Tree-walking automata

Mikołaj Bojańczyk
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Plan

-A tree-walking automaton

-Expressive power

-Pebble automata and logic

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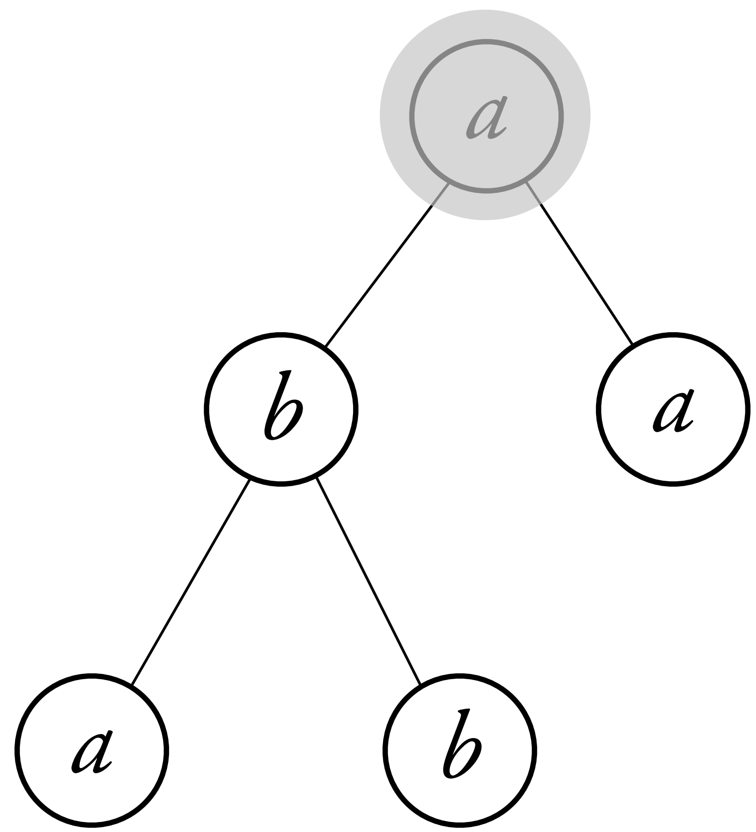
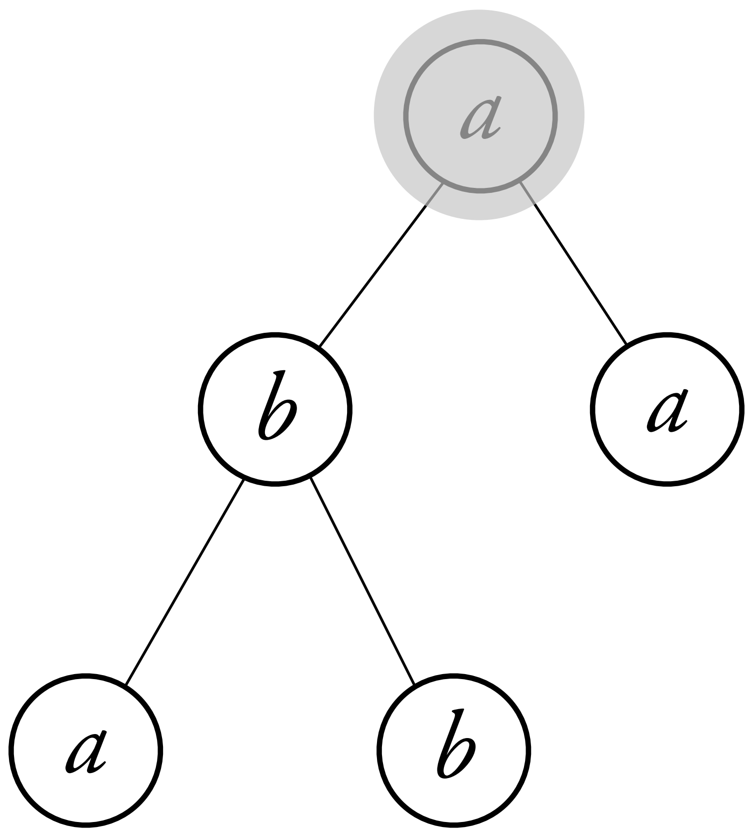
Trees are finite, binary and labeled



Trees are finite, binary and labeled

A tree-walking automaton is sequential and two-way.



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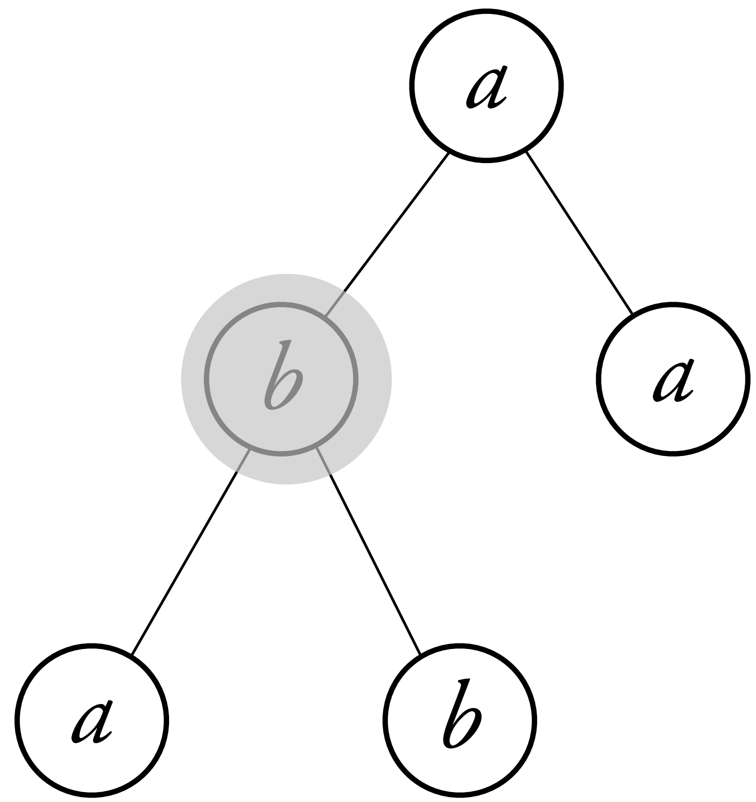

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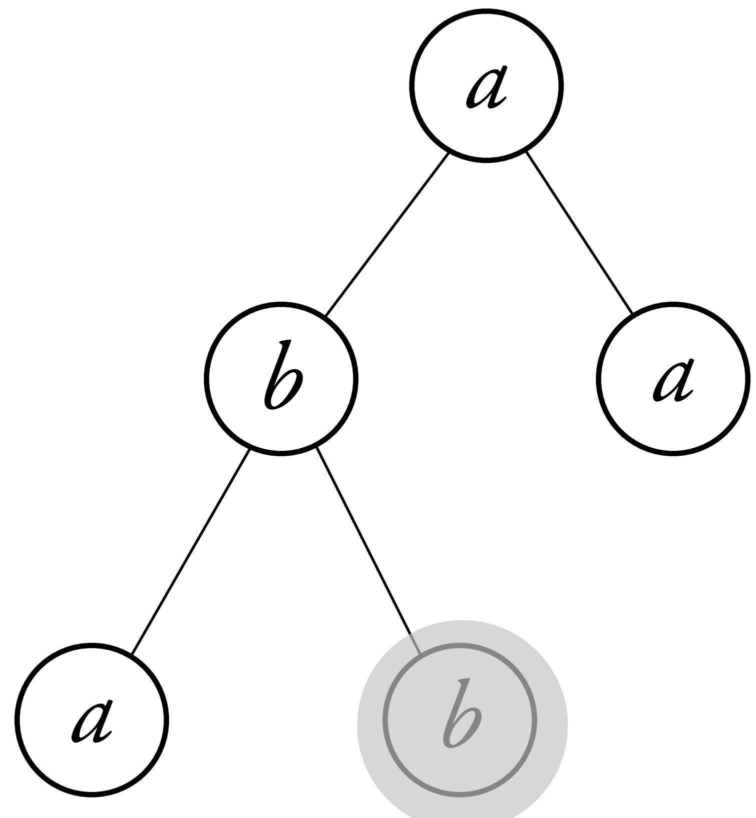

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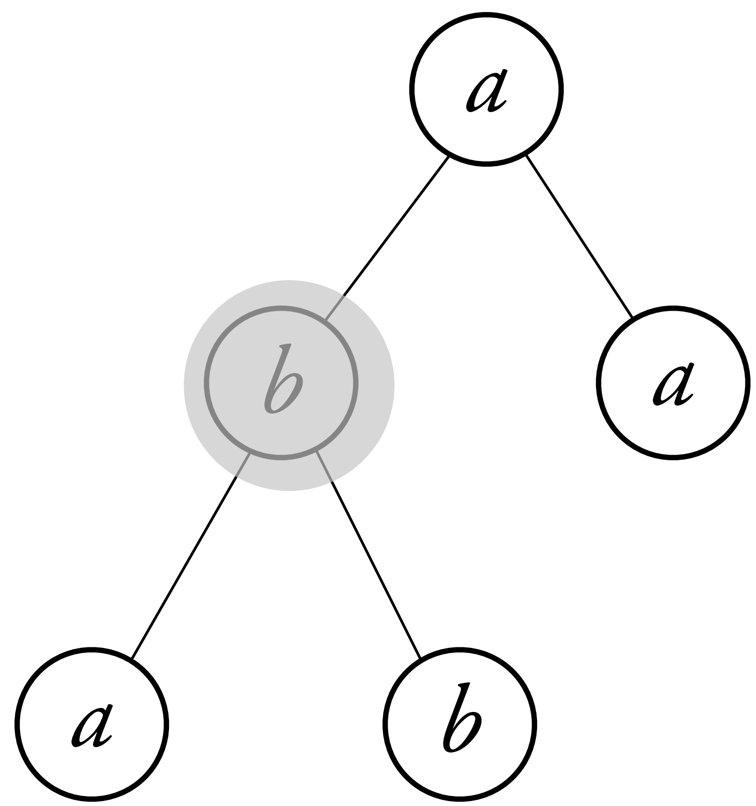

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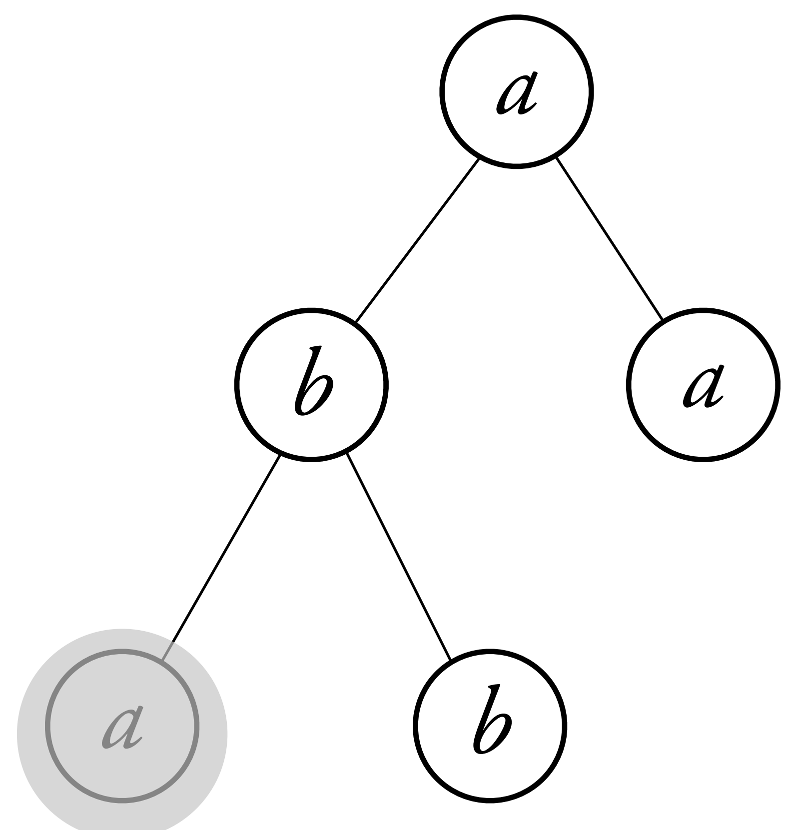
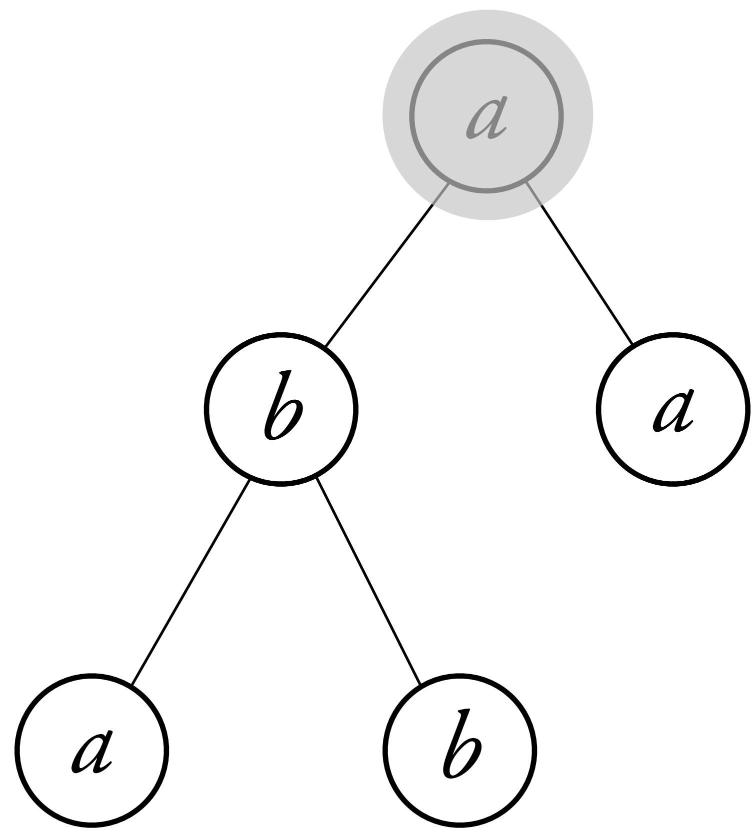


If the state is p and the node is the root with label a, then move to the left child and change state to q.



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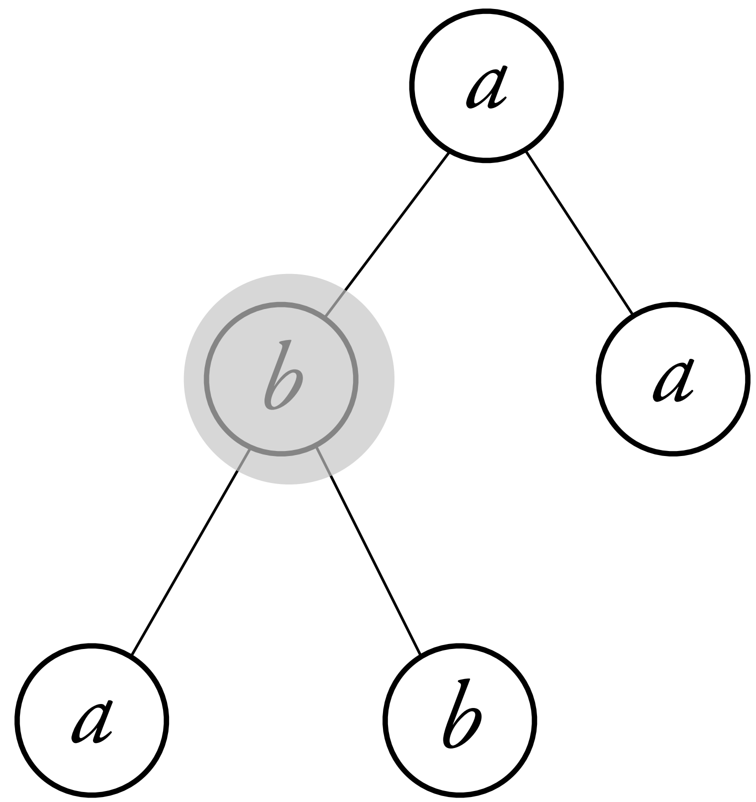

test

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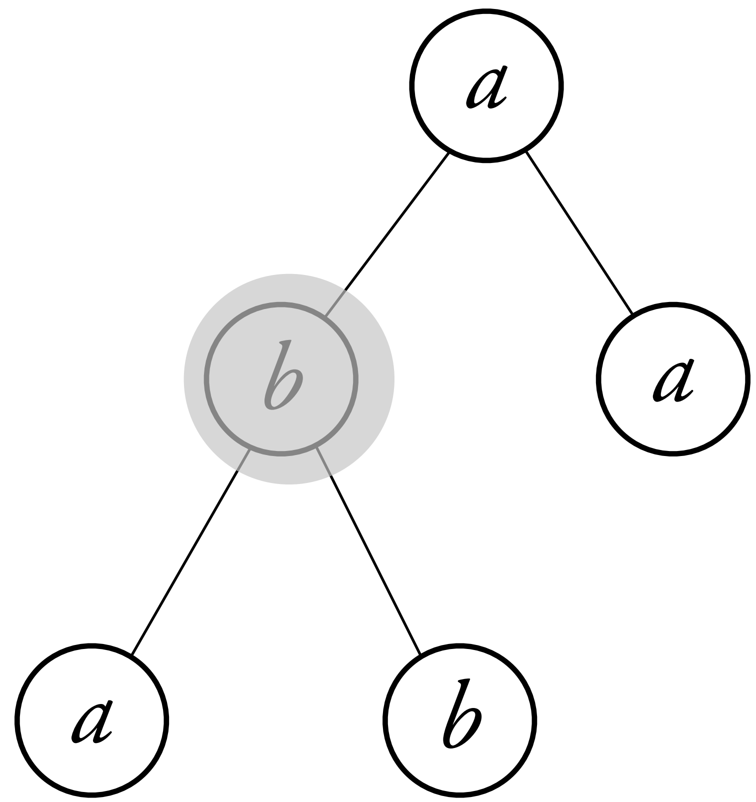
command



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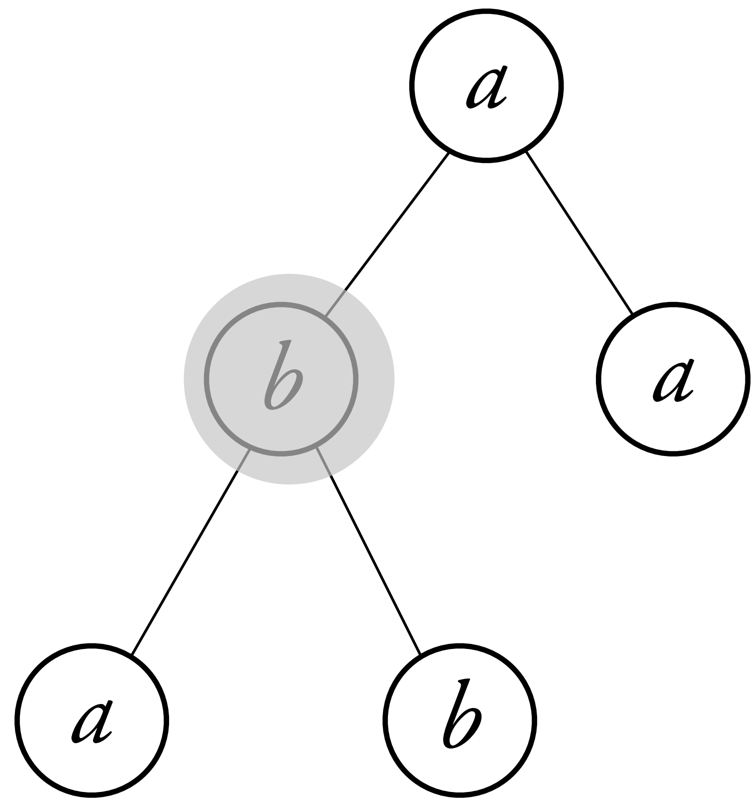
Tests are boolean combinations of: has label a, is right/left child, is leaf



Commands are: move left/right/up, accept, reject

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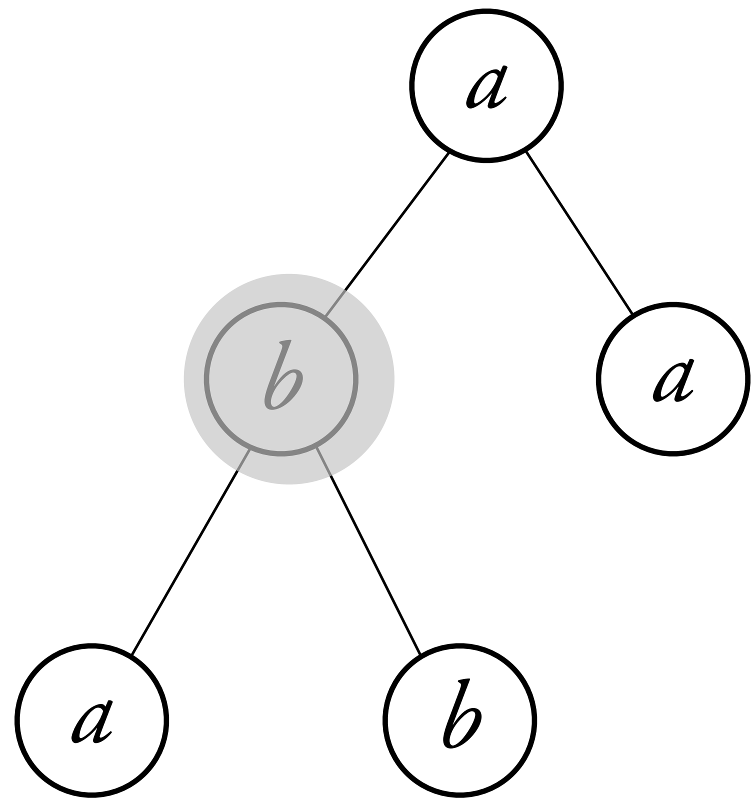

Commands are:
move left/right/up,
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Def. A tree walking-automaton is a tuple $\langle Q, q_I, \Sigma, \delta \rangle$

