. Using Theorem 2, we prove that an MSO transduction can compute an optimum-width tree decomposition given only the graph.

▶ **Corollary 3.** *For every $k \in \{0, 1, 2, \ldots\}$ there is an* MSO *transduction from graphs to tree decompositions such that for every input graph G:*

- *if G has treewidth at most k, then there is at least one output; and*
- *every output is a tree decomposition of G of optimum width.*

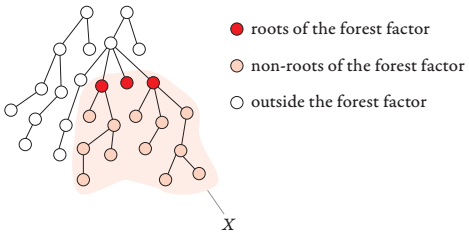
Proof. Theorem 2.4 of [6] says that for every $k \in \{0, 1, 2, \ldots\}$ there is an MSO transduction with exactly the properties stated in the statement, except that when the input has treewidth k, then the output tree decompositions have width at most $f(k)$, for some function $f\colon \mathbb{N} \to \mathbb{N}$. By composing this transduction with the transduction given by Theorem 2, applied to $f(k)$, we obtain the claim. ◀

Structure of the paper. The rest of this paper is devoted to the proof of Theorem 2. In Section 3 we formulate the Dealternation Lemma. Intuitively, this result says that any optimum-width tree decomposition s can be adjusted without increasing the width so that it behaves nicely with respect to the input suboptimal decomposition t in the following sense: for every subtree of t, the vertices appearing in the bags of this subtree are partitioned into few "connected blocks" in s. This part holds the essence of typical sequences of Bodlaender and Kloks [4], but in the full version of the paper (see `https://arxiv.org/abs/1701.06937`) we give a self-contained proof in order to achieve stronger assertions and highlight the key combinatorial properties we use later on. Then, we prove a corollary of the Dealternation Lemma, which we call the Conflict Lemma. This result intuitively states that some optimum-width tree decomposition of the graph can be interpreted in the given suboptimum-width tree decomposition. This intuition is formalized in the last section, where we introduce formally MSO transductions and use the combinatorial property given by the Conflict Lemma to prove Theorem 2.

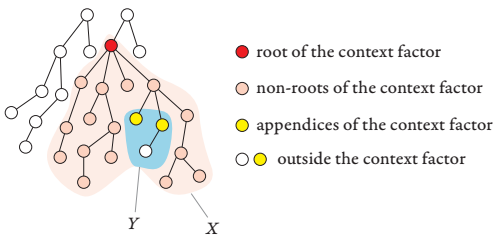
3 Dealternation

This section is devoted to the Dealternation Lemma, which intuitively says that for a tree decomposition t of bounded, though possibly suboptimal width, there always exists an optimum-width decomposition in which every subtree of t is broken into small number of "pieces". We begin by defining factors, which is our notion of "pieces" of a tree decomposition.

Factors and factorizations. Intuitively, a factor is a set of nodes in a forest that respects the tree structure. We define three kinds of factors: tree factors, forest factors, and context factors. A *tree factor* in a forest is a set of nodes obtained by taking all (not necessarily strict) descendants of some node, which is called the *root* of the tree factor. Define a *forest factor* to be a nonempty union of tree factors whose roots are siblings. These roots are called the *roots* of the forest factor. In particular, a tree factor is also a forest factor, with one root.



a forest factor



a context factor

A *context factor* is the difference $X - Y$ for a tree factor X and a forest factor Y, where the root of X is a strict ancestor of every root of Y. For a context factor $X - Y$, its root is defined to be the root of X, while the roots of Y are called the *appendices*. Note that a context factor always contains a unique node that is the parent of all its appendices.

Forest factors and context factors will be jointly called *factors*. The following lemma can be proved by a straightforward case study, and hence we leave its proof to the reader.