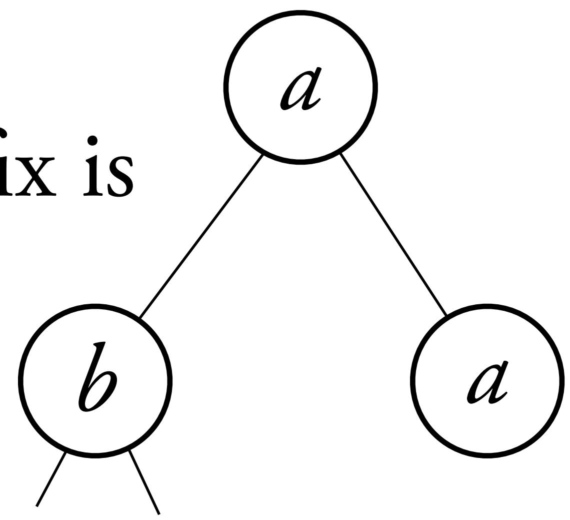
states

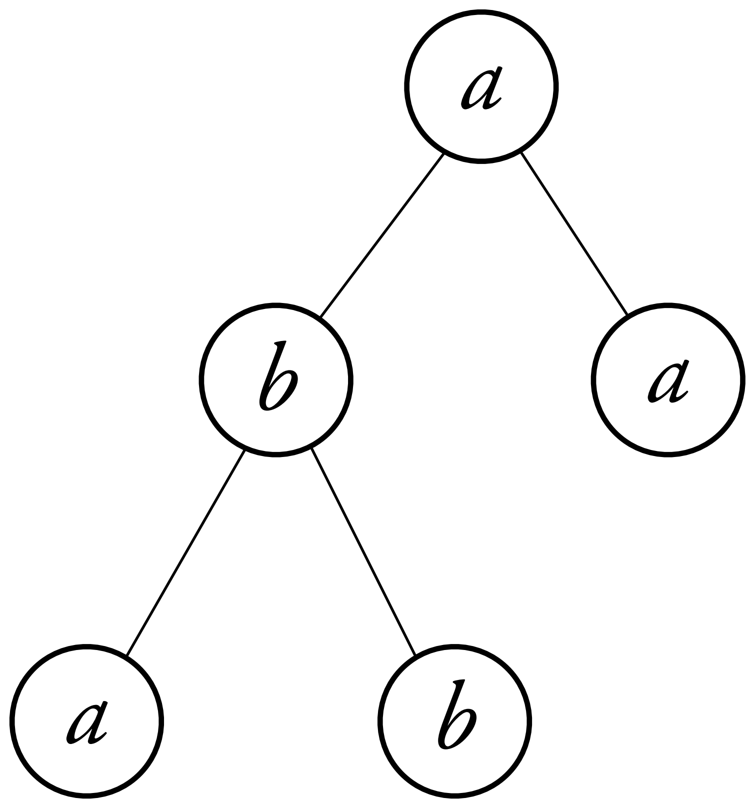
initial state

alphabet

transitions

Example.
Check if the prefix is





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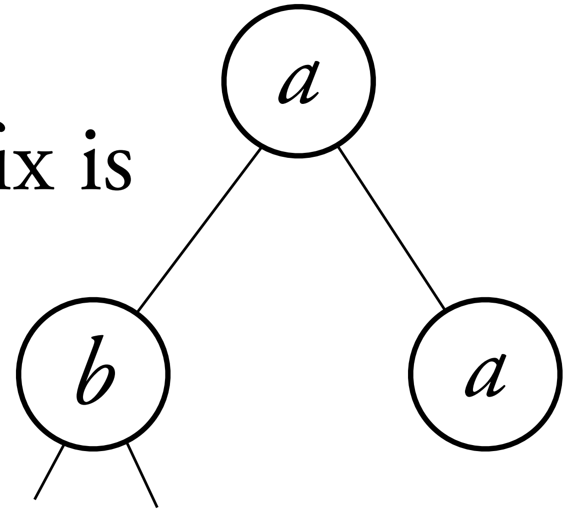
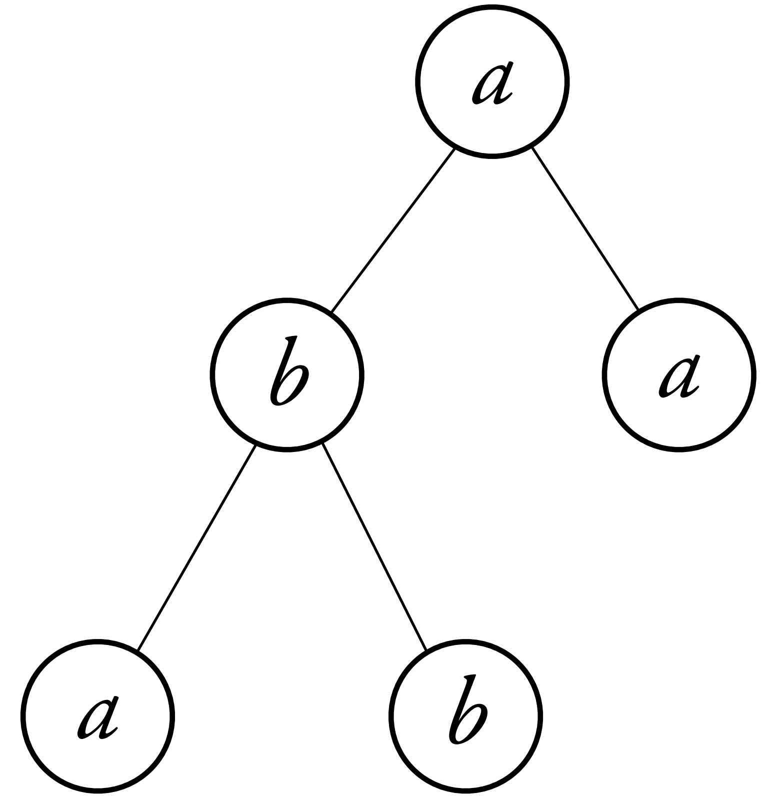
Check if the prefix is 



In state p, label a and root, move left, change state to q
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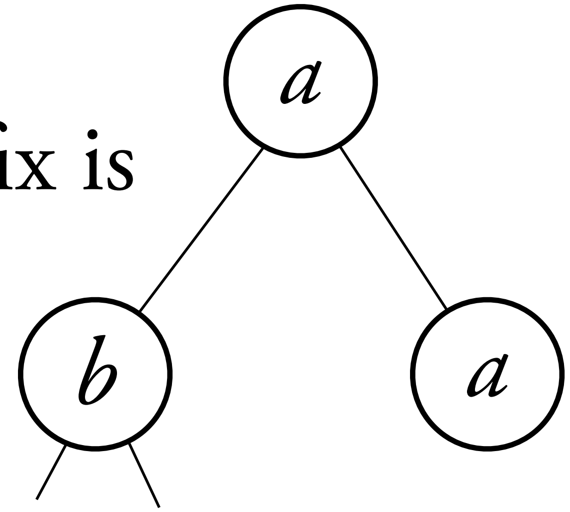

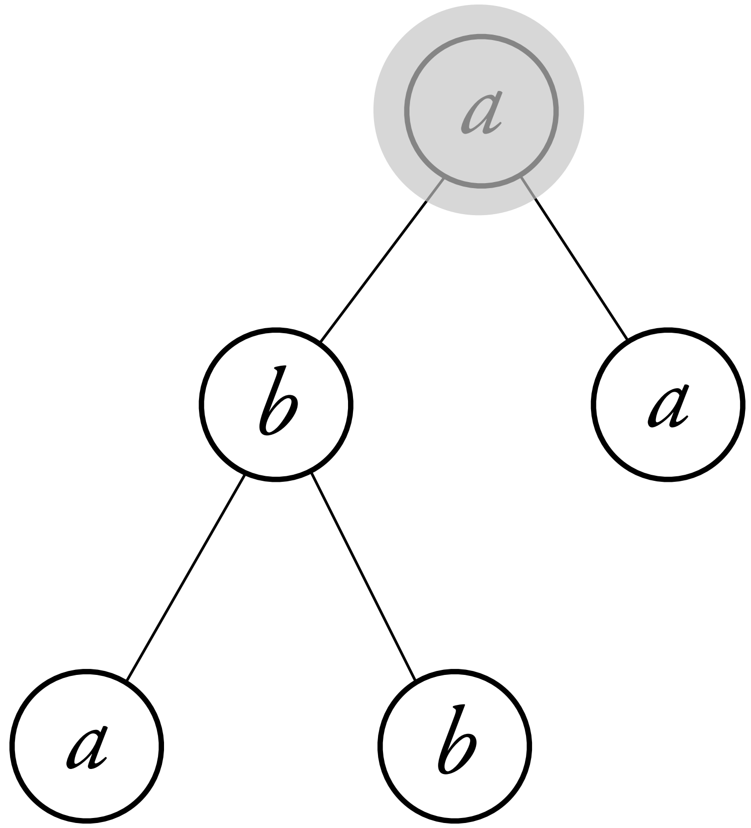

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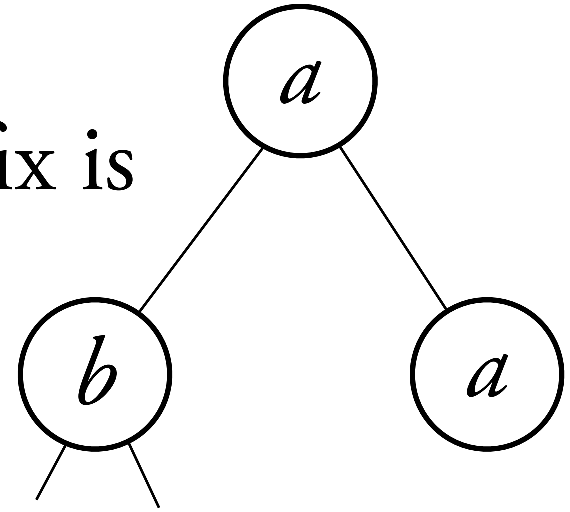

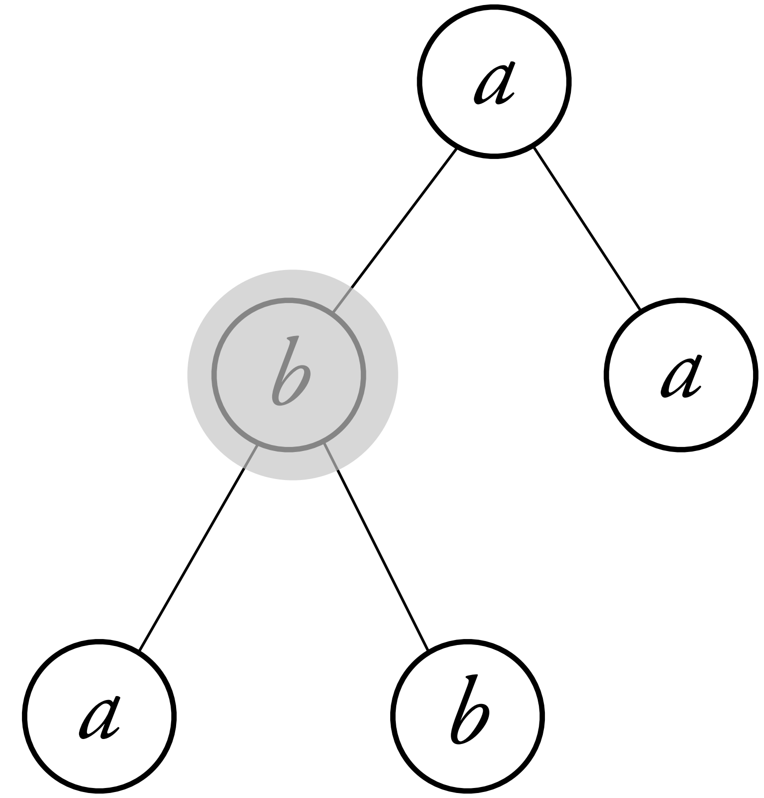

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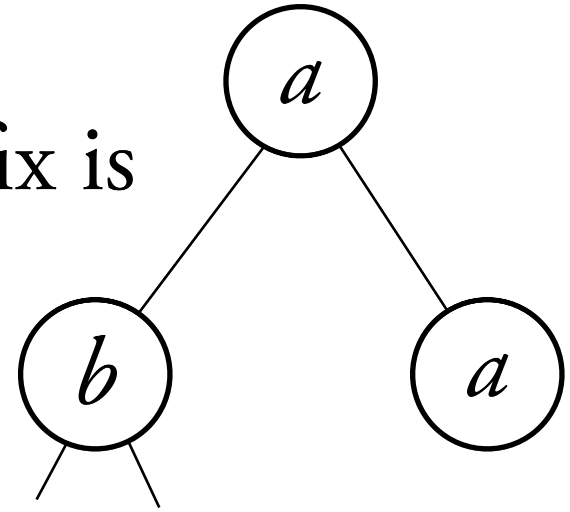

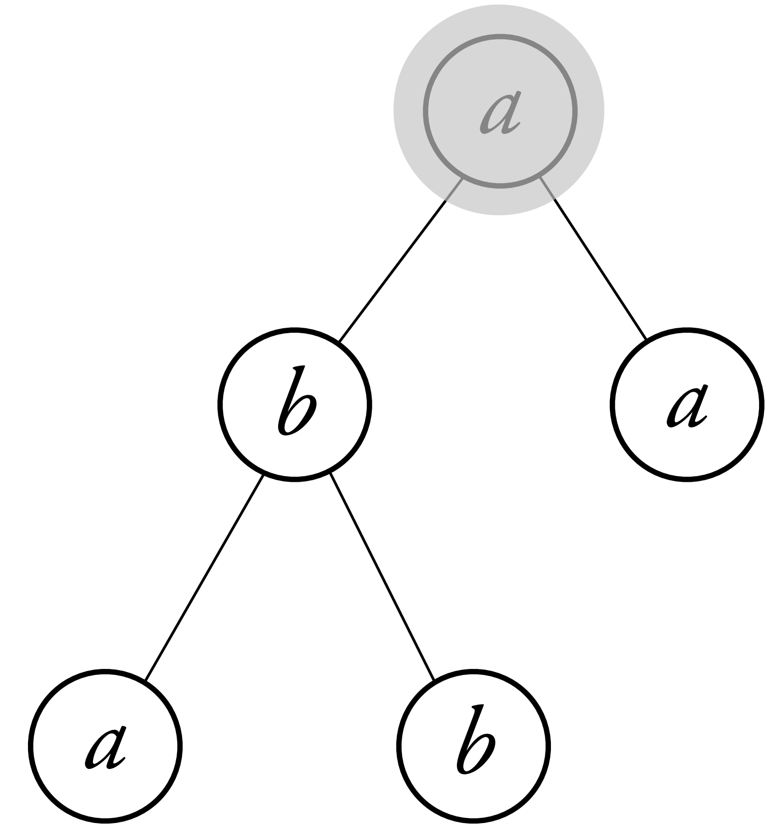

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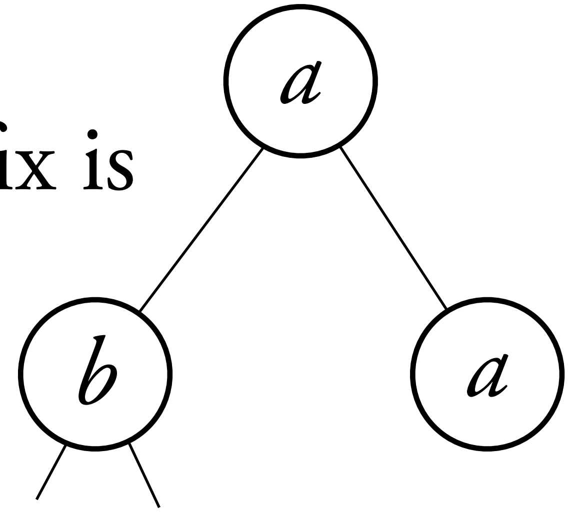

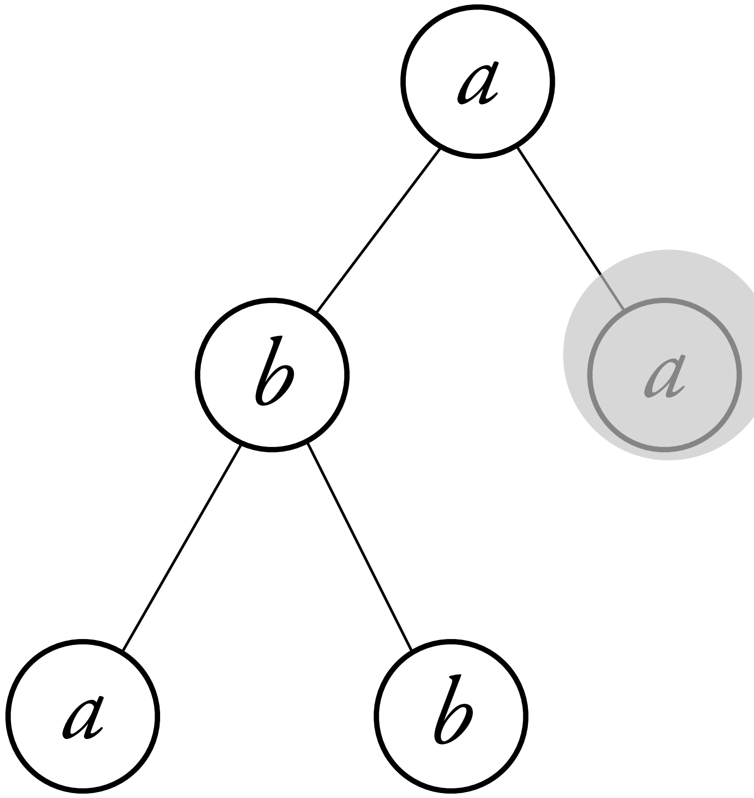

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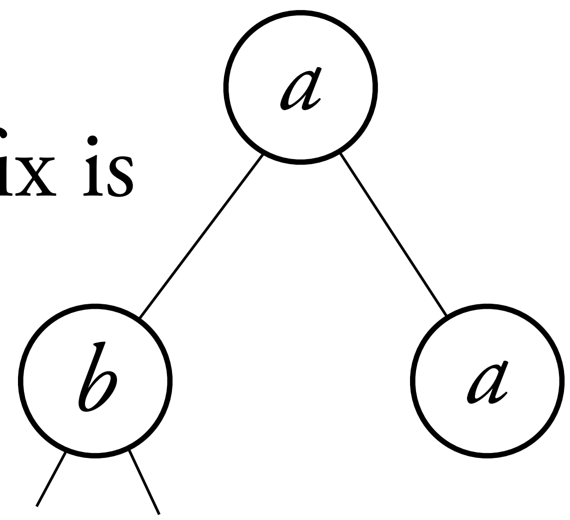

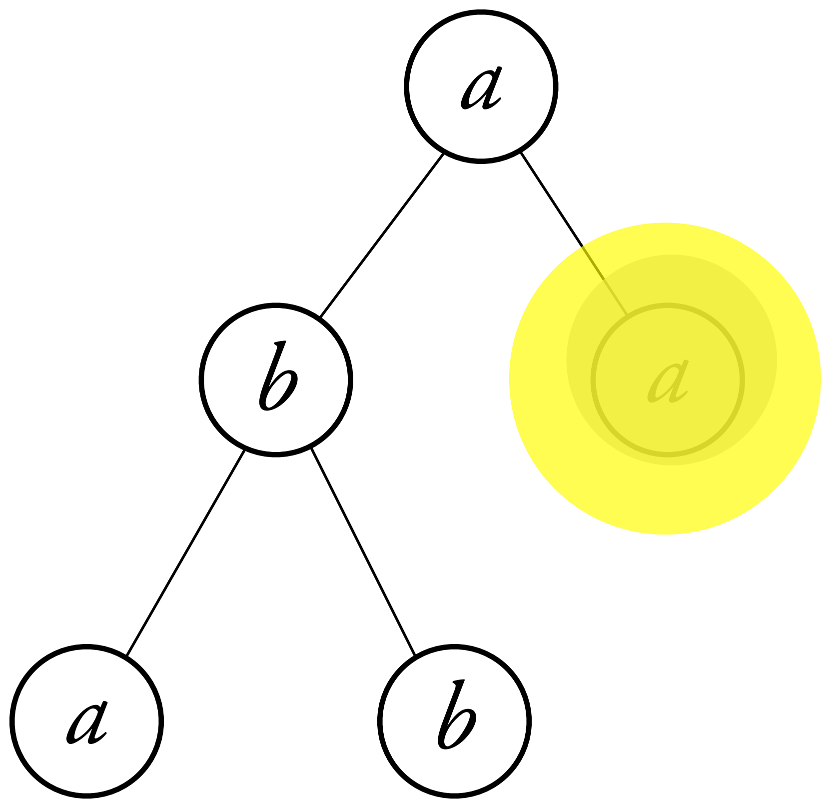

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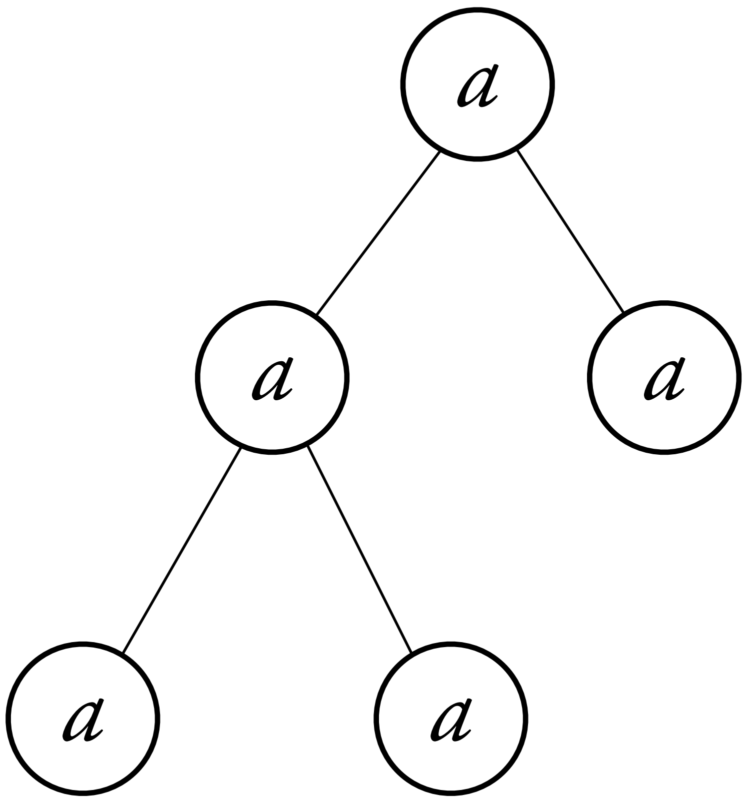
Example.
Some node has label b

In state p, move left.
In state p, move right.
In state p, label b, accept.



Example.
All nodes have label a

Example.
Some node has label *b*

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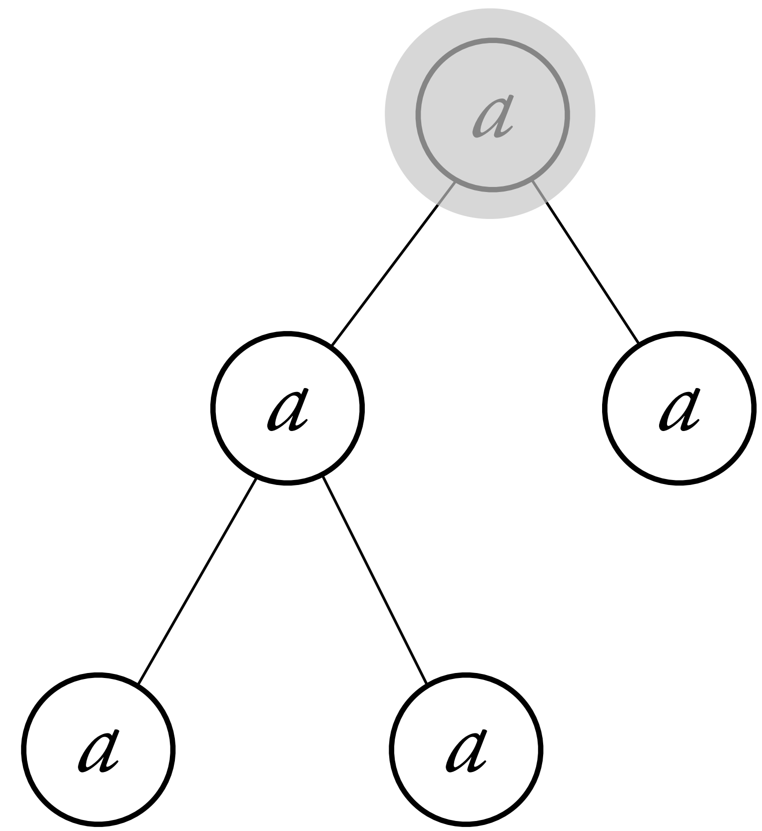

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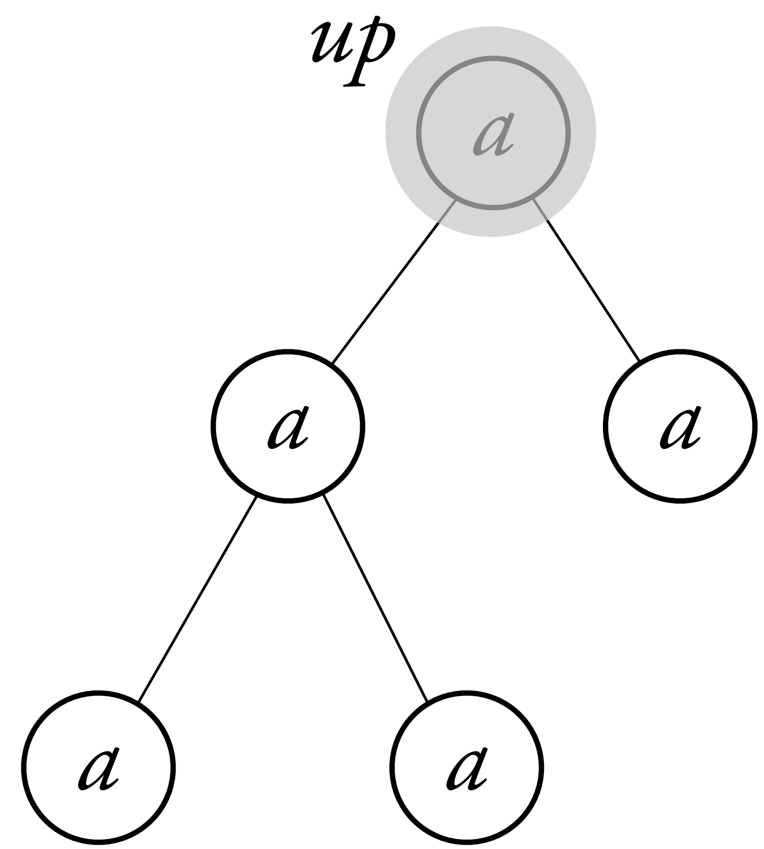

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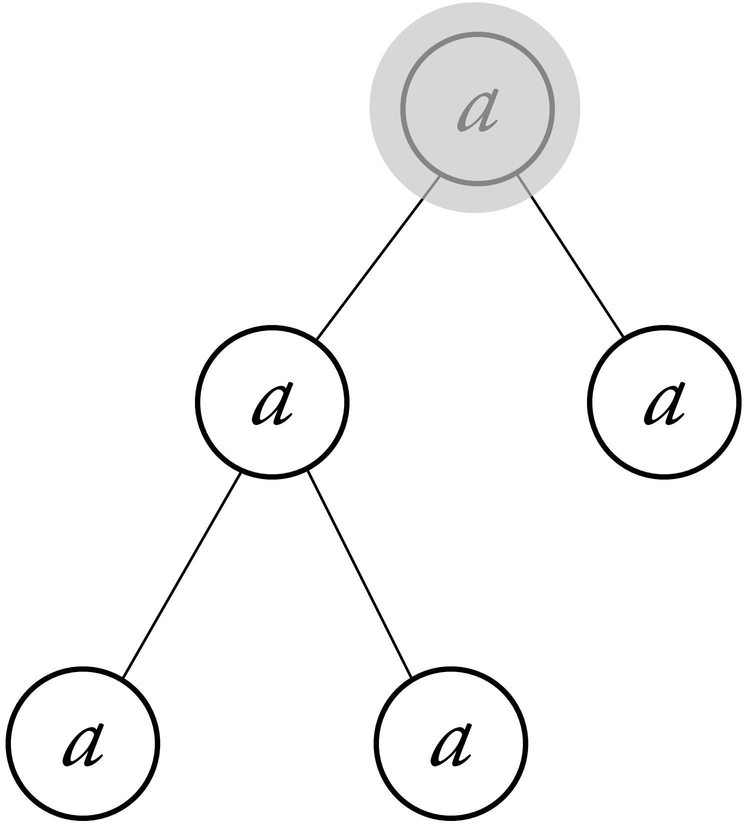

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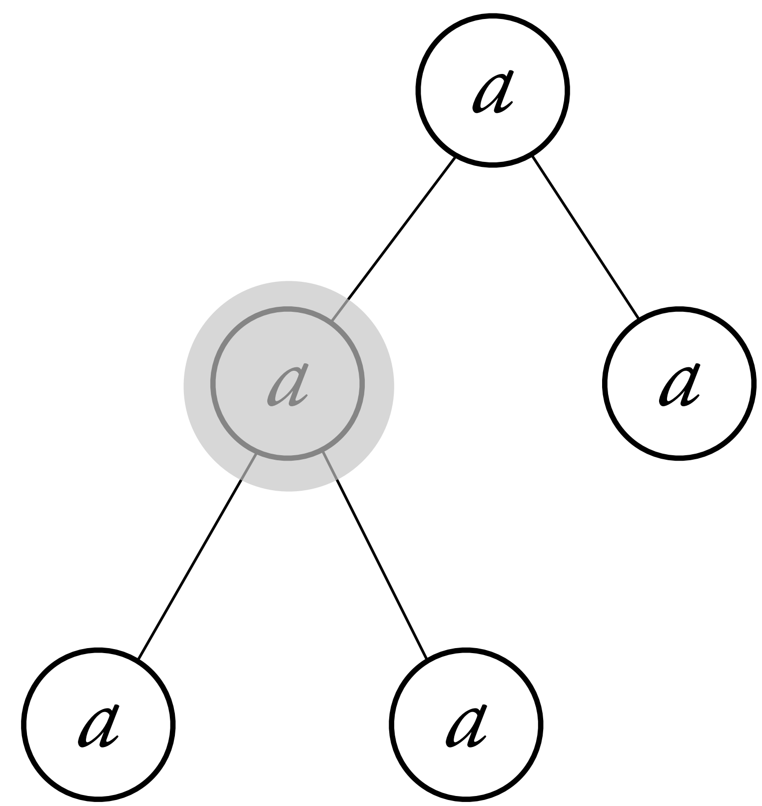

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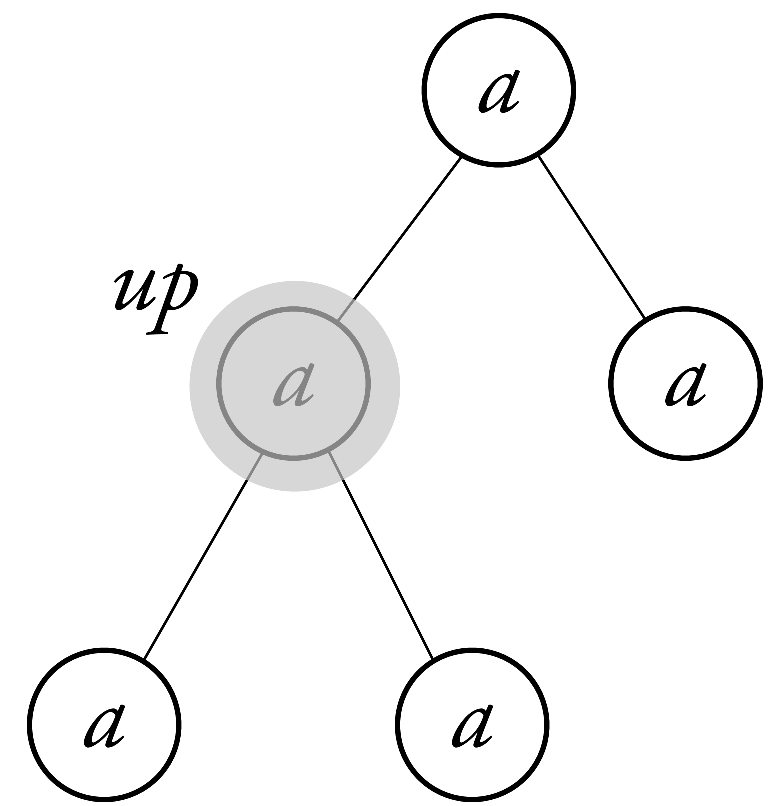

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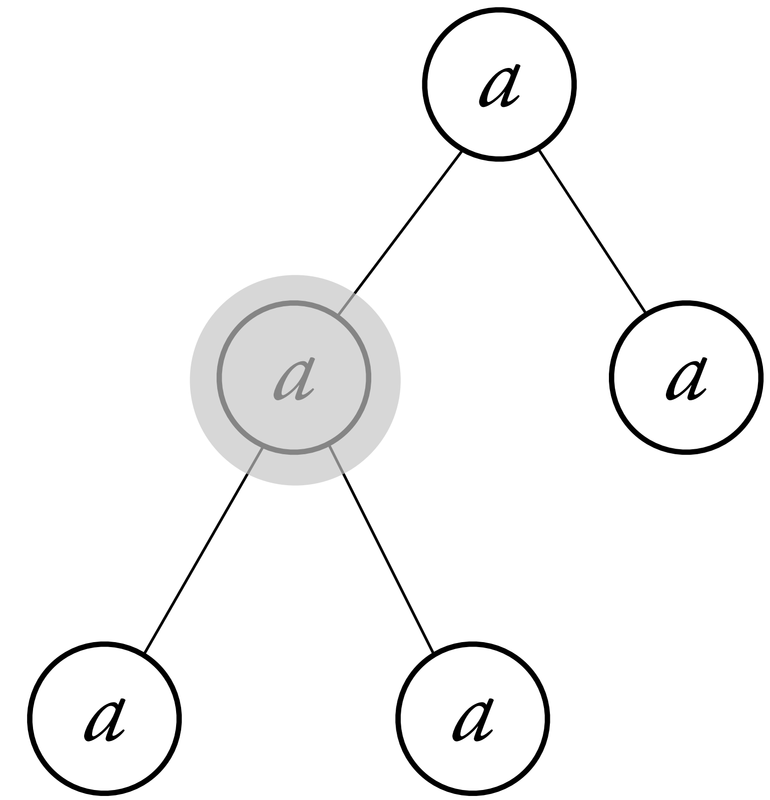

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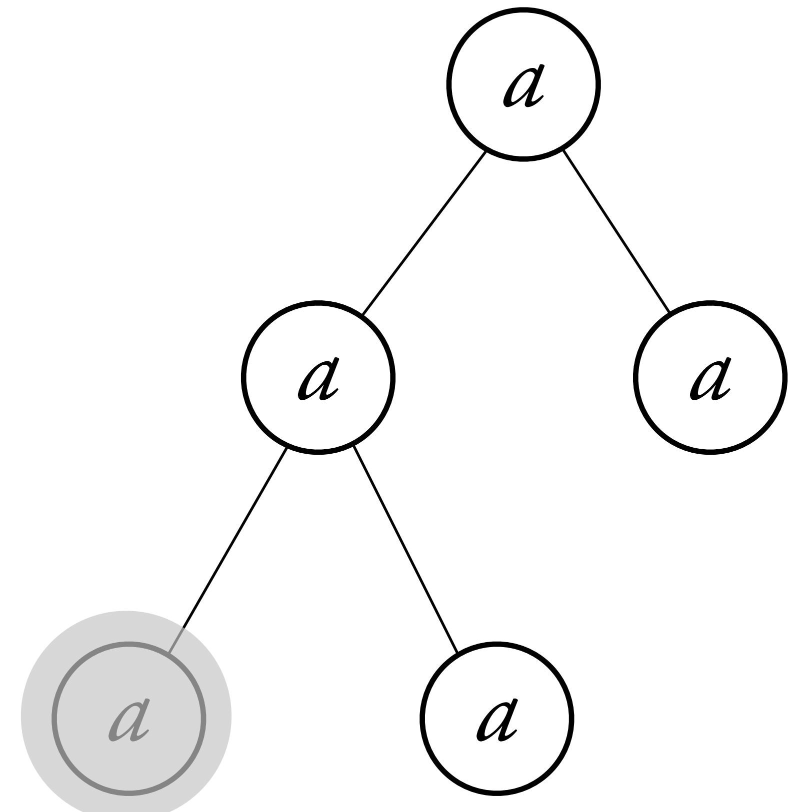

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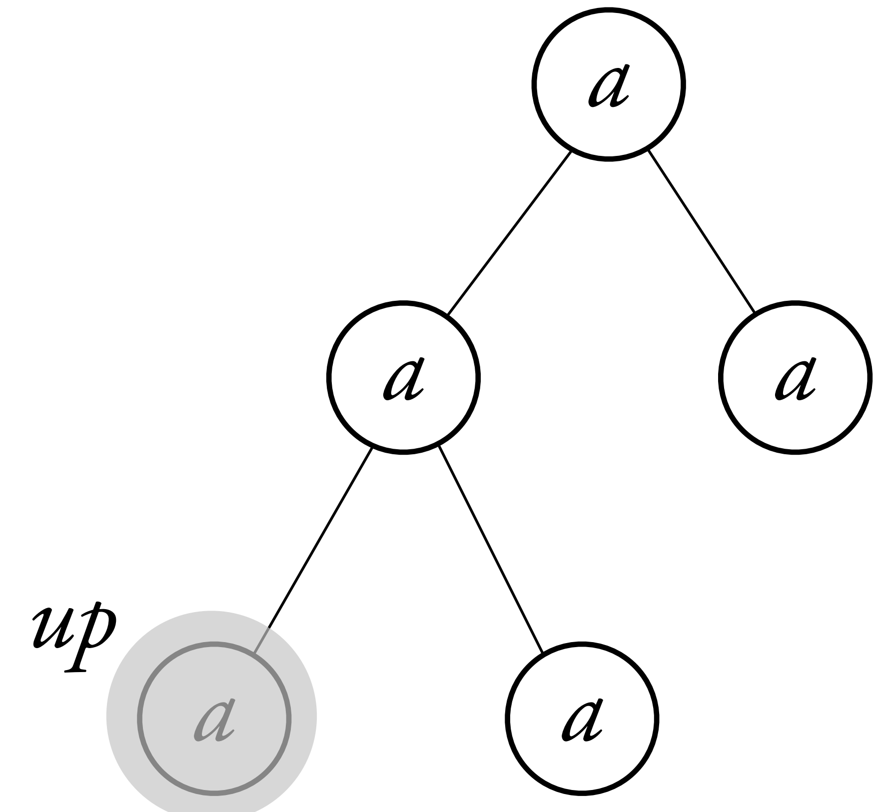

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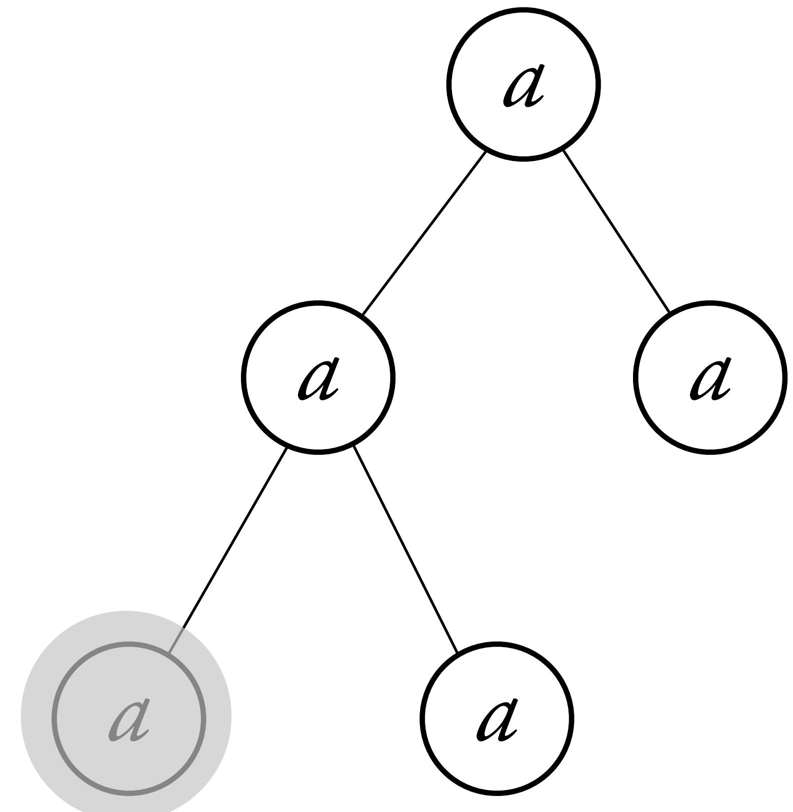

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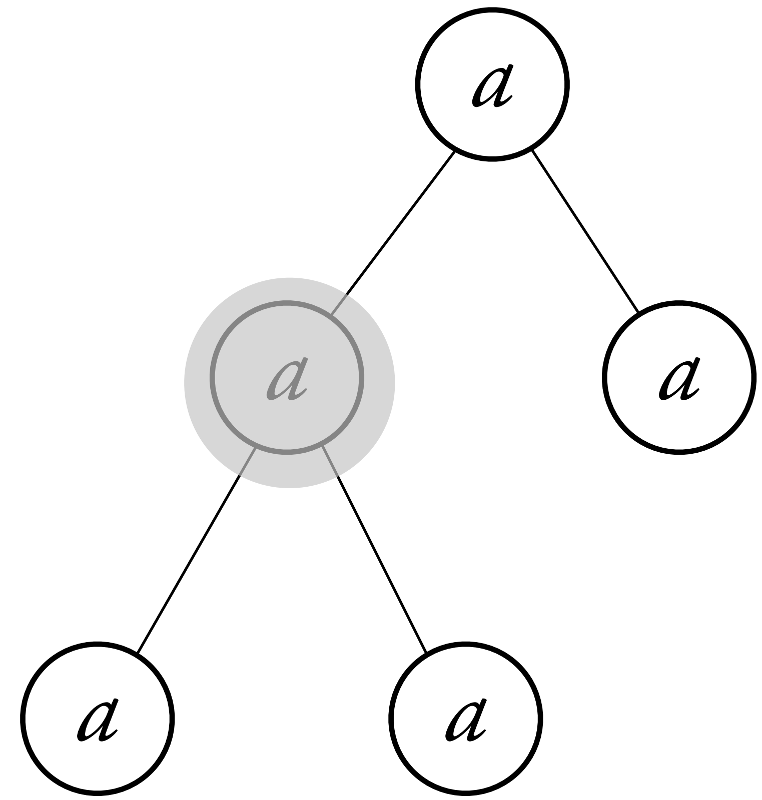

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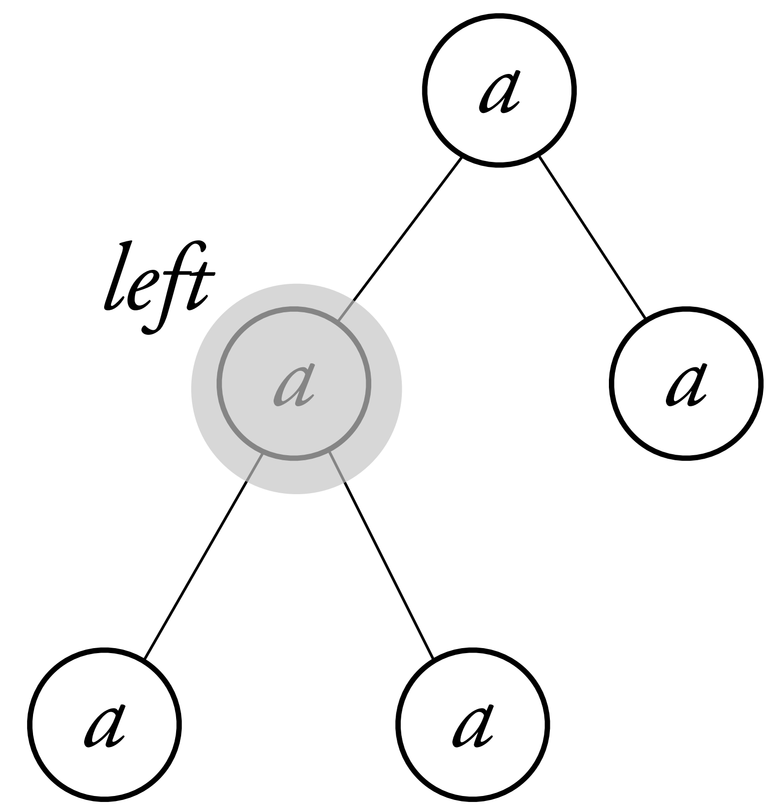

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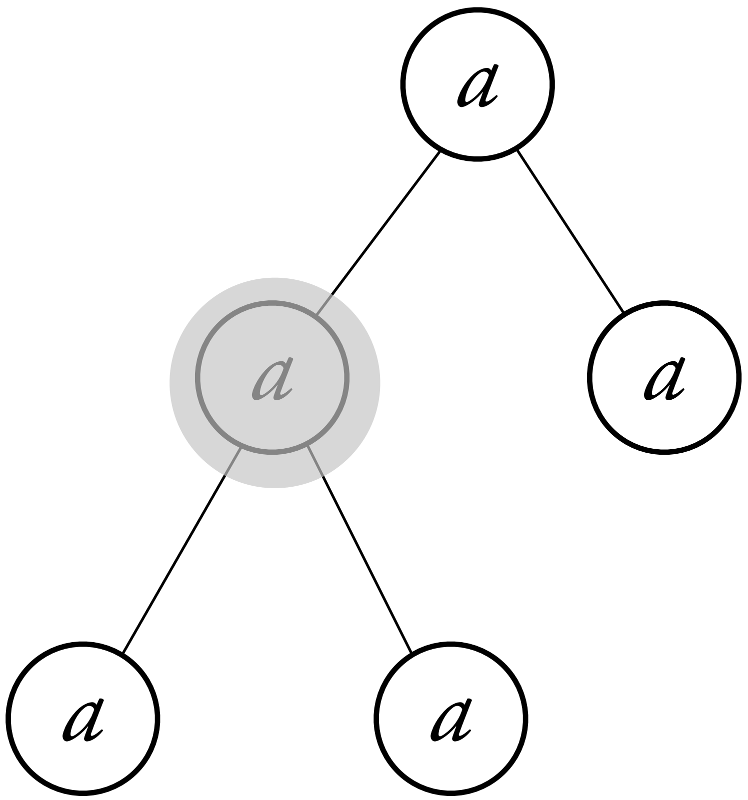

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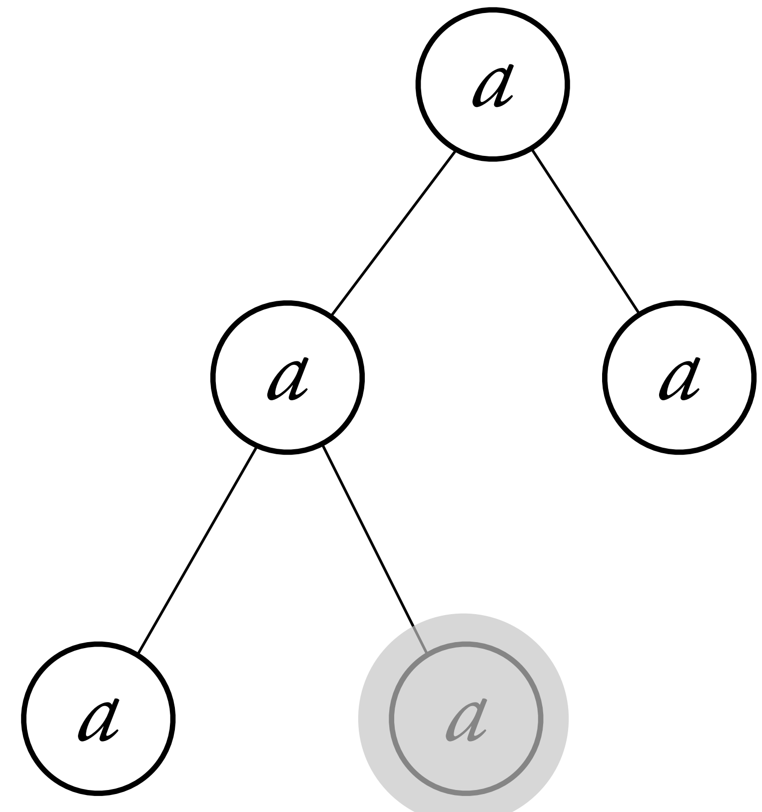

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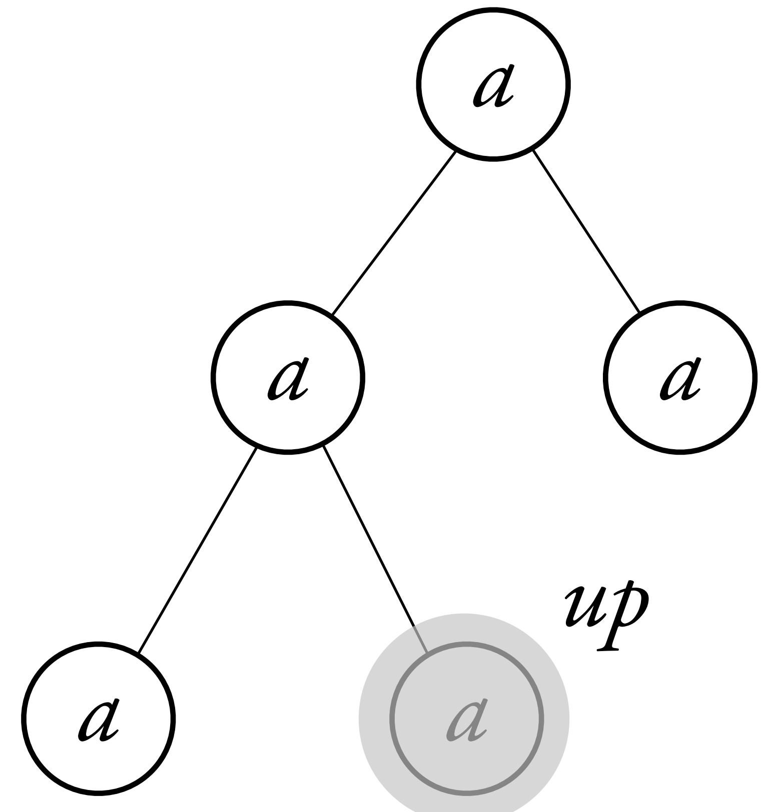