▶ **Lemma 4.** *The union of two intersecting factors in the same forest is also a factor.*

For a subset U of nodes of a forest, a U-factor is a factor that is entirely contained in U. A *factorization* of U is a partition of U into U-factors. A U-factor is *maximal* if no other U-factor contains it as a strict subset.

▶ **Lemma 5.** *For every subset of nodes U in a forest, the maximal U-factors form a factorization of U.*

Proof. Every node of U is contained in some factor, e.g., a singleton factor (which has forest or context type depending on whether the node is a leaf or not). Thus, every node of U is also contained in some maximal U-factor. On the other hand, two different maximal U-factors must be disjoint, since otherwise by Lemma 4, their union would also be a U-factor, contradicting maximality. ◀

The set of all maximal U-factors will be called the *maximal factorization* of U, and will be denoted by $\mathsf{fact}(U)$. We specify the forest in the subscript whenever it is not clear from the context. Lemma 5 asserts that $\mathsf{fact}(U)$ is indeed a factorization of U. Note that the maximal factorization of U is the coarsest in the following sense: in every factorization of U, each of its factors is contained in some factor of $\mathsf{fact}(U)$. In particular, the maximal factorization has the smallest number of factors among all factorizations of U.

In the sequel, we will need the following simple result about relation between the maximal factorizations of a set and of its complement. Its proof is a part of the proof of the Dealternation Lemma, and can be found in the full version of the paper.

▶ **Lemma 6.** *Suppose (U, W) is a partition of the node set of a rooted forest F, and let k be the number of factors in the maximal factorization of W. Then the maximal factorization of U has at most $k + 1$ forest factors and at most $2k - 1$ context factors.*

Separation forests. The general definition of a tree decomposition is flexible and allows for multiple combinatorial adjustments. Here, we will rely on a normalized form that we call *separation forests*, which are essentially tree decompositions where all the margins have size exactly 1. The definition of treewidth via separation forests resembles the definition of pathwidth via the so-called *vertex separation number* [14].

▶ **Definition 7.** Suppose G is a graph. A *separation forest* of G is a rooted forest F on the same vertex set as G such that G is contained in the ancestor-descendant closure of F; that is, whenever uv is an edge of G, then u is an ancestor of v in F or vice versa.

Separation forests are used to define the graph parameter *treedepth*, which is equal to the minimum depth of a separation forest of a graph. To define treewidth, we need to take a different measure than just the depth, as explained next.

Suppose F is a separation forest of G. Endow F with the following bag function $\beta(\cdot)$. For any vertex u of G, assign to u the bag $\beta(u)$ consisting of u and all the ancestors of u in F that have a neighbor among the descendants of u in F. The following claim follows by verifying the definition of a tree decomposition; we leave the easy proof to the reader.

▶ **Claim 8.** *If F is a separation forest of G and $\beta(\cdot)$ is defined as above, then (F, β) is a tree decomposition of G. Moreover, for every vertex u of G, the margin of u in (F, β) is $\{u\}$.*

The tree decomposition (F, β) defined above is said to be *induced* by the separation forest F. Observe that if $t = (F, \beta)$ is induced by F, then for any vertex u, the component of u in t consists of all the descendants of u in F.

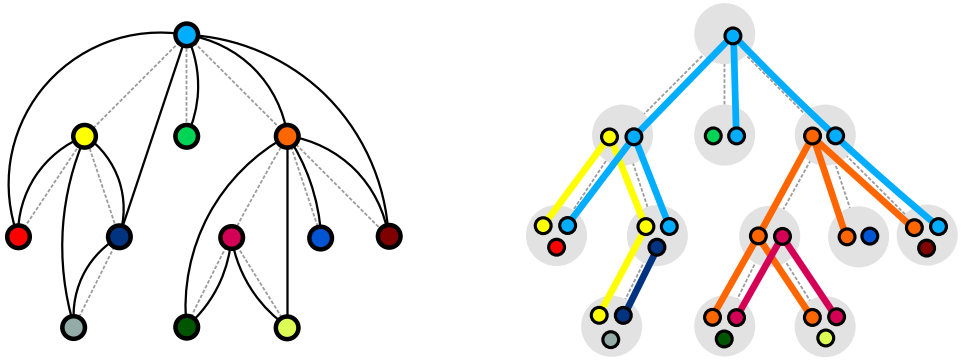


Figure 1 Construction of the induced tree decomposition from a separation forest. The graph edges are depicted in black, the child-parent relation of the forest is depicted as dashed grey lines.