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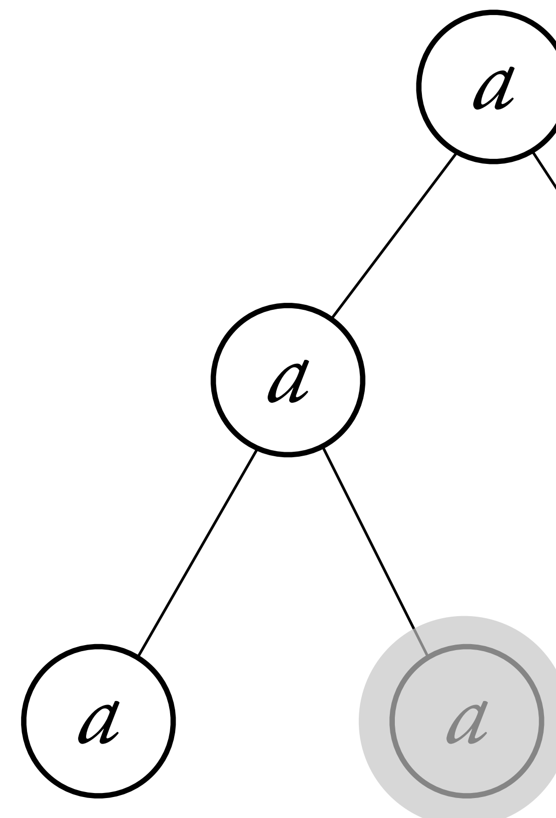

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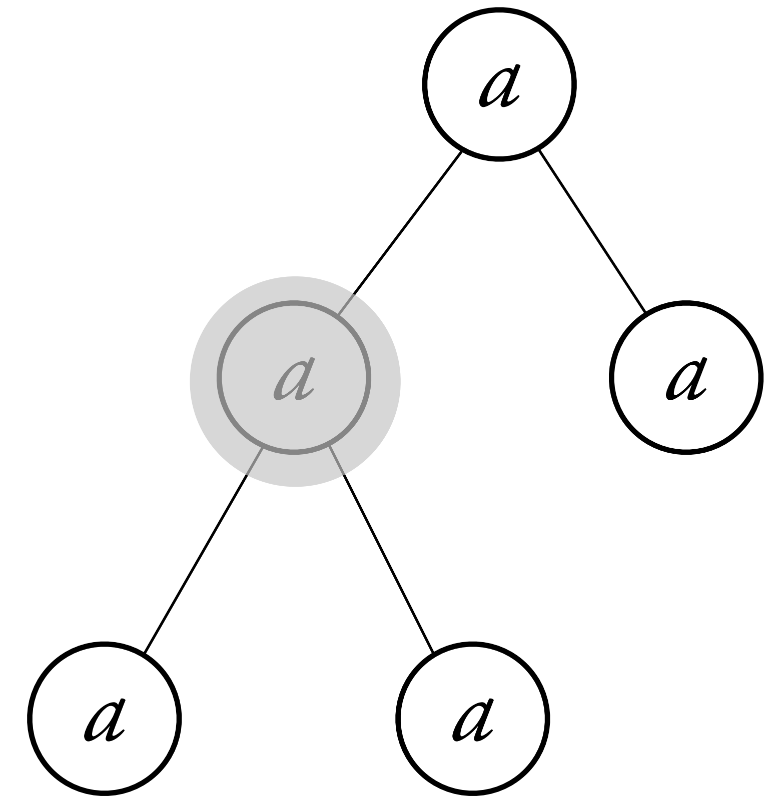

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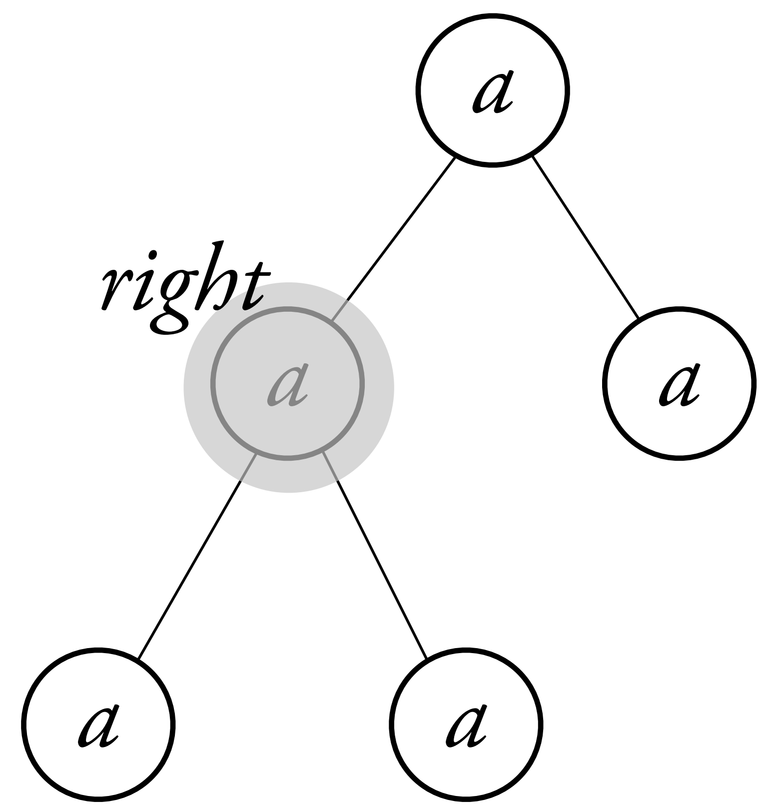

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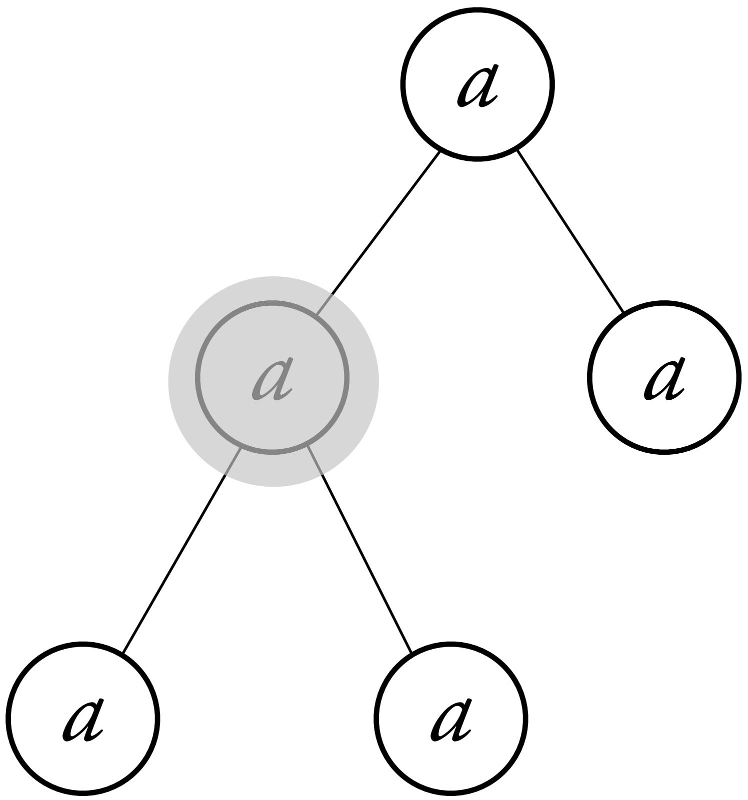

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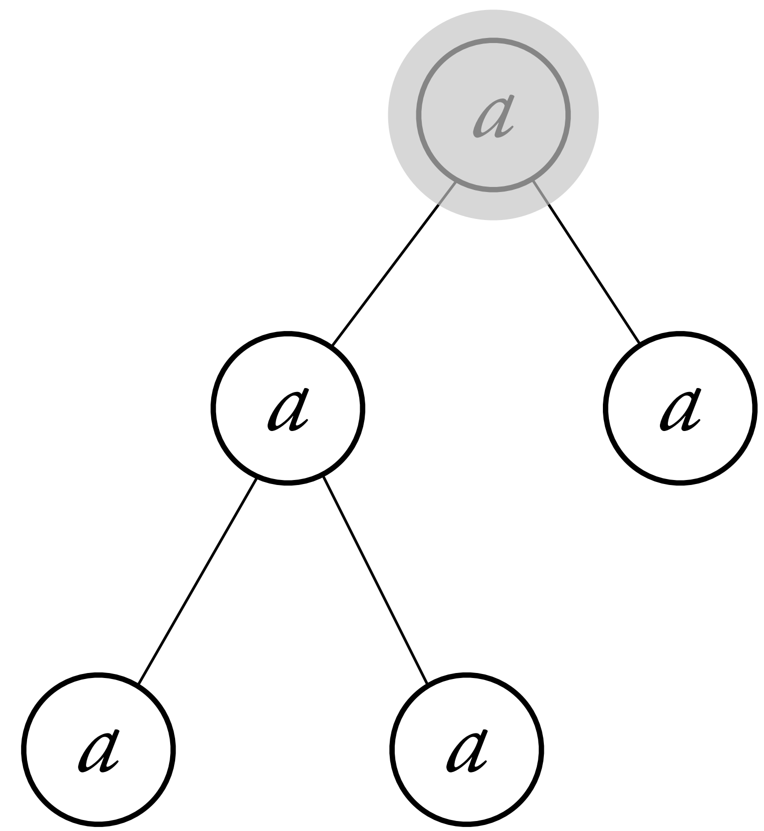

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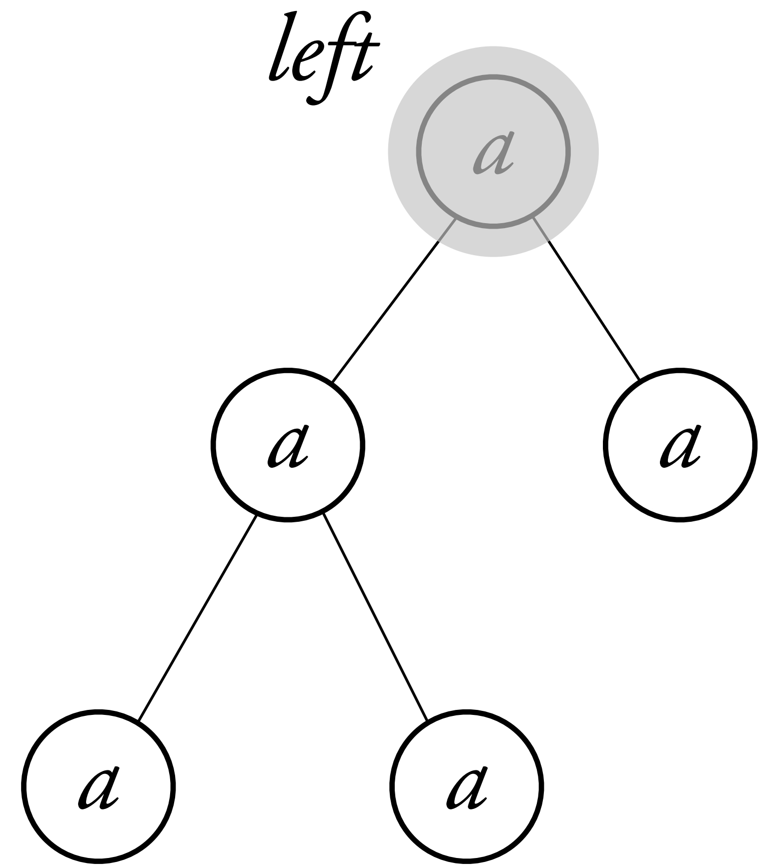

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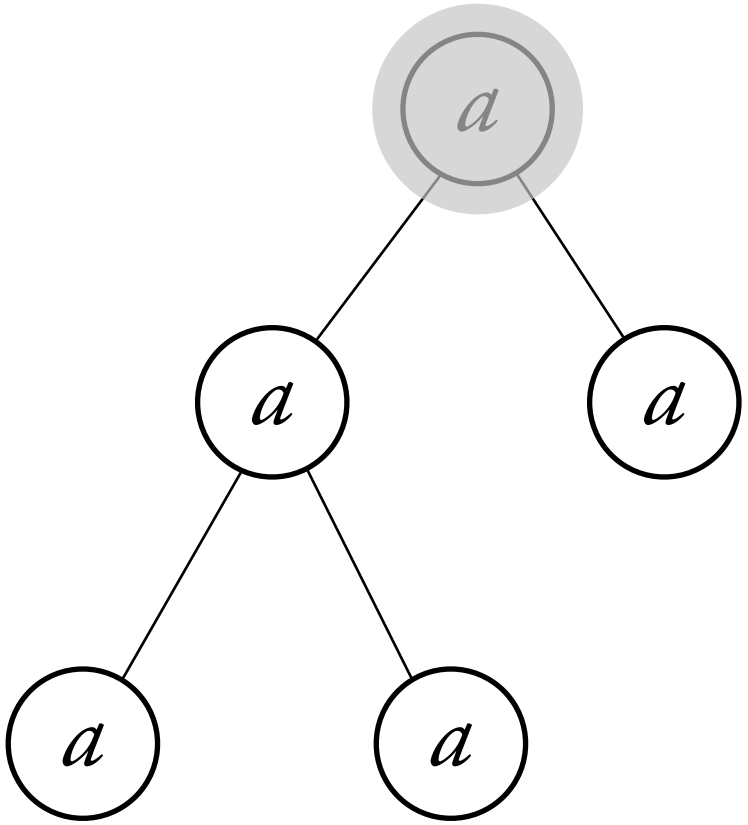

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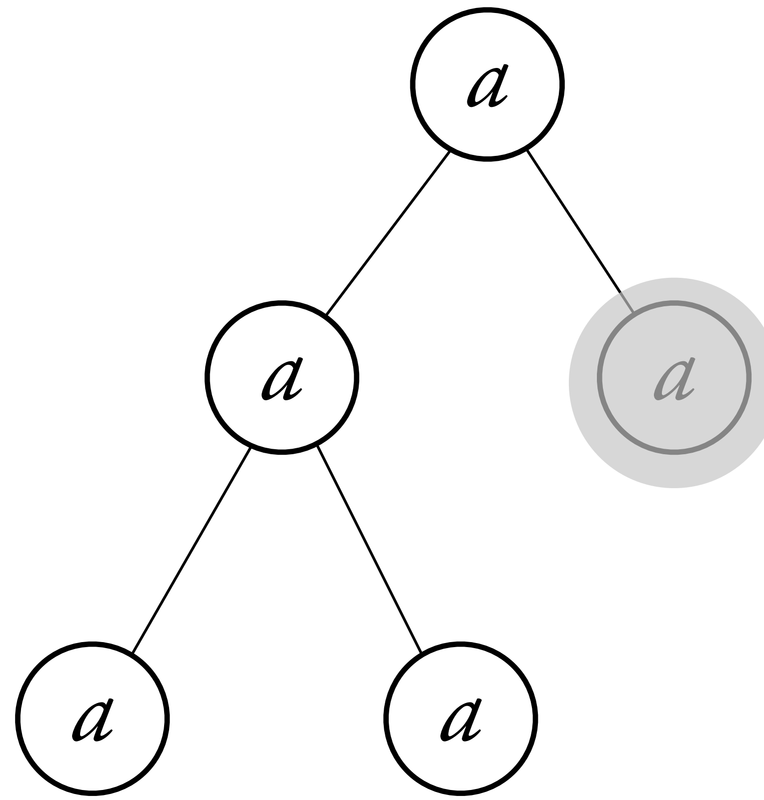

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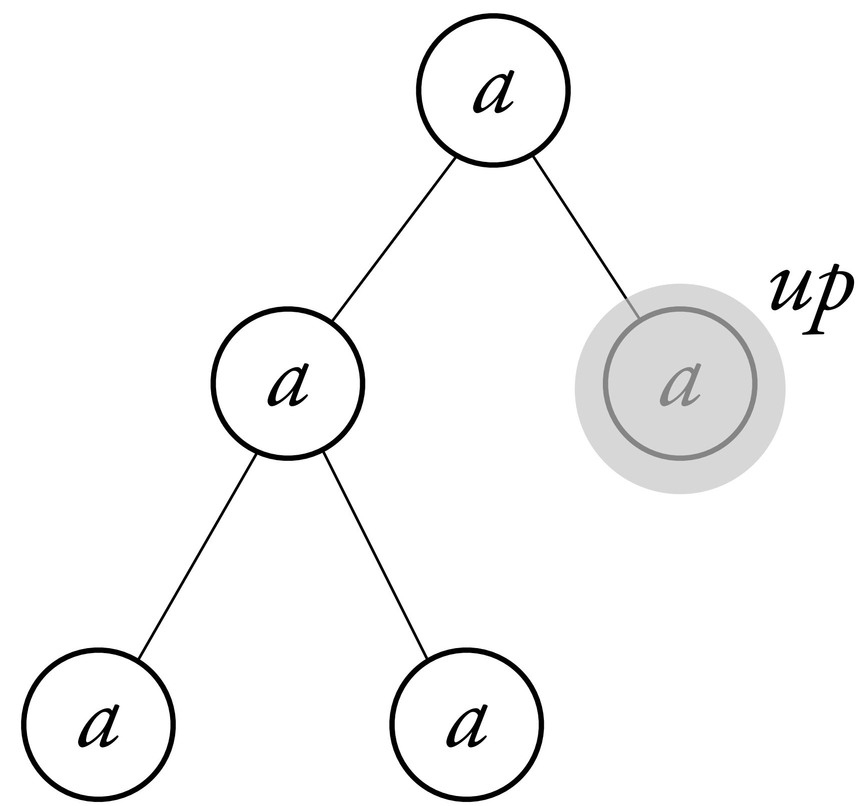

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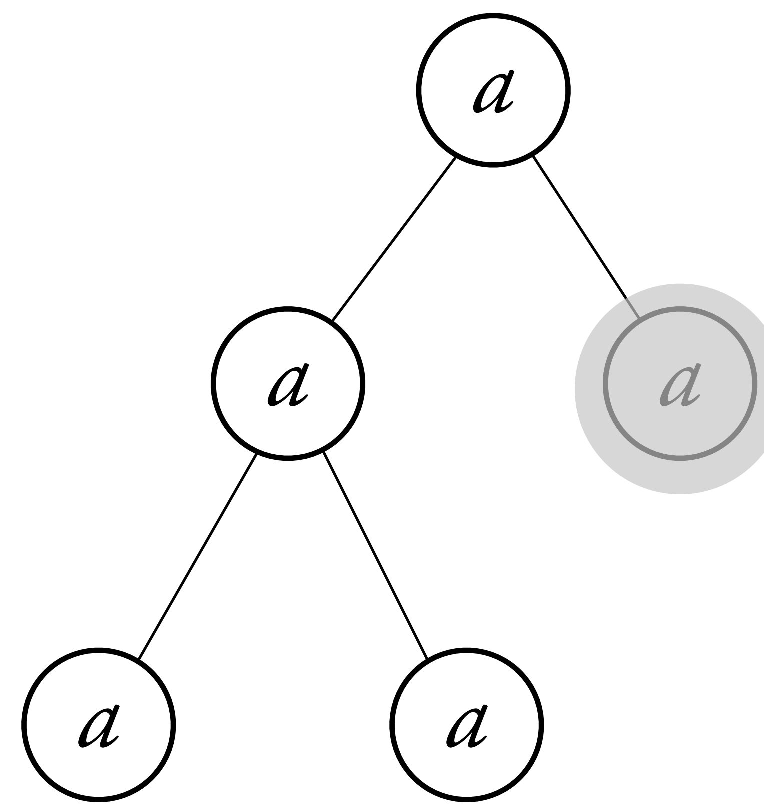

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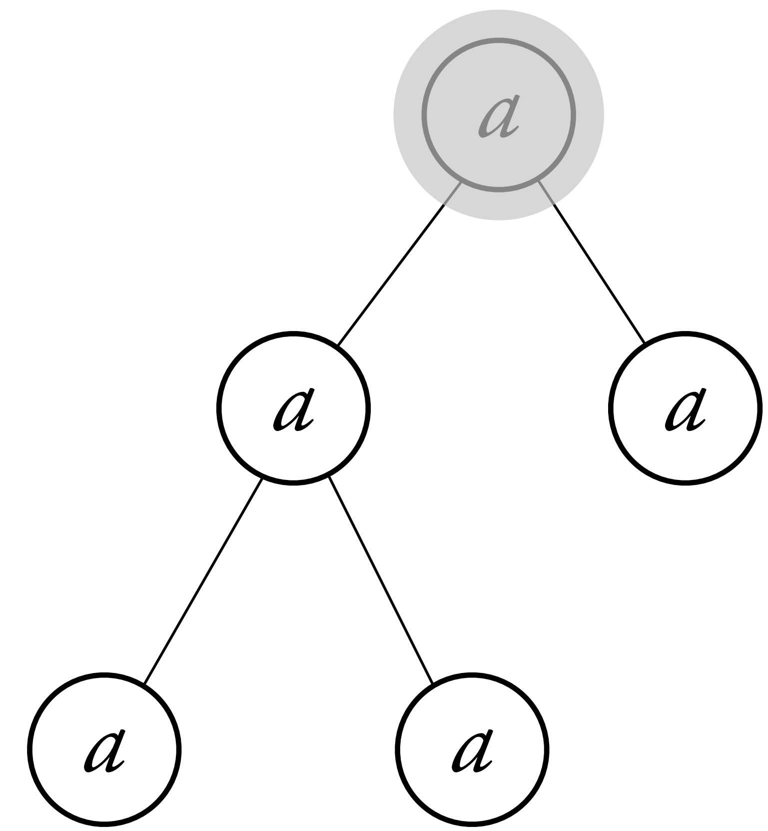

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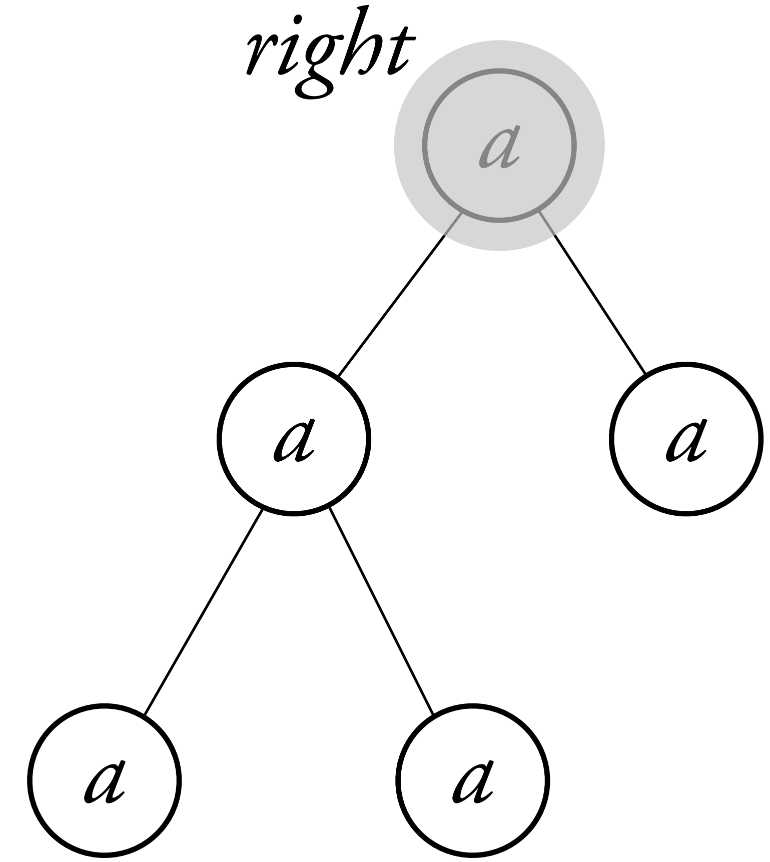