One can reformulate the construction given above as follows. First, put every vertex u into its bag $\beta(u)$. Then, examine every neighbor v of u, and if v is a descendant of u in F, then add u to every bag on the path from v to u in F. Thus, every vertex u is "smeared" onto a subtree of F, where u is the root of this subtree and its leaves correspond to those neighbors of u that are also its descendants in F. This construction is depicted in Figure 1.

The *width* of a separation forest is simply the width of the tree decomposition induced by it. Consequently, the width of a separation forest is never smaller than the treewidth of a graph. The next result shows that in fact there is always a separation forest of optimum width. The proof follows by a simple surgery on an optimum-width tree decomposition, and can be found in the full version of the paper.

▶ **Lemma 9.** *For every graph G there exists a separation forest of G whose width is equal to the treewidth of G.*

Dealternation Lemma. We are finally ready to state the Dealternation Lemma.

▶ **Lemma 10** (Dealternation Lemma)**.** *There exist functions $f(k) \in \mathcal{O}(k^2)$ and $g(k) \in \mathcal{O}(k^3)$ such that the following holds. Suppose that t is a tree decomposition of a graph G of width k. Then there exists an optimum-width separation forest F of G such that:*
(D1) *for every node x of t, the maximal factorization $\mathsf{fact}_F(\alpha_t(x))$ has at most $f(k)$ factors;*
(D2) *for every node x of t, there are at most $g(k)$ children of x in the set*

$$\{y\colon y \text{ is a node of } t \text{ with at least one context factor in } \mathsf{fact}_F(\alpha_t(y))\}.$$

Note that in the statement of the Dealternation Lemma, the vertex set of G is at the same time the node set of the forest F. Thus, $\mathsf{fact}_F(\alpha_t(x))$ denotes the maximal factorization of $\alpha_t(x)$, treated as a subset of nodes of F.

The proof of the Dealternation Lemma uses essentially the same core ideas as the correctness proof of the algorithm of Bodlaender and Kloks [4]. We include our proof in the full version of the paper for several reasons. First, unlike in [4], in our setting we cannot assume that t has binary branching, as is the case in [4]. In fact, condition (D2) is superfluous when t has binary branching. Second, our formulation of the Dealternation Lemma highlights the key combinatorial property, which is expressed as the existence of a single separation forest F that behaves nicely with respect to the input decomposition t. This property is somehow implicit [4], where the existence of nicely-behaved optimum-width tree decompositions is argued along performing dynamic programming. For this reason, we find the new formulation more explanatory and potentially interesting on its own.

4 Using the Dealternation Lemma

In this section we use the Dealternation Lemma to show that an optimum-width separation forest of a graph can be interpreted in a suboptimum-width tree decomposition. For this, we need to develop a better understanding of the combinatorial insight provided by the Dealternation Lemma, which is expressed via an auxiliary graph, called the *conflict graph*.

Suppose G is a graph, t is a tree decomposition of G of width k, and F is a separation forest of G. Let ϕ be the mapping that sends each vertex u of G to the unique node of t that contains u in its margin. For a vertex u of G, we define the *stain* of u, denoted S_u, which is a subgraph of the underlying forest of t, as follows. For every child v of u in F, find the unique path in t between $\phi(u)$ and $\phi(v)$. Then stain S_u consists of the node $\phi(u)$ and the union of these paths. Note that if u is a leaf of F, then the stain S_u consists only of the node $\phi(u)$. Define the *conflict graph* $H(t, F)$ as follows. The vertices of $H(t, F)$ are the vertices of G, and vertices u and v are adjacent in $H(t, F)$ if and only their stains S_u and S_v have a node in common. The main result of this section can be formulated as follows.