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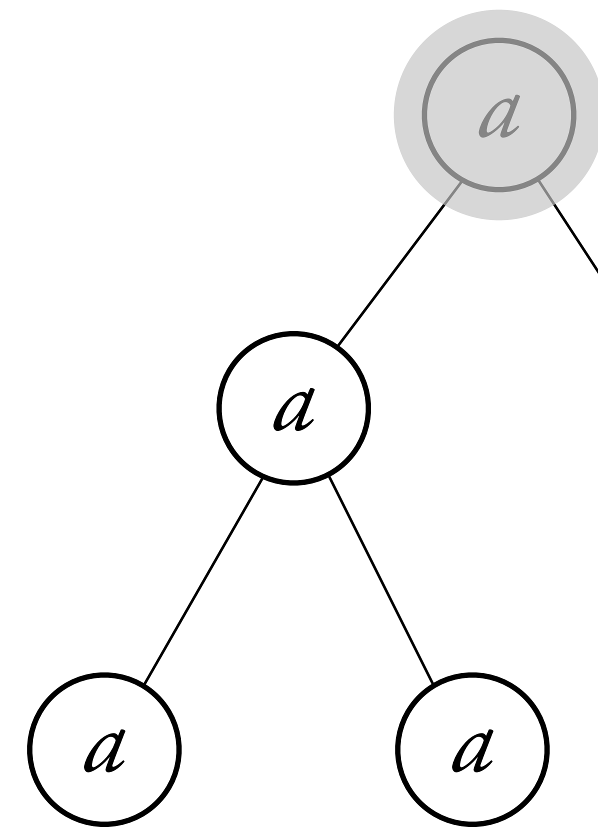

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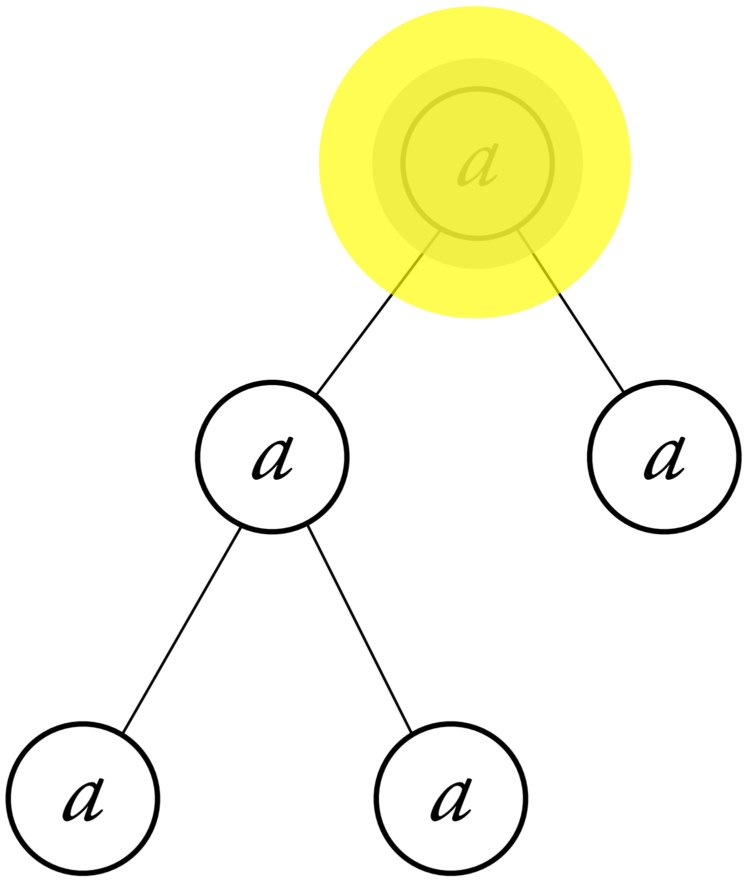

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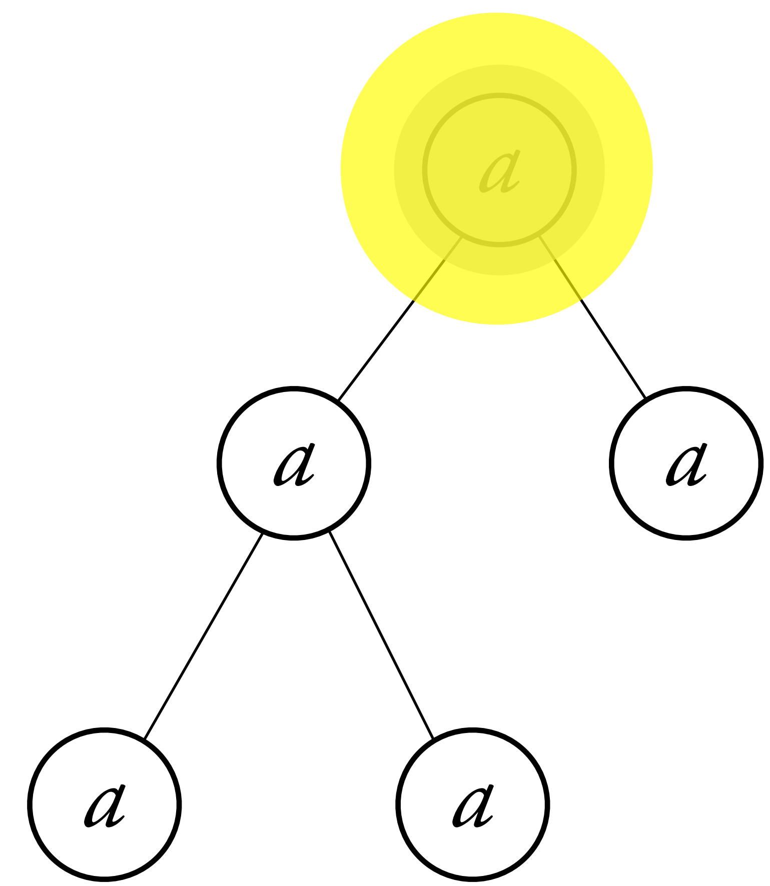
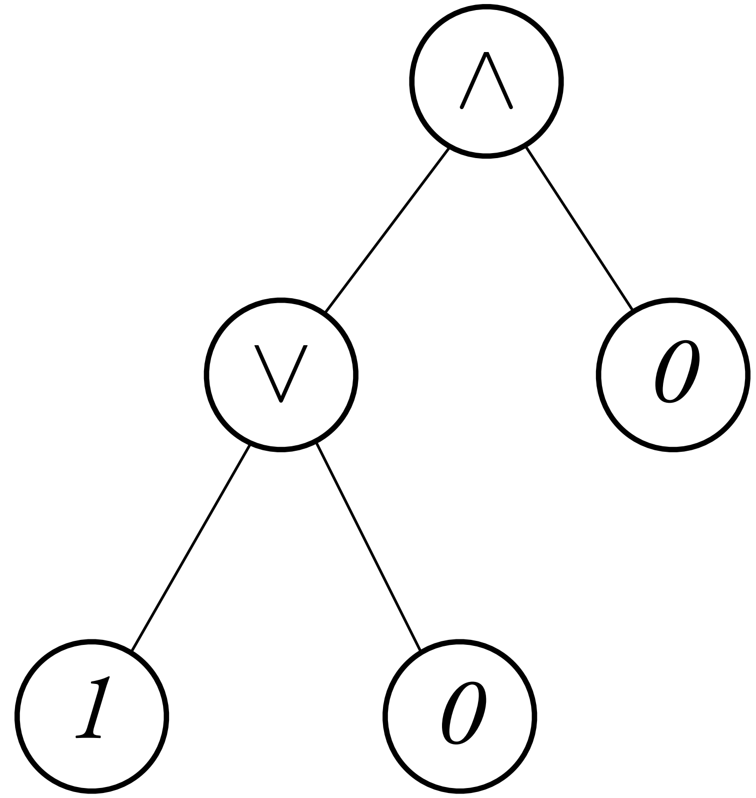
Complemenation is difficult!

Open problem:
Are nondeterministic tree-walking automata closed under complementation?



Example.
All nodes have label a

A clever tree-walking automaton



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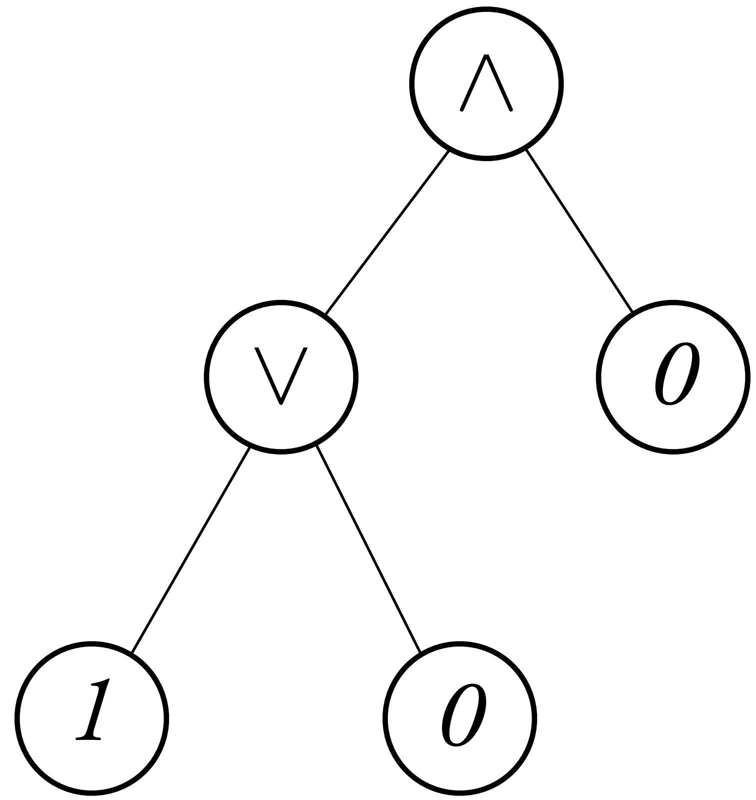
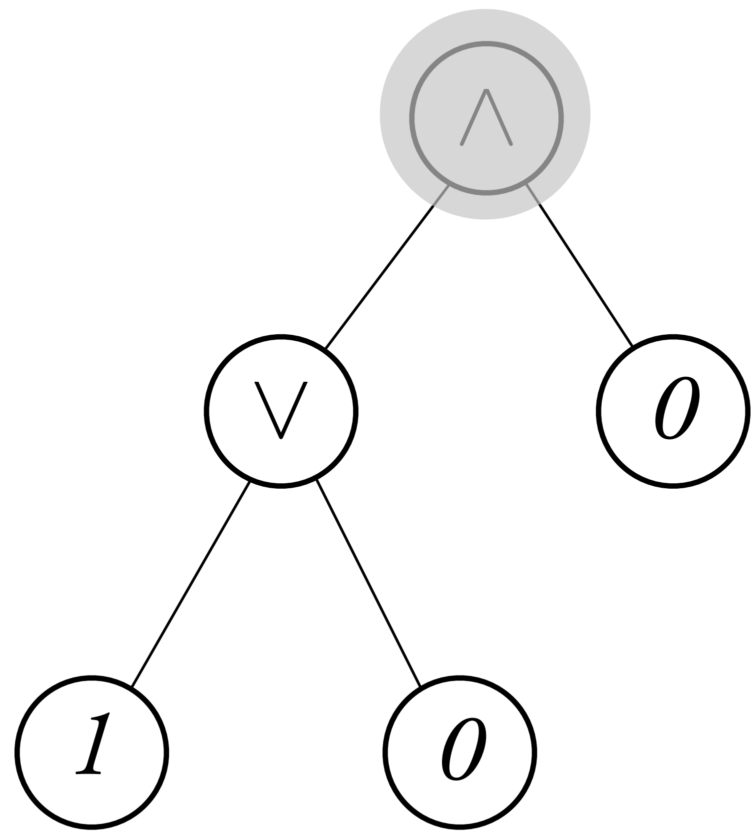


States: $\{q\} \cup \left(\underbrace{\{left, right\} \times \{0, 1\}}\right)$

first time

just evaluated evaluated left/right subtree to 0/1

A clever tree-walking automaton



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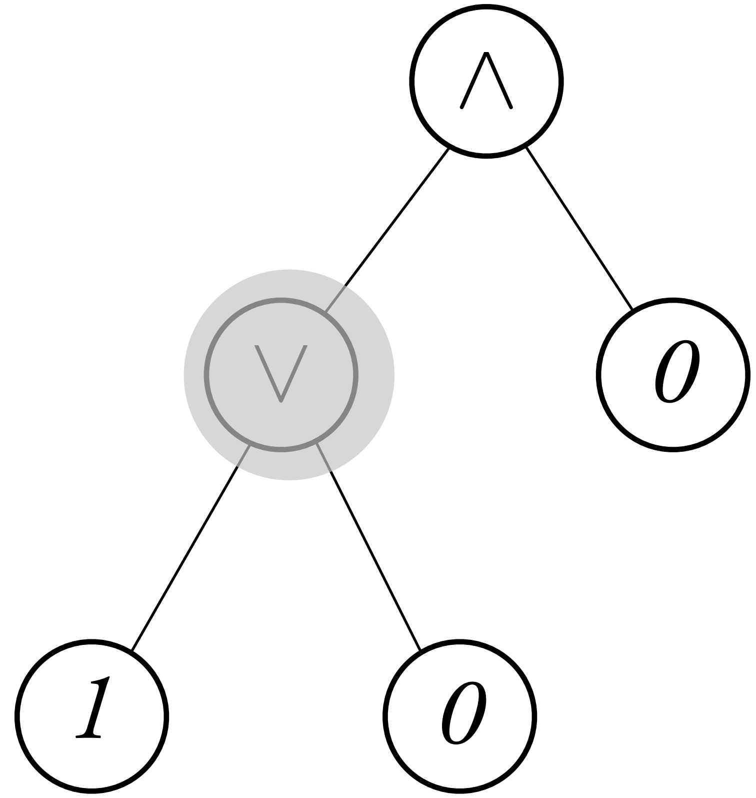

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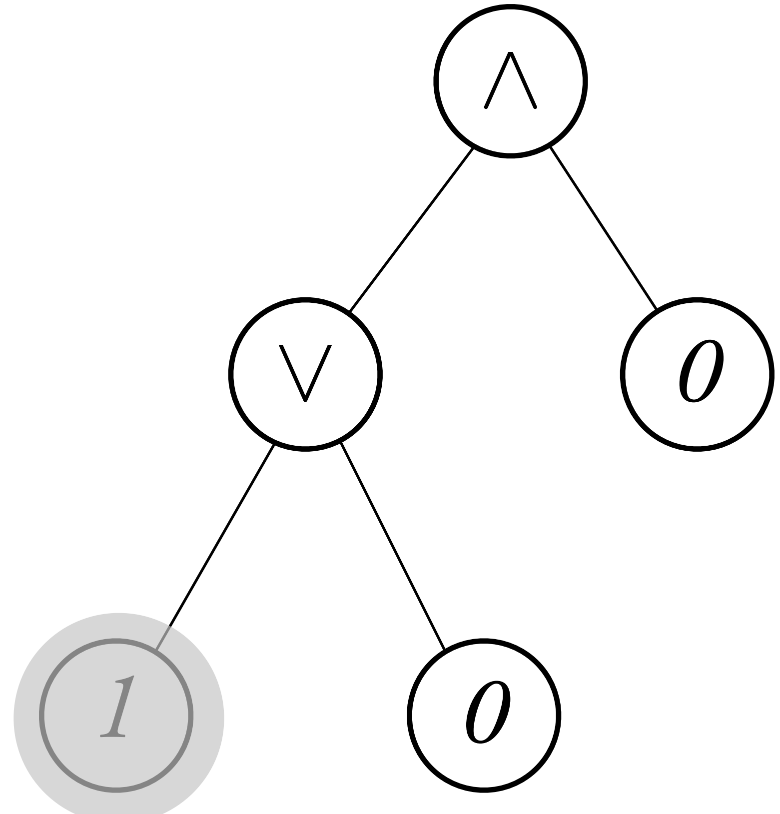

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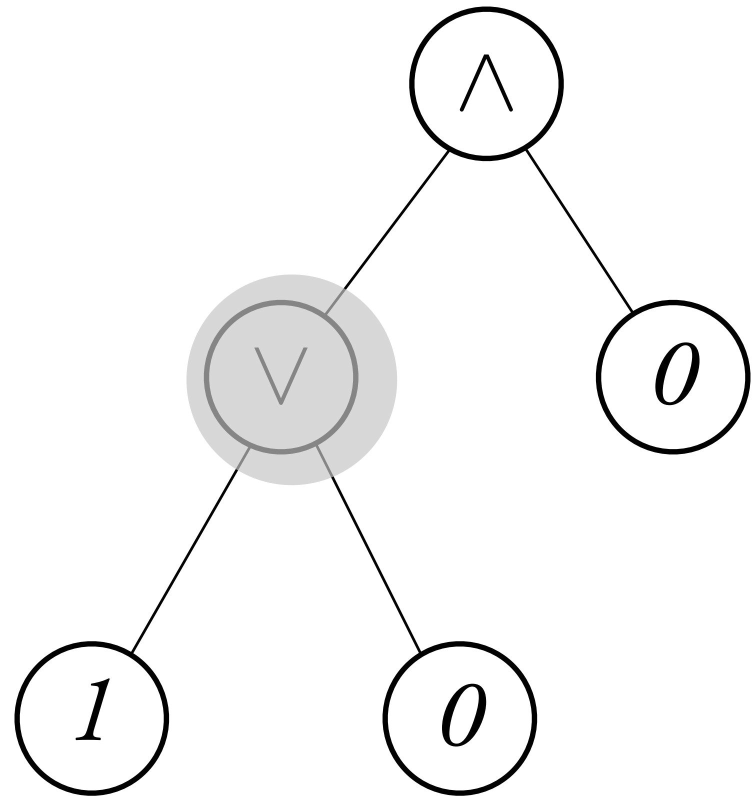

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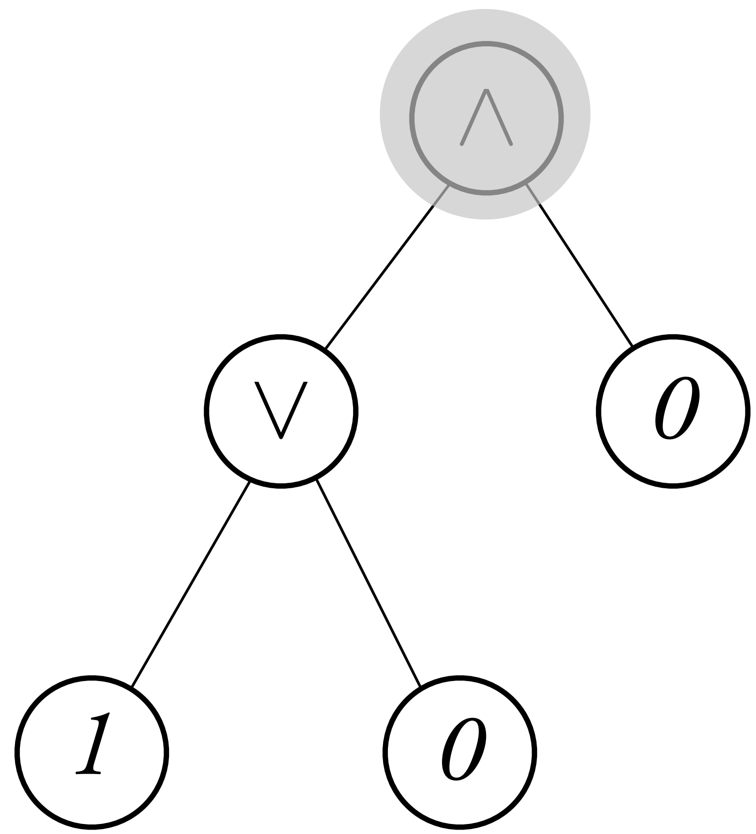

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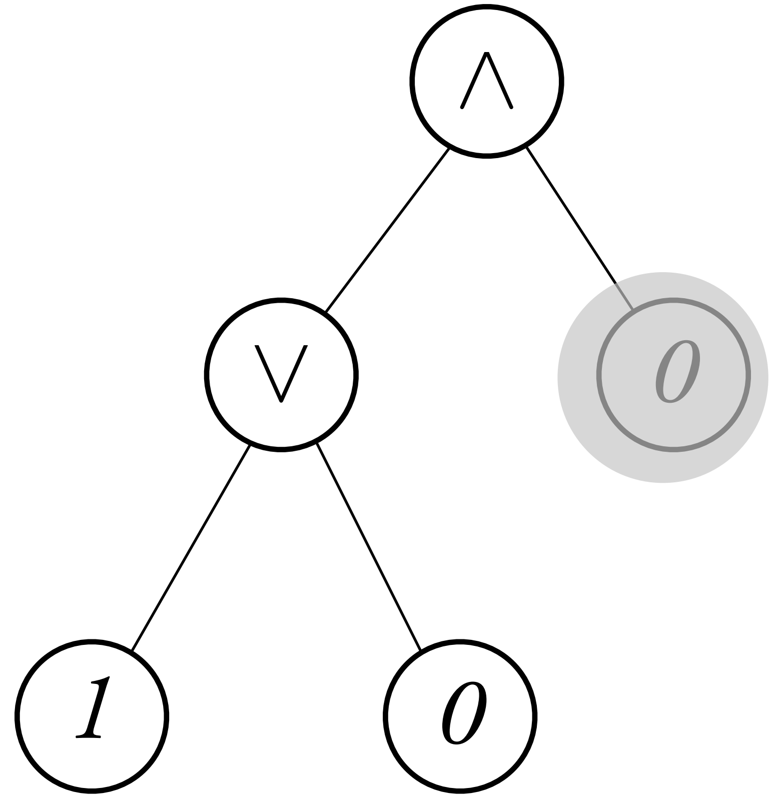

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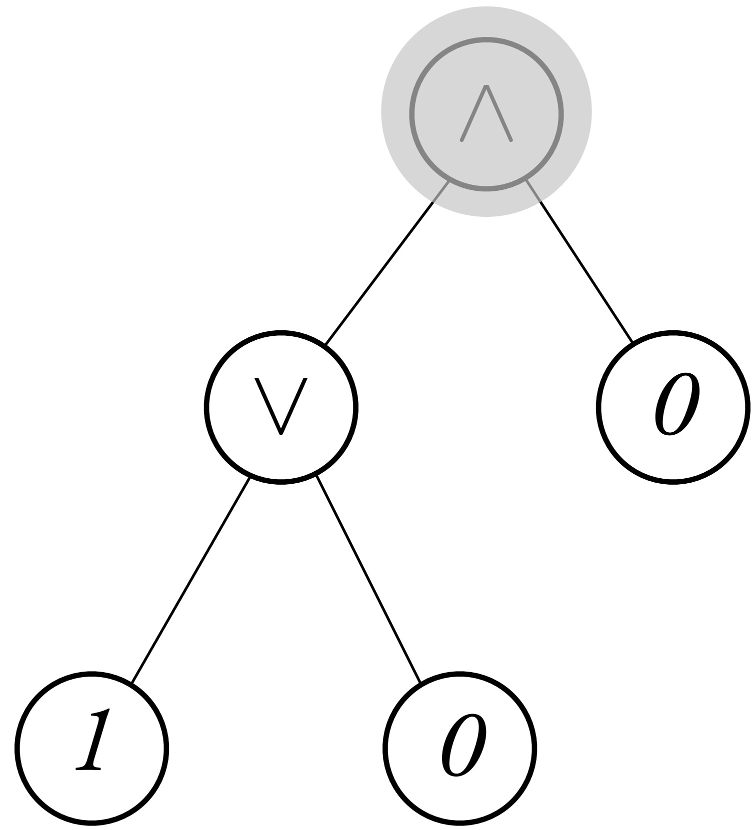

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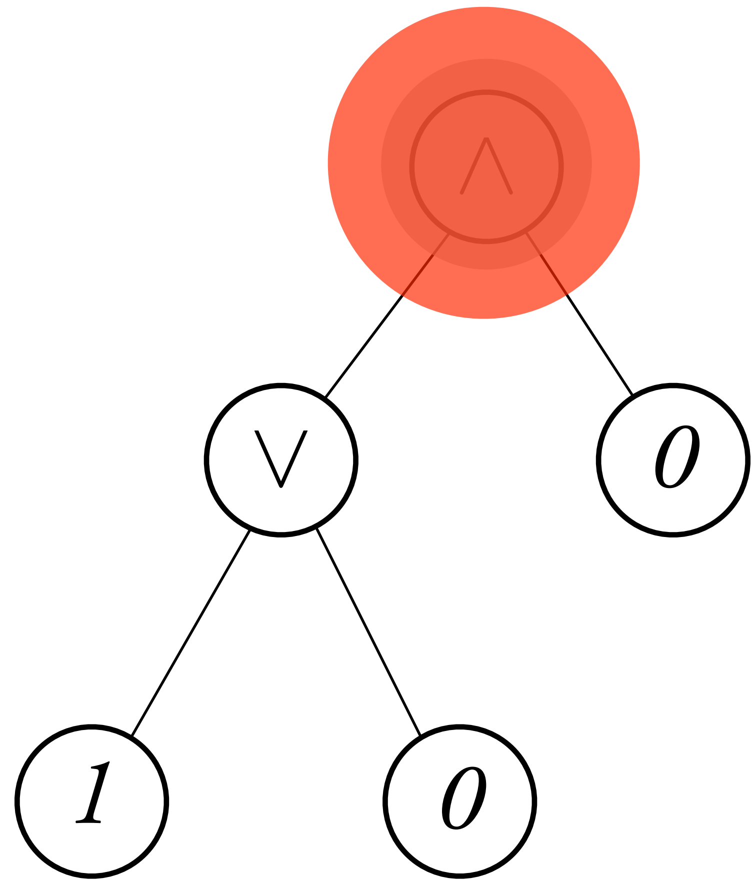

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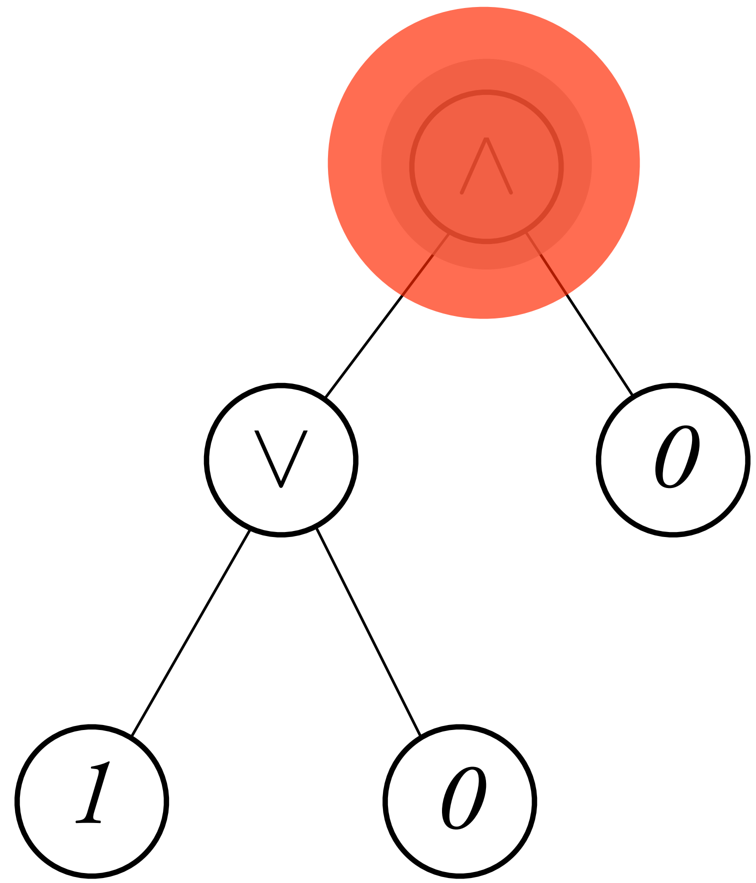
still works with negation, but what about XOR?



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Open problem:
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Theorem. [Muscholl, Samuelides, Segoufin]
Deterministic tree-walking automata are closed under complementation.

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Theorem. [Muscholl, Samuelides, Segoufin]
Deterministic tree-walking automata are closed under complementation.

Lemma.
Every deterministic tree-walking automaton is equivalent to one that ends every run with a *reject* or *accept* command.

Plan

-A tree-walking automaton

-Expressive power

-Pebble automata and logic

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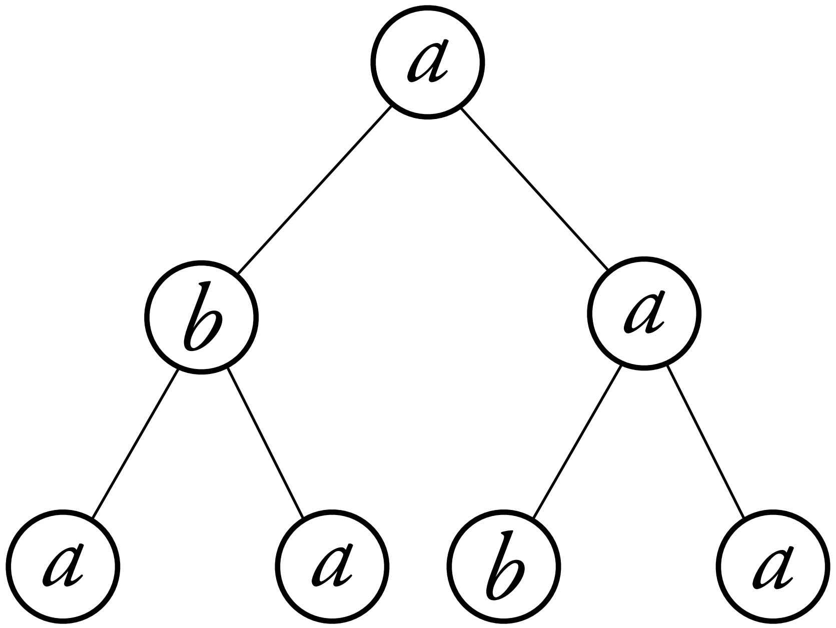
How do tree-walking automata relate to “real” tree automata?



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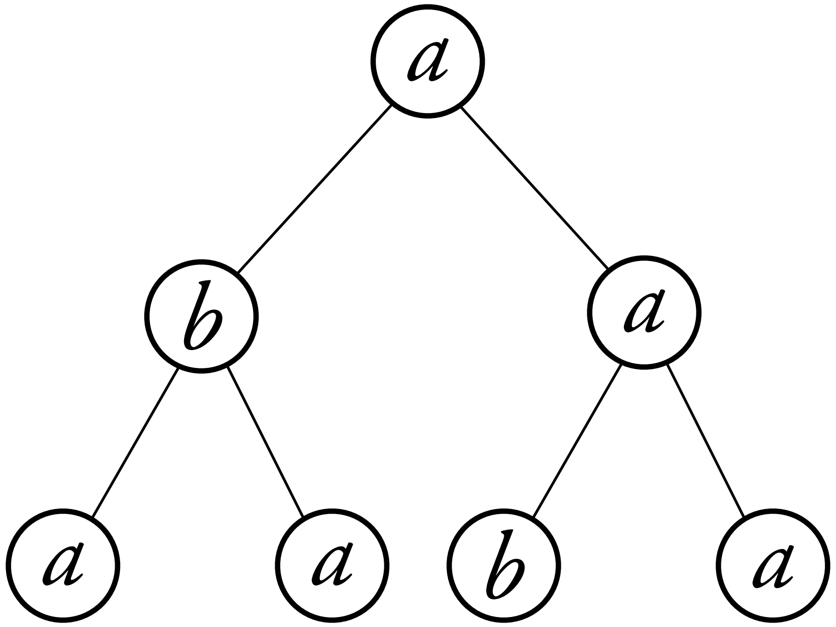

Standard model is a *branching automaton*.

Here we use bottom-up deterministic branching automata.

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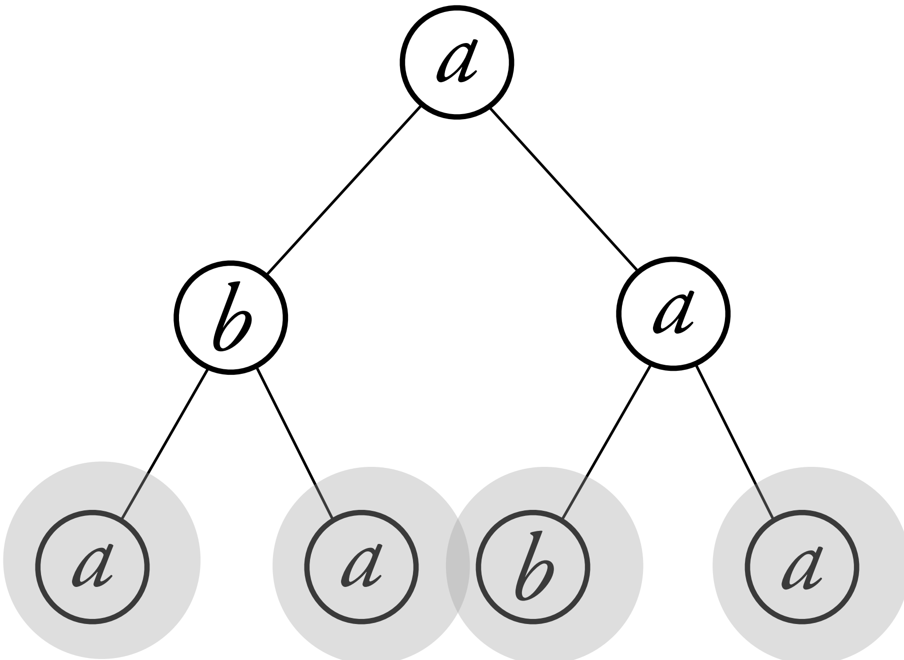

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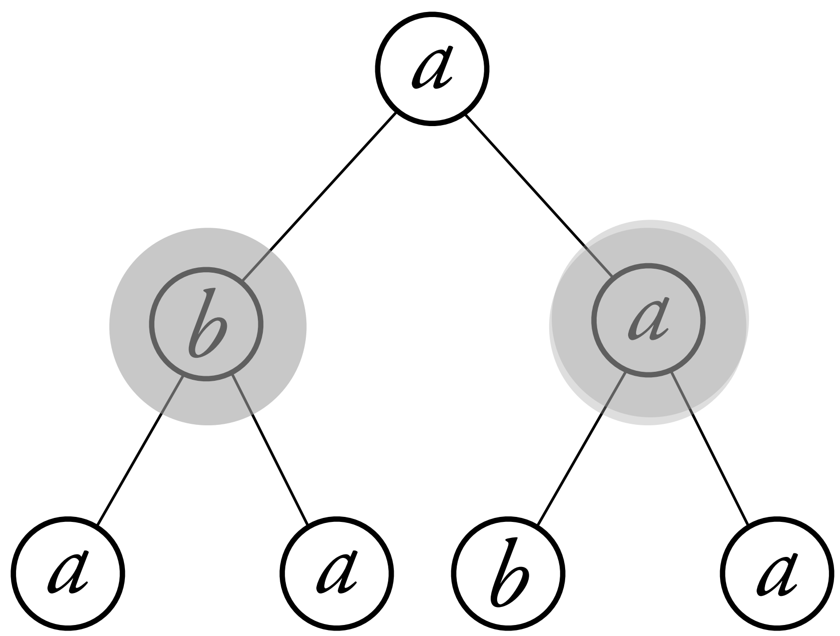

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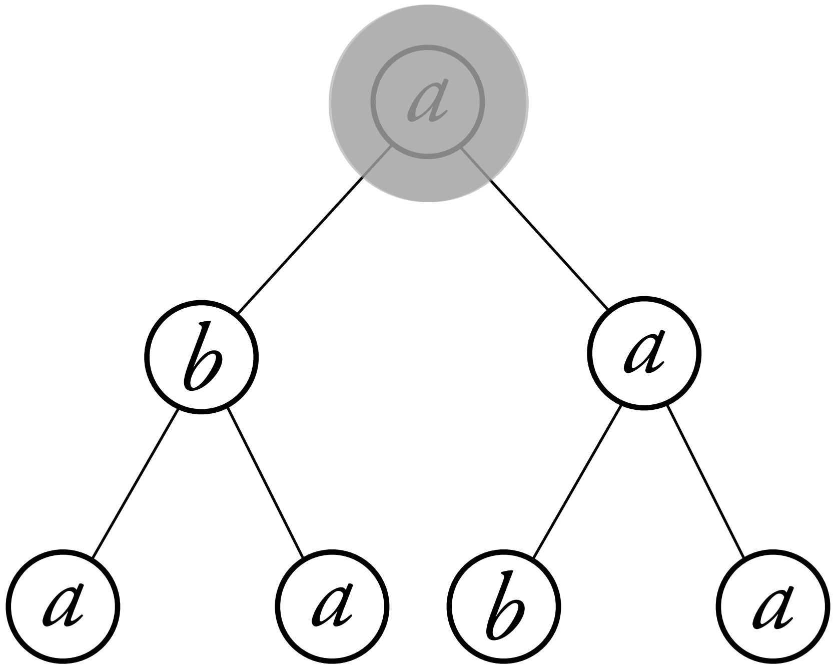

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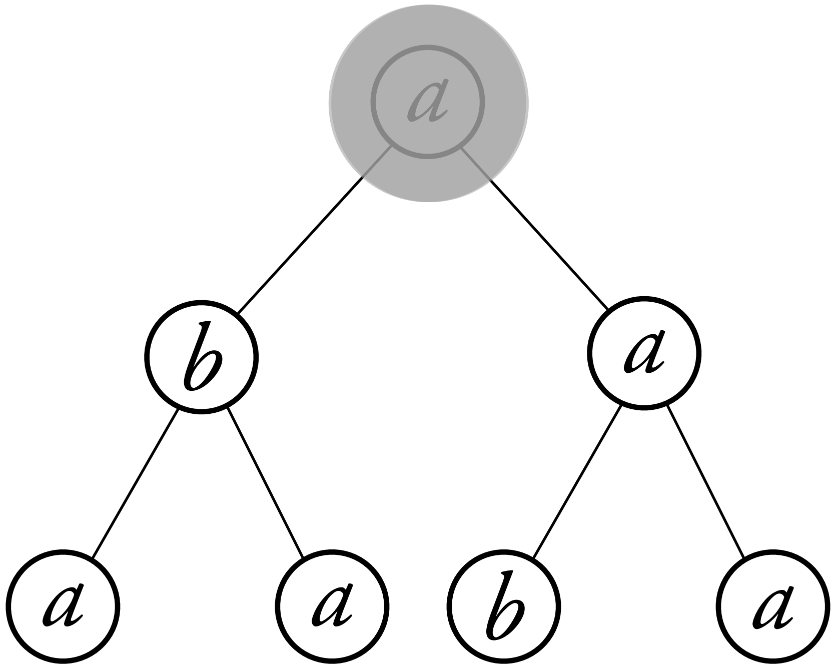

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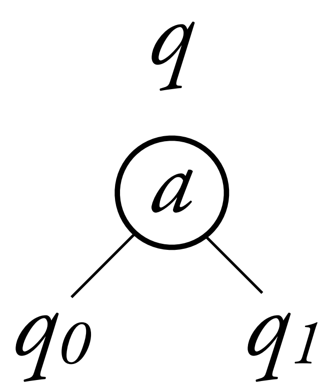

If the root label is *a*, the left subtree has value q_0 , and the right subtree has value q_1 , then the whole tree has value q.

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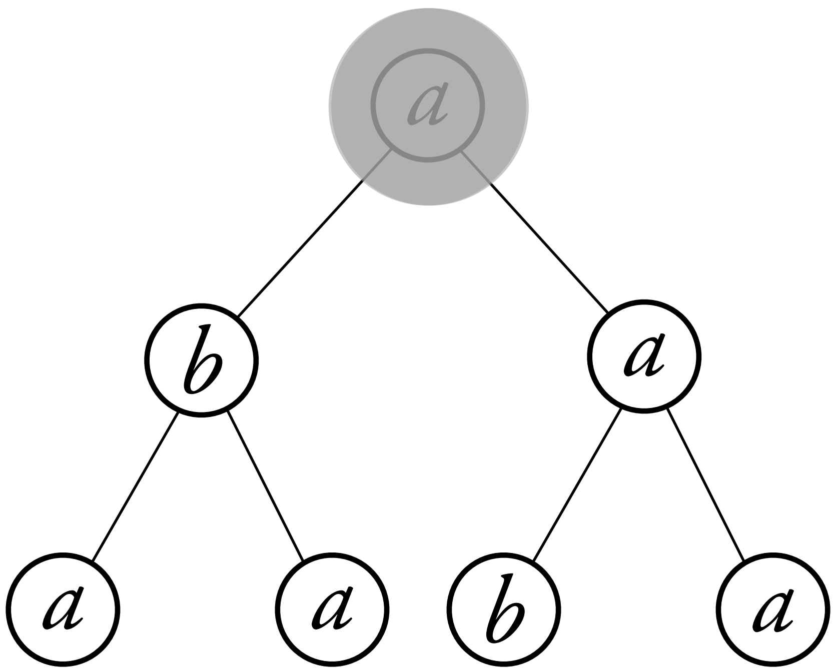

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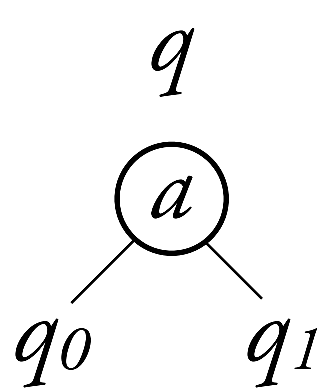
$$\langle Q, q_I, \Sigma, \delta, F \rangle$$



If the root label is a, the left subtree has value q_0, and the right subtree has value q_1, then the whole tree has value q.

Branching automata are closed under union, intersection, complementation, projection etc.

Def. A tree language is called *regular* if it is recognized by a branching automaton.

Question: how do tree-walking automata relate to regular languages?

$$\mathrm{TWA} \subseteq \mathrm{REG}$$

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To a tree-walking automaton

$$\langle Q, q_I, \Sigma, \delta \rangle$$

we associate a branching automaton that accepts the same trees.

States $P(Q \times Q)$

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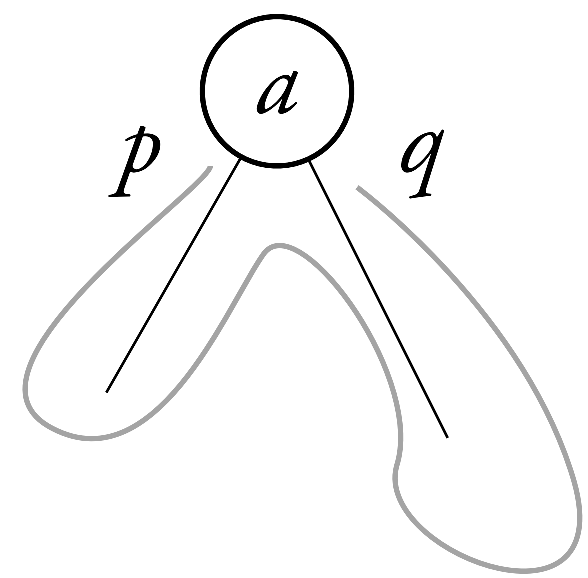

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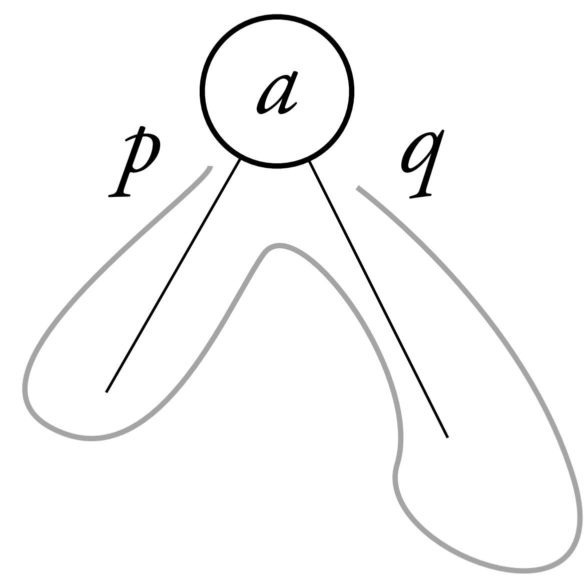

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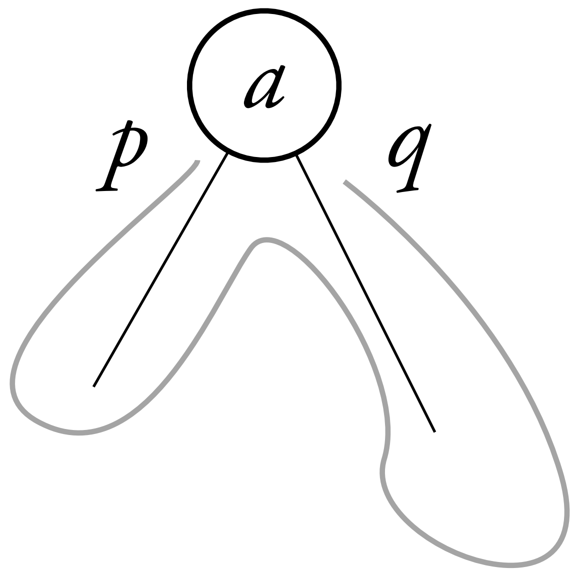

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Corollary. Emptiness for tree-walking automata is in EXPTIME.

Theorem. Emptiness for tree-walking automata is EXPTIME-complete.

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Hardness. Reduction from APSPACE.

For an alternating Turing machine that uses n memory cells, we write a tree-walking automaton with equivalent emptiness and $O(n)$ states.

The tree-walking automaton accepts computation trees. It does a DFS through the tree and checks that each branching is correct.