▶ **Lemma 11** (Conflict Lemma). *There is a function $h(k) \in \mathcal{O}(k^5)$ such that if t and F are as in the Dealternation Lemma, then their conflict graph $H(t, F)$ admits a proper coloring with $h(k)$ colors.*

Recall here that a proper coloring of a graph is a coloring of its vertex set such that no two adjacent vertices receive the same color. The rest of this section is devoted to the proof of the Conflict Lemma. From now on, we assume that G, t, F are as in the Dealternation Lemma, and we denote $H = H(t, F)$.

Observe that the conflict graph H is an intersection graph of a family of subtrees of a forest. It is well-known (see, e.g., [11]) that this property precisely characterizes the class of chordal graphs (graphs with no induced cycle of length larger than 3), so H is chordal. Chordal graphs are known to be perfect (again see, e.g., [11]), hence the chromatic number of a chordal graph (the minimum number of colors needed in a proper coloring) is equal to the size of the largest clique in it. On the other hand, subtrees of a forest are known to satisfy the so-called Helly property: whenever $\mathcal{F}$ is some family of subtrees such that the subtrees in $\mathcal{F}$ pairwise intersect, then in fact there is a node of the forest that belongs to all the subtrees in $\mathcal{F}$. This means that the largest clique in an intersection graph of a family of subtrees of a forest can be obtained by taking all the subtrees that contain some fixed node. Therefore, to prove the Conflict Lemma it is sufficient to prove the following claim.

▶ **Claim 12.** *There exists a function $h(k) \in \mathcal{O}(k^5)$ such that every node of t belongs to at most $h(k)$ of the stains $\{S_u \colon u \in V(G)\}$.*

In the remainder of this section we prove Claim 12. Fix any node x of t, and let $y_1, y_2, \ldots, y_p$ be its children in t. Consider the following partition of the vertex set of G:

$$\Pi = (\alpha_t(y_1), \alpha_t(y_2), \ldots, \alpha_t(y_p), \mu_t(x), V(G) \setminus \alpha_t(x))$$

Define a factorization Φ of the whole node set of F as follows: for each set X from the partition Π, take its maximal factorization $\mathsf{fact}_F(X)$, and define Φ to be the union of these maximal factorizations. Thus, Φ is a factorization that refines the partition Π. Since the number of children y_i is unbounded, we cannot expect that Φ has a small number of factors, but at least it has a small number of context factors.

▶ **Claim 13.** *Factorization Φ contains at most $g(k) \cdot f(k) + 2f(k) + k$ context factors, where f and g are as in the Dealternation Lemma.*

Proof. By the Dealternation Lemma, the maximal factorization in F of each of the sets $\alpha_t(y_1), \ldots, \alpha_t(y_p), \alpha_t(x)$ has at most $f(k)$ factors. Moreover, only at most $g(k)$ of the sets $\alpha_t(y_1), \ldots, \alpha_t(y_p)$ can have a context factor in their maximal factorizations. Hence, the maximal factorizations of sets $\alpha_t(y_1), \ldots, \alpha_t(y_p)$ introduce at most $g(k) \cdot f(k)$ context factors to the factorization Π. Since the maximal factorization of $\alpha_t(x)$ has at most $f(k)$ factors as well, by Lemma 6 we deduce that the maximal factorization of $V(G) \setminus \alpha_t(x)$ has at most $2f(k) - 1$ context factors. Finally, the cardinality of $\mu_t(x)$ is at most $k + 1$, so in particular its maximal factorization has at most $k + 1$ factors in total. Summing up all these upper bounds, we conclude that Φ has at most $g(k) \cdot f(k) + 2f(k) + k$ context factors. ◀

With Claim 13 in hand, we complete now the proof of Claim 12. Take any vertex u such that x belongs to the stain S_u. This means that either

(i) u belongs to the margin of x, or

(ii) u does not belong to the margin of x, but u has a child v in F such that the unique path in t between $\phi(u)$ and $\phi(v)$ passes through x.