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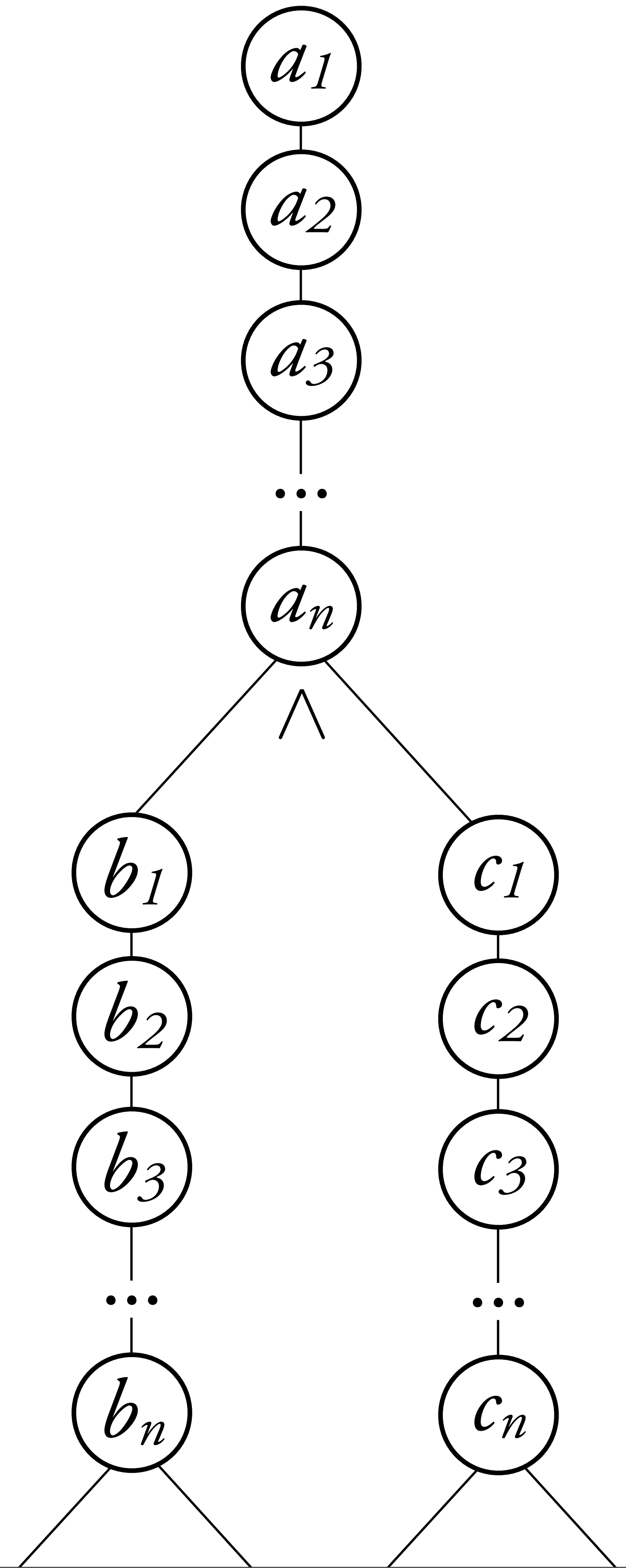

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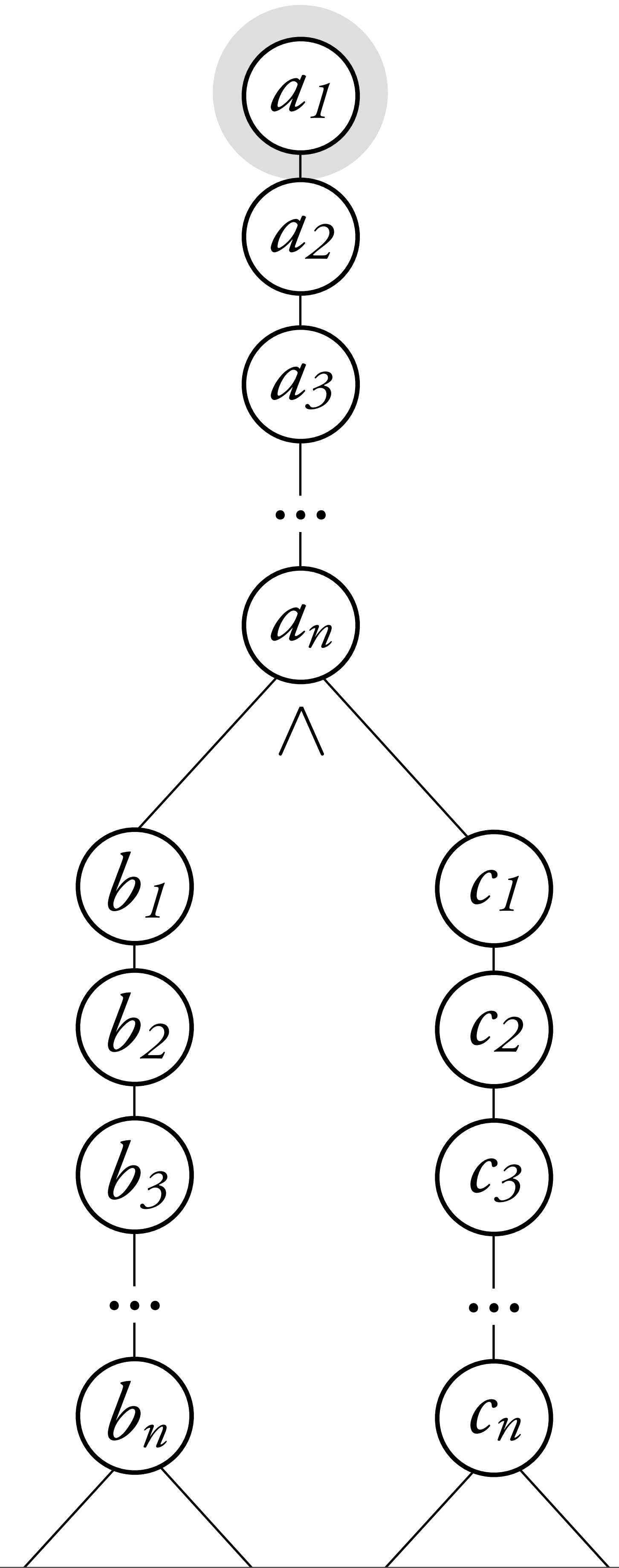

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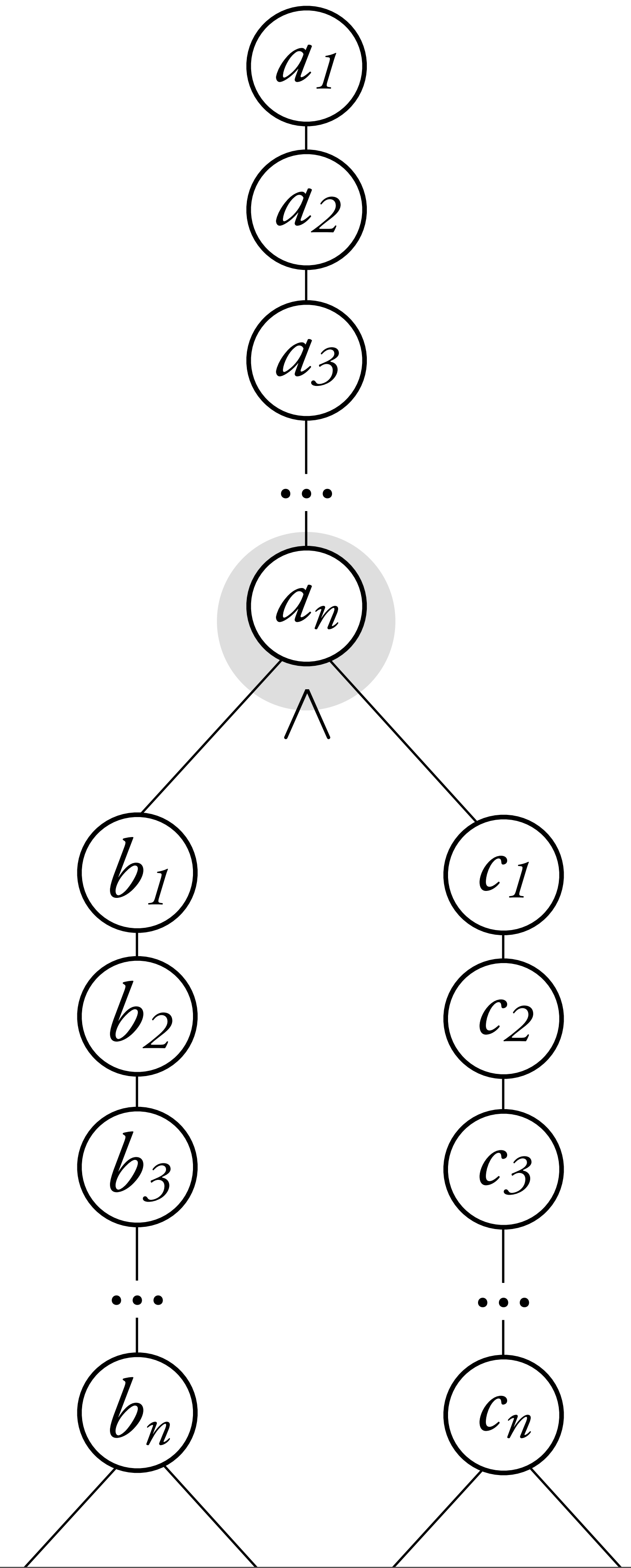

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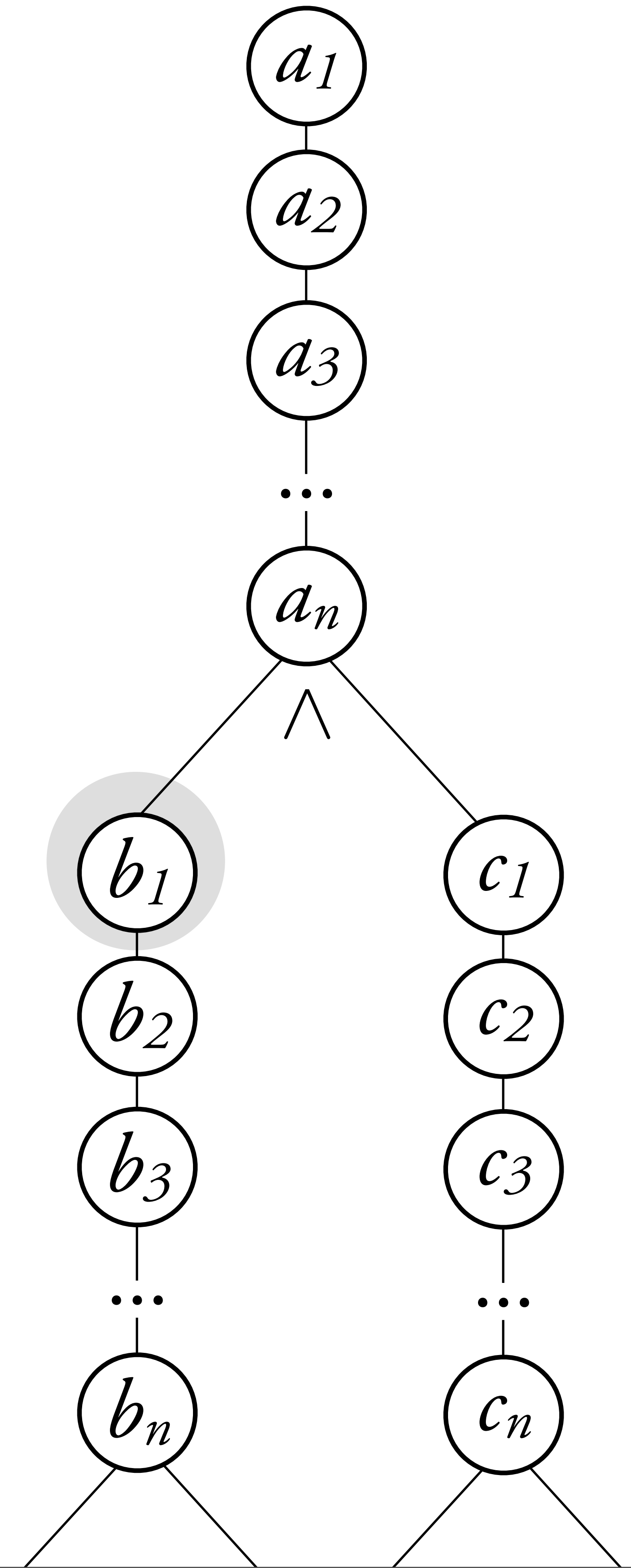

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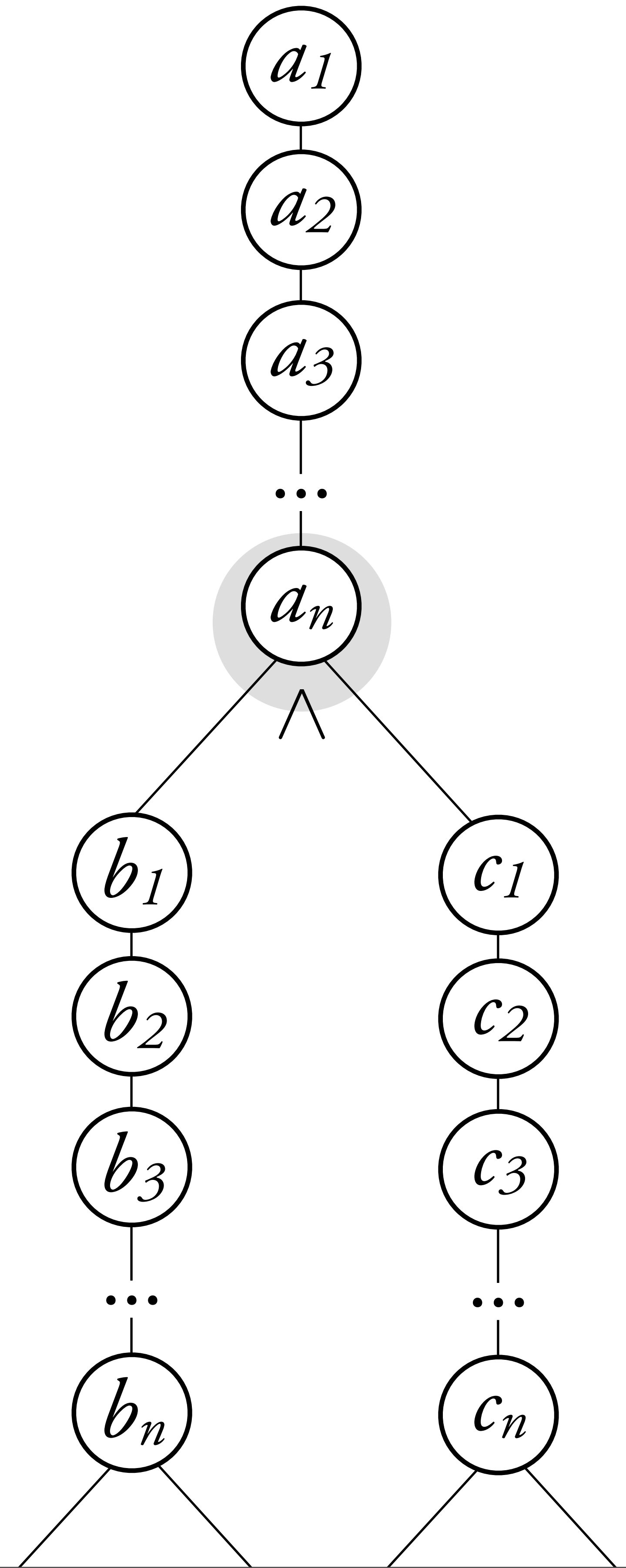

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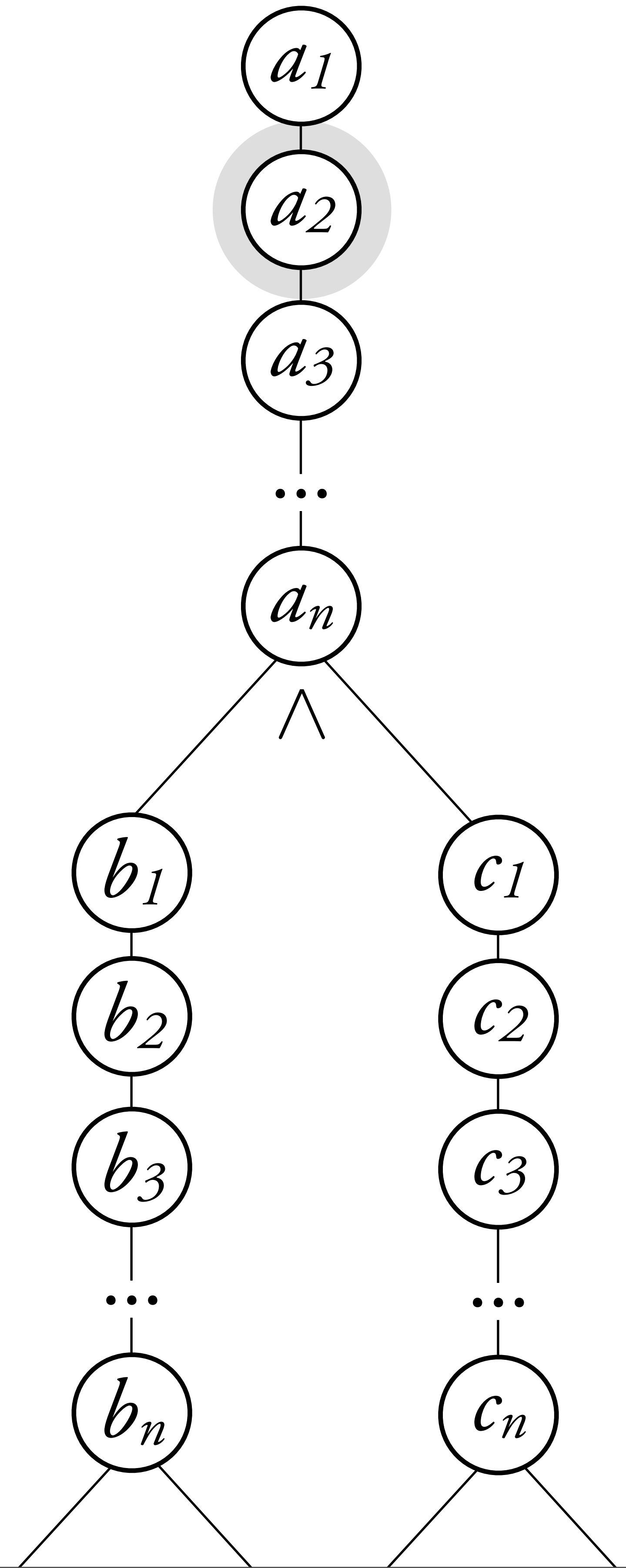

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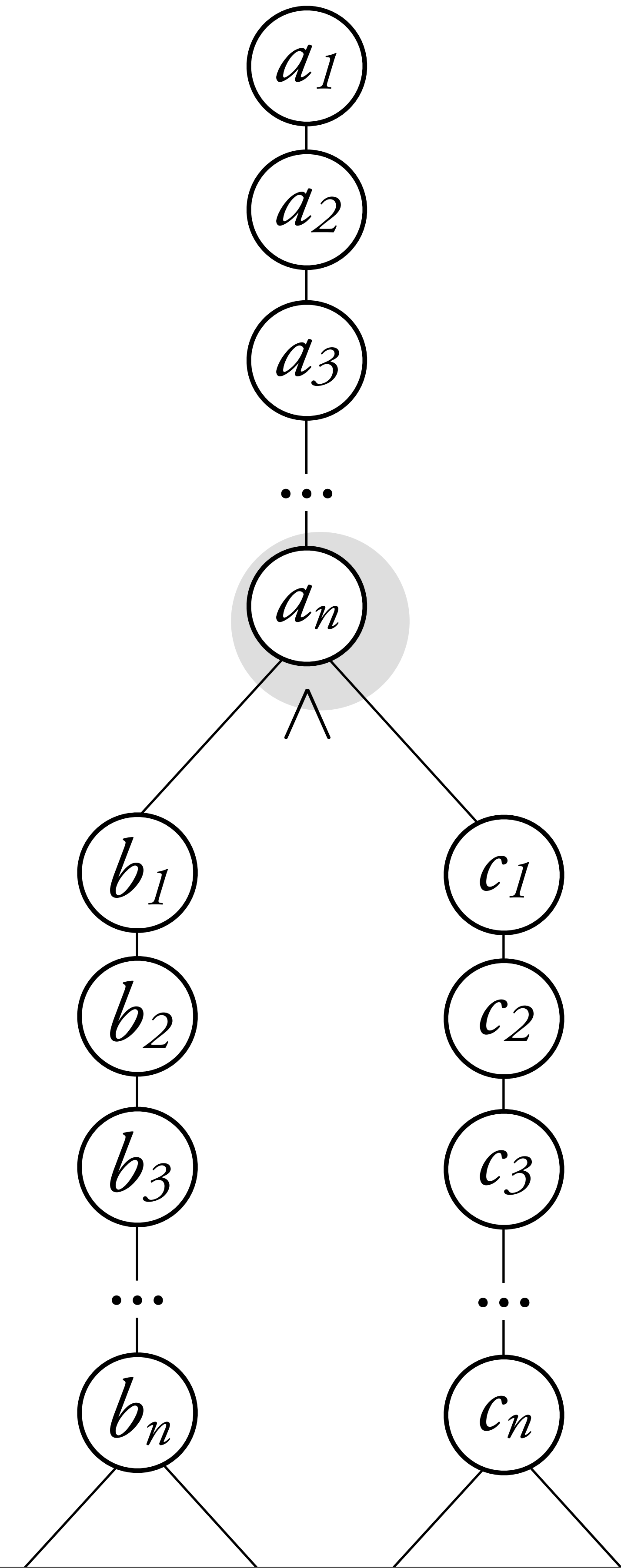

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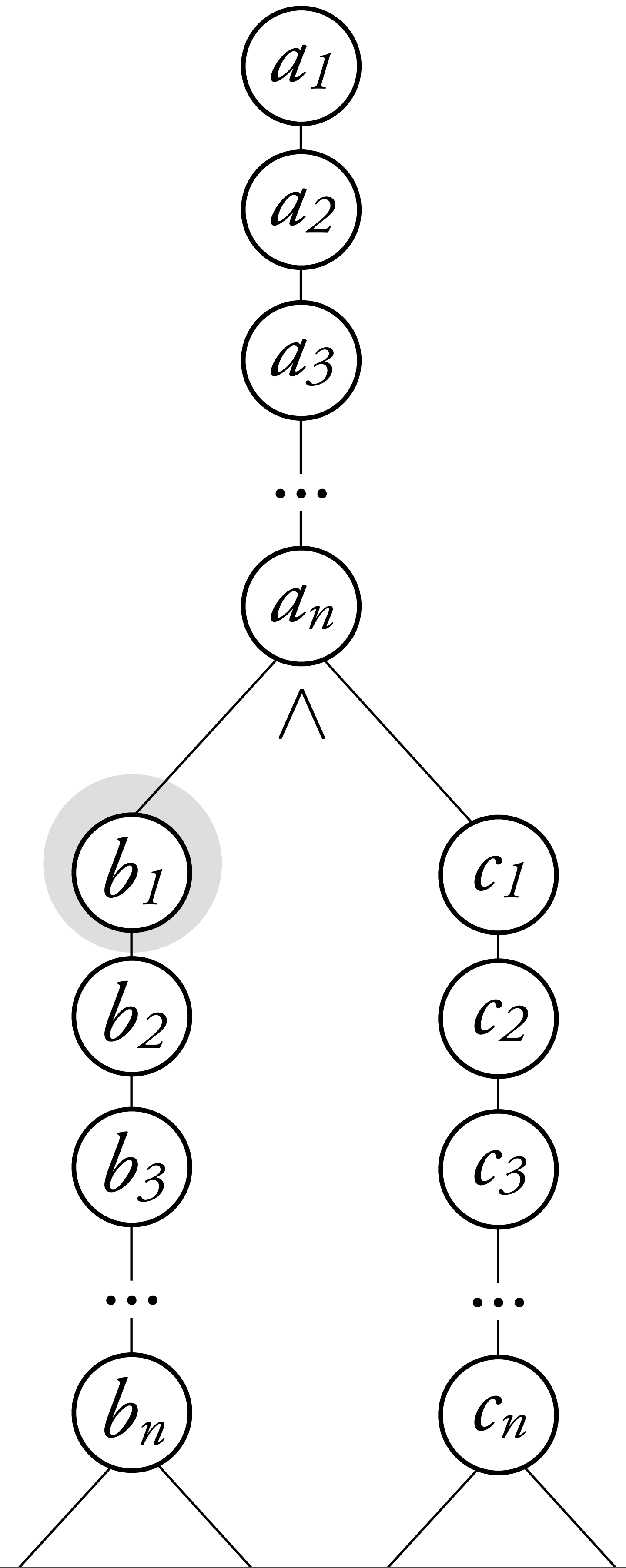