The number of vertices u satisfying 1 is bounded by the size of the margin of x, which is at most $k + 1$, hence we focus on vertices u that satisfy 2. Observe that condition 2 in particular means that u and v belong to different parts of partition Π, so also to different factors of factorization Φ. Since u is the parent of v in F, this means that the unique factor of Φ that contains u must be a context factor, and u must be the parent of its appendices. Consequently, the number of vertices u satisfying 2 is upper bounded by the number of context factors in factorization Φ, which is at most $g(k) \cdot f(k) + 2f(k) + k$ by Claim 13. We conclude that the number of stains S_u containing x is at most

$$h(k) := g(k) \cdot f(k) + 2f(k) + 2k + 1;$$

in particular $h(k) \in \mathcal{O}(k^5)$. This concludes the proof of Claim 12, so also the proof of the Conflict Lemma is complete.

5 Constructing the transduction

We now use the understanding gathered in the previous sections to give an MSO transduction that takes a tree decomposition of a graph of suboptimum width, and produces an optimum-width tree decomposition. First, we need to precisely define MSO transductions.

MSO transductions. Formally, an MSO transduction is any transduction that can be obtained by composing a finite number of transductions of the following kinds. Note that kind 1 is a partial function, kinds 2, 3, 4 are functions, and kind 5 is a relation.

1. **Filtering.** For every MSO sentence φ over the input vocabulary there is transduction that filters out structures where φ is satisfied. Formally, the transduction is the partial identity whose domain consists of the structures that satisfy the sentence. The input and output vocabularies are the same.
2. **Universe restriction.** For every MSO formula $\varphi(x)$ over the input vocabulary with one free first-order variable there is a transduction, which restricts the universe to those elements that satisfy φ. The input and output vocabularies are the same, the interpretation of each relation in the output structure is defined as the restriction of its interpretation in the input structure to tuples of elements that remain in the universe.
3. **MSO interpretation.** This kind of transduction changes the vocabulary of the structure while keeping the universe intact. For every relation name R of the output vocabulary,

there is an MSO formula $\varphi_R(x_1, \ldots, x_k)$ over the input vocabulary which has as many free first-order variables as the arity of R. The output structure is obtained from the input structure by keeping the same universe, and interpreting each relation R of the output vocabulary as the set of those tuples $(x_1, \ldots, x_k)$ that satisfy φ_R.

4. **Copying.** For $k \in \{1, 2, \ldots\}$, define k-copying to be the transduction which inputs a structure and outputs a structure consisting of k disjoint copies of the input. Precisely, the output universe consists of k copies of the input universe. The output vocabulary is the input vocabulary enriched with a binary predicate copy that selects copies of the same element, and unary predicates $\mathsf{layer}_1, \mathsf{layer}_2, \ldots, \mathsf{layer}_k$ which select elements belonging to the first, second, etc. copies of the universe. In the output structure, a relation name R of the input vocabulary is interpreted as the set of all those tuples over the output structure, where the original elements of the copies were in relation R in the input structure.
5. **Coloring.** We add a new unary predicate to the input structure. Precisely, the universe as well as the interpretations of all relation names of the input vocabulary stay intact, but the output vocabulary has one more unary predicate. For every possible interpretation of this unary predicate, there is a different output with this interpretation implemented.

We remark that the above definition is easily equivalent to the one used in [6], where filtering, universe restriction, and MSO interpretation are merged into one kind of a transduction.

Proving the main result. We are finally ready to prove our main result, Theorem 2. The proof is broken down into several steps. The first, main step shows that an MSO transduction can output optimum-width separation forests. Here, a separation forest of a graph G is encoded by enriching the relational structure encoding G with a single binary relation interpreted as the child relation of F. Note that the definition of a separation forest is MSO-expressible: there is an MSO sentence that checks whether the additional relation indeed encodes a separation forest of the graph.

▶ **Lemma 14.** *For every $k \in \{0, 1, 2, \ldots\}$, there is an* MSO *transduction from tree decompositions to separation forests such that for every input tree decomposition t:*

- *every output is a separation forest of the underlying graph of t; and*
- *if t has width at most k, then there is at least one output that is a separation forest of optimum width.*

Proof. Observe that the verification whether the width of t is at most k can be expressed by an MSO sentence, so we can first use filtering to filter out any input tree decomposition t whose width is larger than k; for such decompositions, the transduction produces no output. Let G be the underlying graph of t, and let ϕ be the mapping that sends each vertex u of G to the unique node of t whose margin contains u. By the Conflict Lemma, there exists some separation forest F of G of optimum width such that the conflict graph $H(t, F)$ admits some proper coloring λ with $h(k)$ colors. The constructed MSO transduction attempts at guessing and interpreting F as follows.

First, using coloring and filtering, we guess the coloring λ, represented as a partition of the vertex set of G. Then, again using coloring and filtering, for every vertex u of G we guess whether u is a root of F, and if not, then we guess the color under λ of the parent of u in F.

Next, for every color c used in λ, we guess the forest

$$M_c := \bigcup_{u \in \lambda^{-1}(c)} S_u,$$

where S_u is the stain of u in t, defined as in Section 4 for the separation forest F. Note that the stains $\{S_u \colon u \in \lambda^{-1}(c)\}$ are pairwise disjoint, because λ is a proper coloring of

the conflict graph $H(t, F)$. Thus, the connected components of M_c are exactly these stains. Observe also that M_c is a subgraph of the decomposition t, so we can emulate guessing M_c in an MSO transduction working over t by guessing the subset of those nodes of t, for which the edge of t connecting the node and its parent belongs to M_c.

Having done all these guesses, we can interpret the child relation of F using an MSO predicate as follows. Fix a pair of vertices u and v, and let c be the guessed color of u under λ. Then one can readily check that u is the parent of v in F if and only if the following conditions are satisfied:

- we have guessed that v is not a root of F,
- we have guessed that the color of the parent of v in F is c, and
- u is the unique vertex of color c such that $\phi(u)$ belongs to the same connected component of M_c as $\phi(v)$.

It can be easily seen that these conditions can be expressed by an MSO formula with two free variables u and v.

Finally, we filter out all the wrong guesses by verifying, using an MSO sentence, whether the interpreted child relation on the vertices of G indeed forms a rooted forest, and whether this forest is a separation forest of G. Obviously, the separation forest F was obtained for at least one of the guesses, and survives this filtering. At the end, we remove the nodes of decomposition t from the structure using universe restriction. ◀

Next, we need to construct the induced tree decomposition out of a separation forest.

▶ **Lemma 15.** *There is an* MSO *transduction from separation forests to tree decompositions that on each input separation forest has exactly one output, which is the tree decomposition induced by the input.*

Proof. We copy the vertex set of the graph two times, and declare the second copies to be the nodes of the constructed tree decomposition. Using the child relation of the input separation forest, we can interpret in MSO the descendant relation in the forest of the decomposition. Finally, the bag relation in the induced tree decomposition, as defined in Section 3, can be easily interpreted using an MSO formula. ◀

Finally, so far the transduction can output tree decompositions of suboptimal width, which should be filtered out. For this, we need the following MSO-expressible predicate.

▶ **Lemma 16.** *For every $k \in \{0, 1, 2, \ldots\}$, there is an* MSO*-sentence over tree decompositions that holds if and only if the given tree decomposition has width at most k and its width is optimum for the underlying graph.*

Proof. Let t be the given tree decomposition of a graph G. Obviously, we can verify using an MSO sentence whether the width of t is at most k. To check that the width of t is optimum, we could use the fact that graphs of treewidth k are characterized by a finite list of forbidden minors, but we choose to apply the following different strategy. Let R_k be the MSO transduction that is the composition of the transductions of Lemmas 14 (for parameter k) and 15. Provided the input tree decomposition t has width at most k, transduction R_k outputs some set of tree decompositions of G among which one has optimum width. Hence, t has optimum width if and only if the output $R_k(t)$ does not contain any tree decomposition of width smaller than t.

The Backwards Translation Theorem for MSO transductions [7] states that whenever T is an MSO transduction and ψ is an MSO sentence over the output vocabulary, then the set of structures on which T outputs at least one structure satisfying ψ, is MSO-definable over

the input vocabulary. Hence, for every $p < k$, there exists an MSO sentence φ_p that verifies whether $R_k(t)$ outputs at least one tree decomposition of width at most p. Therefore, we can check whether t has optimum width by making a disjunction over all ℓ with $0 \leq \ell \leq k$ of the sentences stating that t has width exactly ℓ and $R_k(t)$ does not output any tree decomposition of width less than ℓ. ◀

Theorem 2 now follows by composing the MSO transductions given by Lemmas 14 and 15, and at the end applying filtering using the predicate given by Lemma 16.

6 Conclusions

In this work we have constructed an MSO transduction that, given a constant-width tree decomposition of a graph, computes a tree decomposition of this graph of optimum width. As we have shown, this transduction can be conveniently composed with the MSO transduction given in [6] to prove that given a graph of constant treewidth, some optimum-width tree decomposition can be computed by means of an MSO transduction.

One direct application of this result is a strengthening of the main result of [6]. There, we have proved that if a class of graphs of treewidth at most k is recognizable (see [6] for omitted definitions), then it can be defined in MSO with modular counting predicates. The main technical component of this proof was Theorem 2.4, which states that for every k there is an MSO transduction from graphs to tree decompositions, which given a graph of treewidth k outputs some its tree decomposition of width bounded by $f(k)$, for some doubly-exponential function f. Then the proof of the main result of [6] used $f(k)$-recognizability, i.e., recognizability within the interface (sourced) graphs with at most $f(k)$ interfaces (sources). By replacing the usage of Theorem 2.4 of [6] with Corollary 3 of this paper, we deduce that only k-recognizability of a class of graphs of treewidth at most k is sufficient to prove that it can be defined in MSO with modular counting predicates. However, this strengthening was already known: Courcelle and Lagergren [8] proved that if a class of graphs of treewidth at most k is k-recognizable, then it is also k'-recognizable for all $k' \geq k$. In fact, the proof technique of Courcelle and Lagergren essentially uses the same technique as Bodlaender and Kloks [4] and as we do in this work; the main technical component of [8] can be interpreted as a variant of our Local Dealternation Lemma (see the full version of the paper).

Finally, we see potential algorithmic applications of our main result. Namely, it seems that the existing results, in particular the literature on constructing answers to MSO queries on trees [1, 9, 13], are likely to imply the following algorithmic statement. Suppose R is an MSO transduction whose domain are rooted forests labelled by a finite alphabet. Then, given a forest t, one can compute in time $f(k) \cdot (n+m)$ any member of the output of R on t, or conclude that this output is empty. Here, n is the size of t, m is the size of the output structure (or 0 if transduction R applied to t yields no output), k is the size of the description of R, and f is some function. Assuming such an algorithmic statement, the algorithmic result of Bodlaender and Kloks [4] (without a specified dependency on k of the running time) would basically follow from applying it to the MSO transduction constructed in this paper. However, such a tool could be of more general use. It would essentially reduce designing constructive dynamic programming algorithms on tree decompositions, which most often is a tedious and complicated task, to describing the corresponding transformations using MSO transductions, similarly as Courcelle's theorem reduces designing dynamic programming algorithm for decision problems to expressing them in MSO. We will explore these algorithmic applications in the journal version of our work.

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