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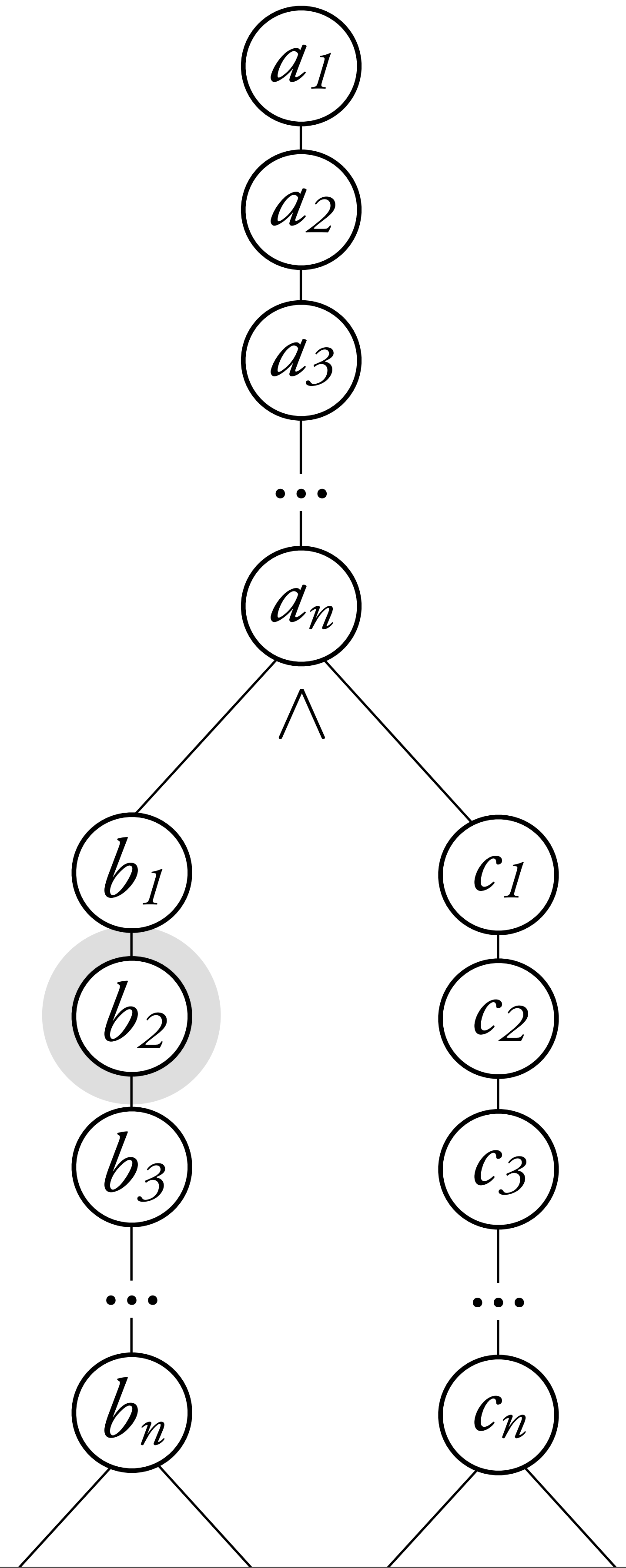

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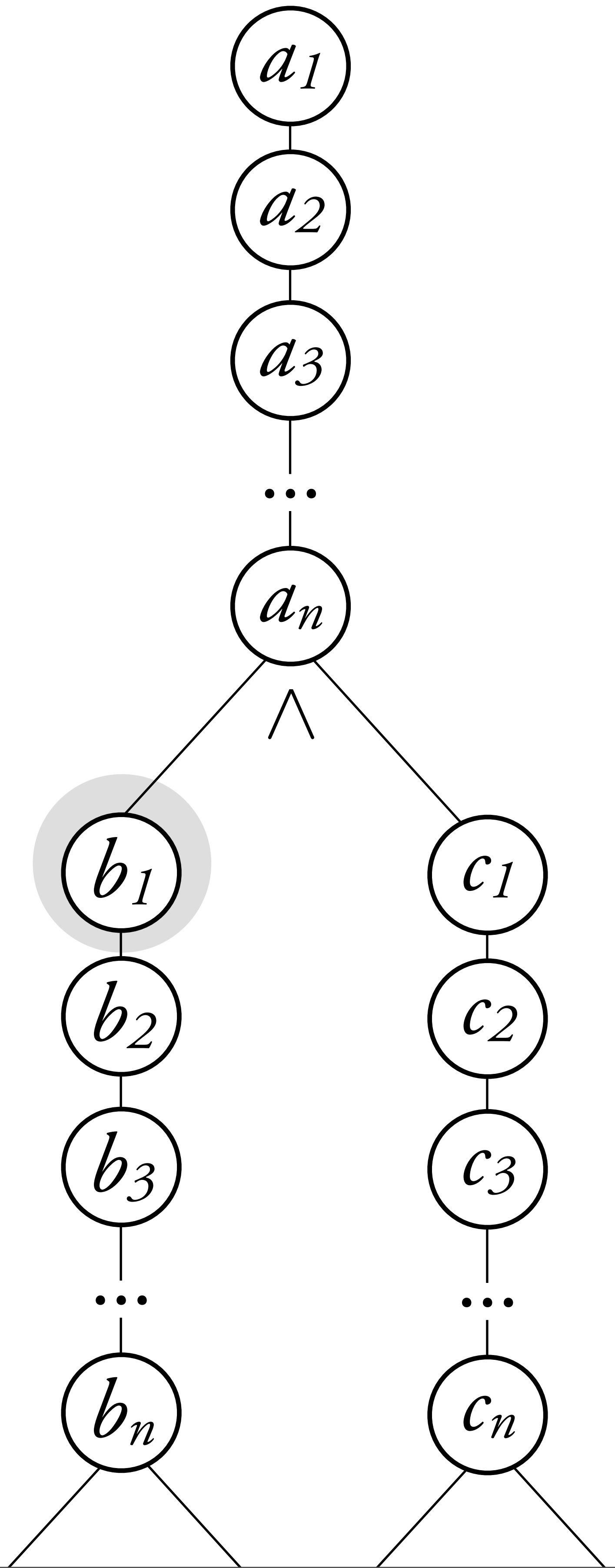

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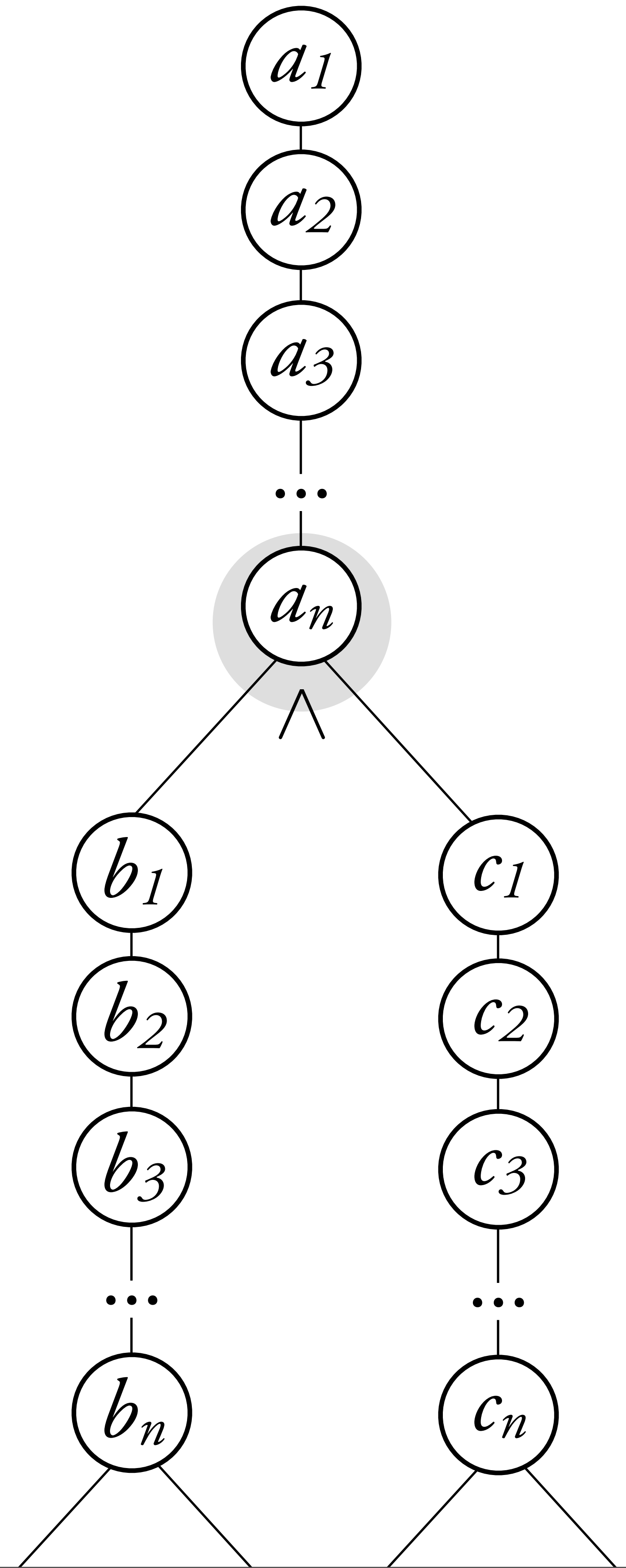

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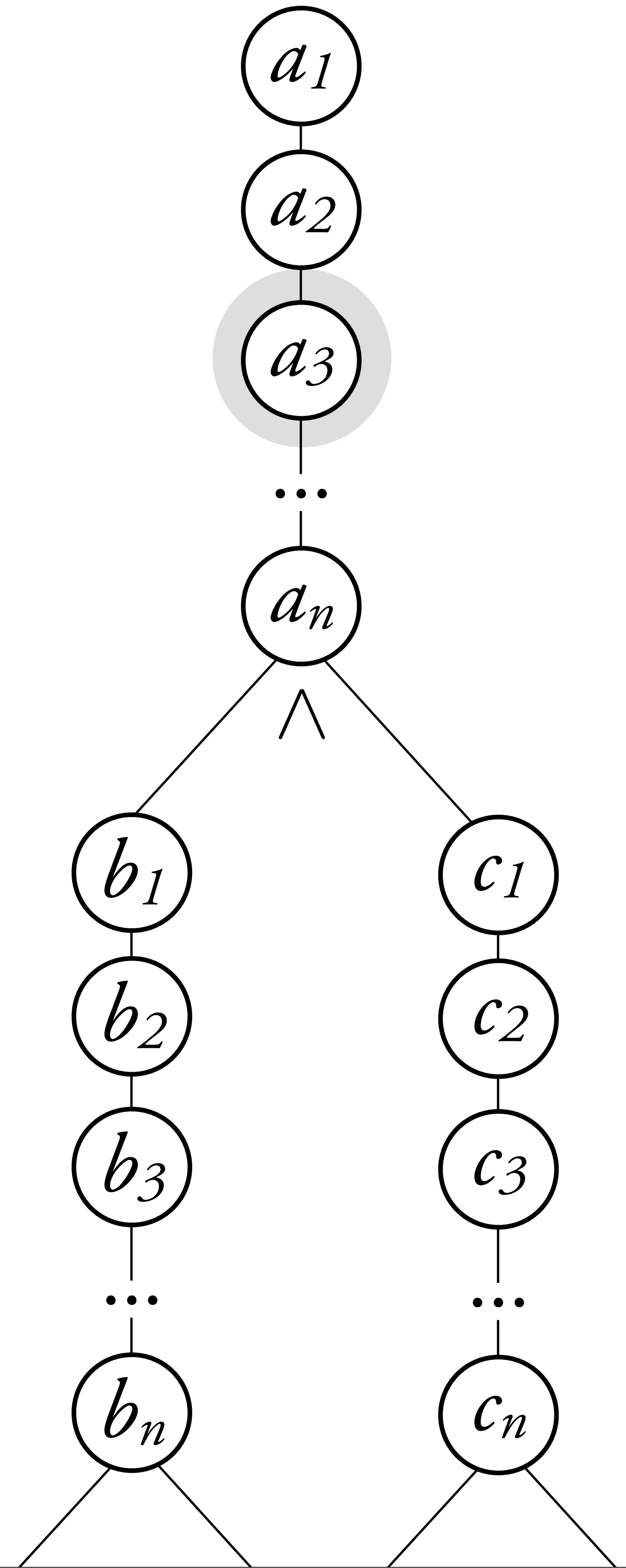

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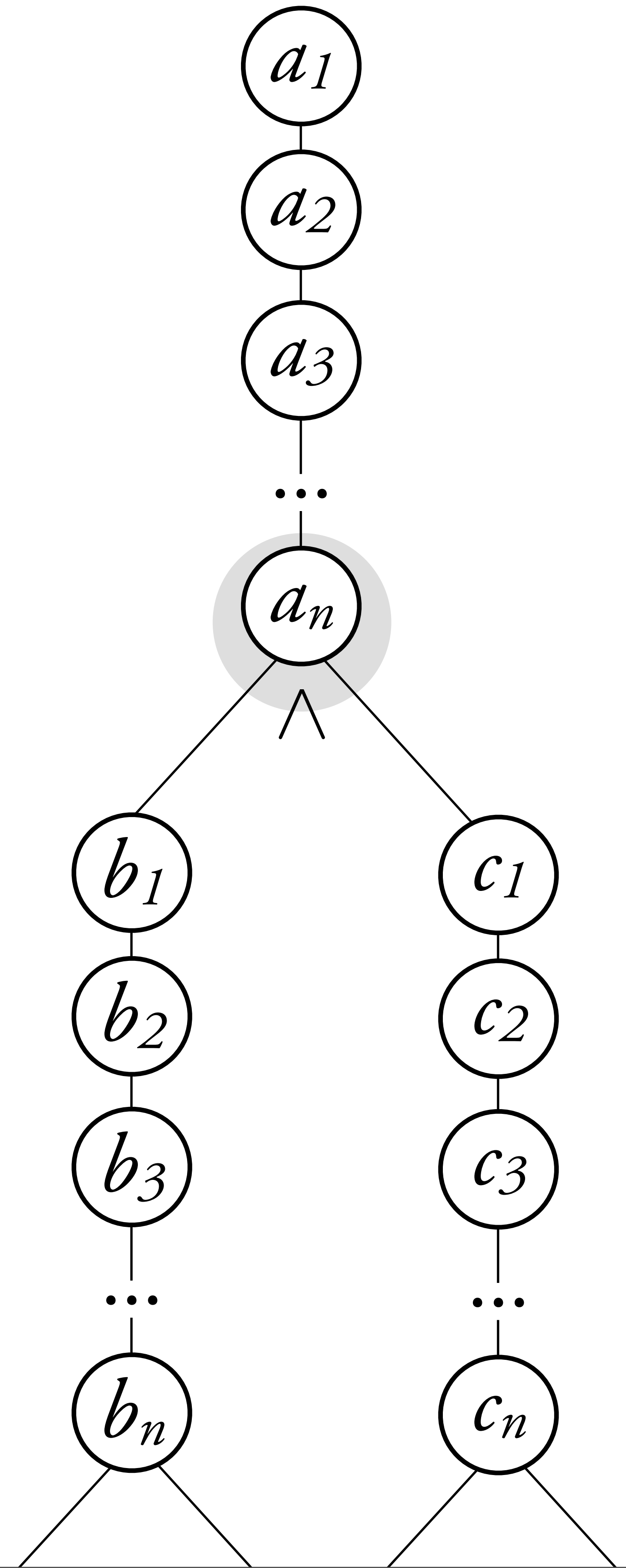

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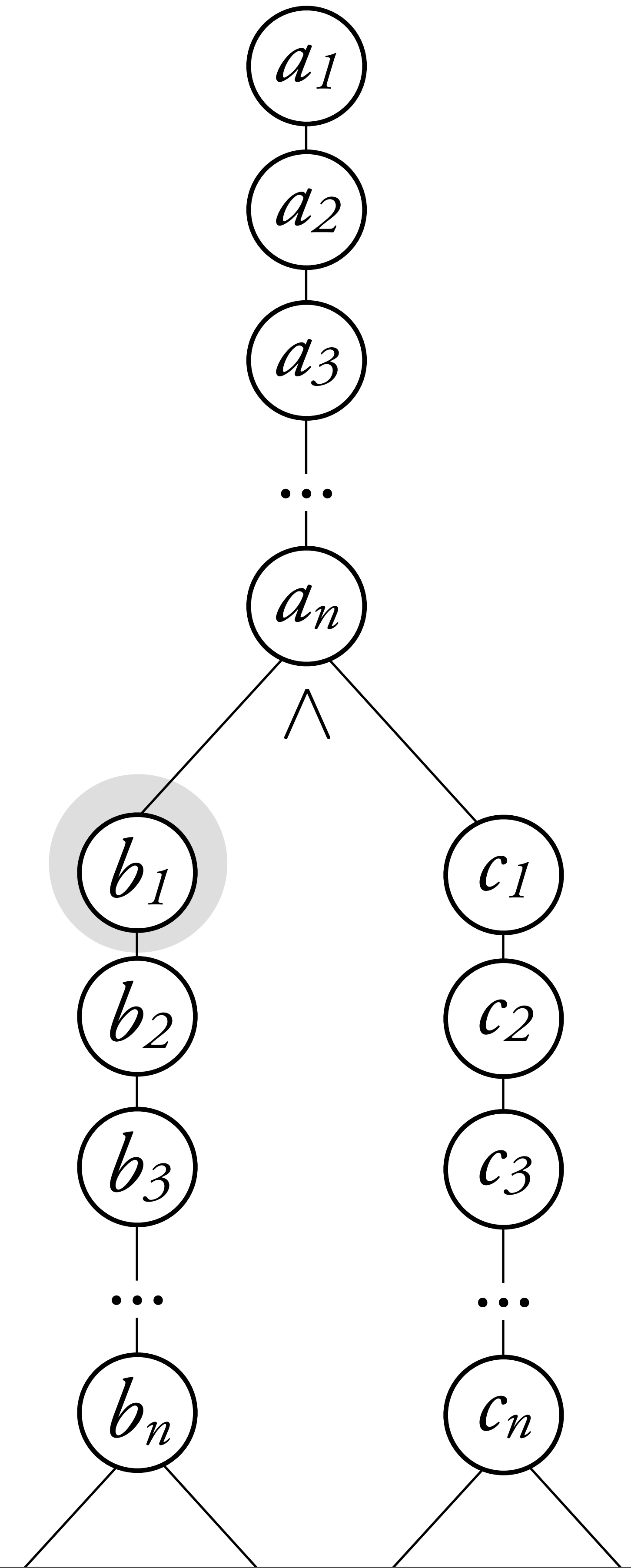

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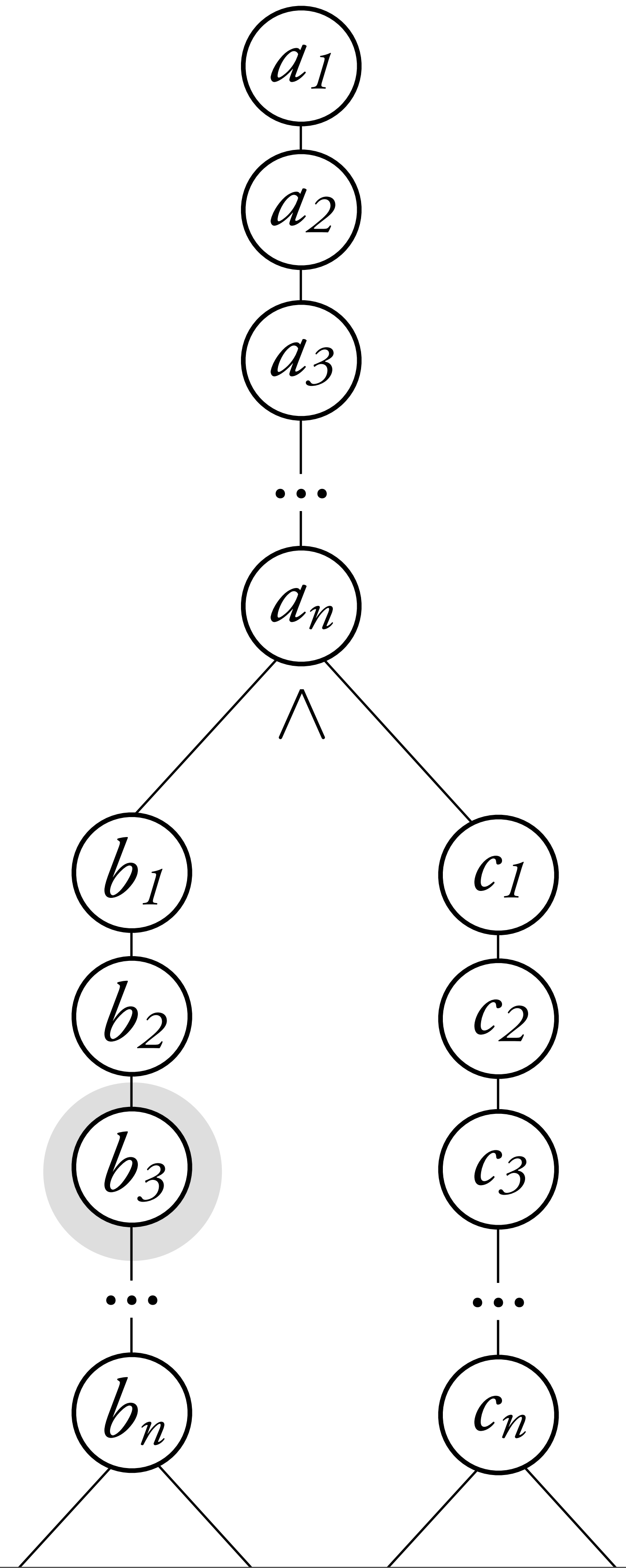

$\mathrm{TWA} \subseteq \mathrm{REG}$

Is the inclusion strict?

Theorem (B., Colcombet '05) The inclusion is strict.

Theorem (B., Colcombet '04)

Tree-walking automata cannot be determinized.

$L \in \mathrm{TWA}$

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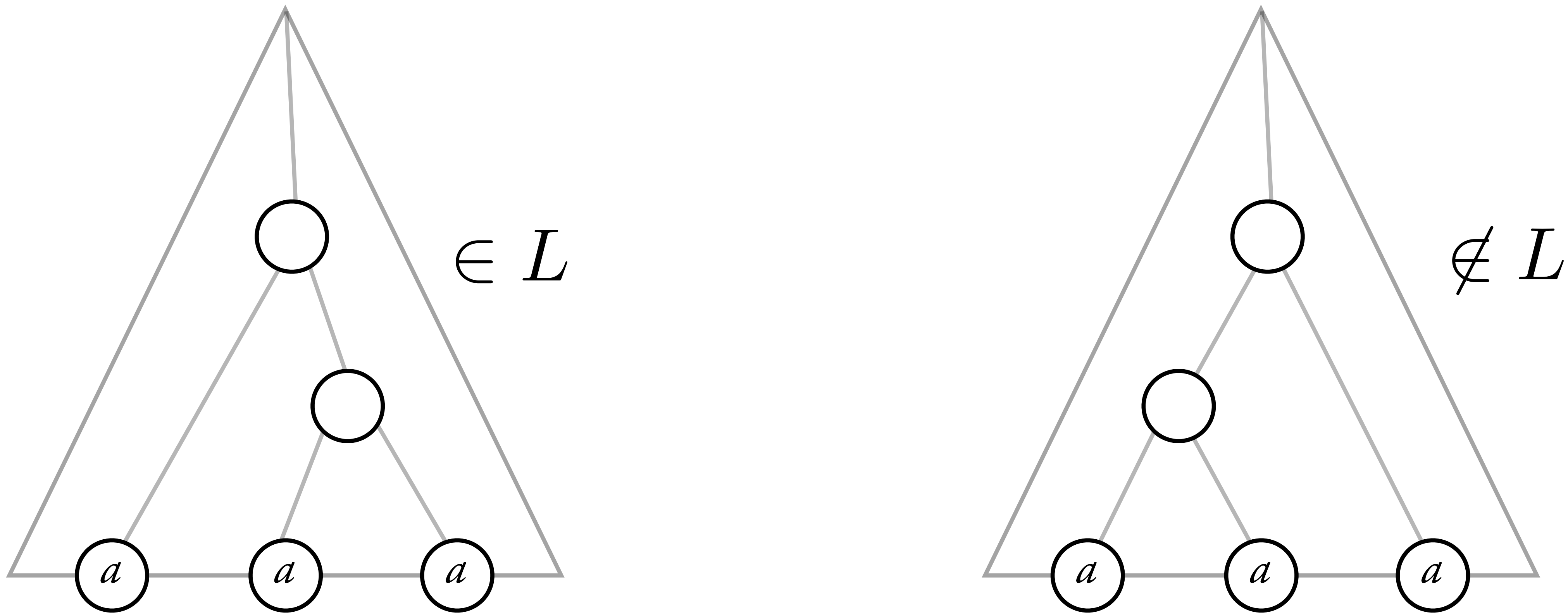

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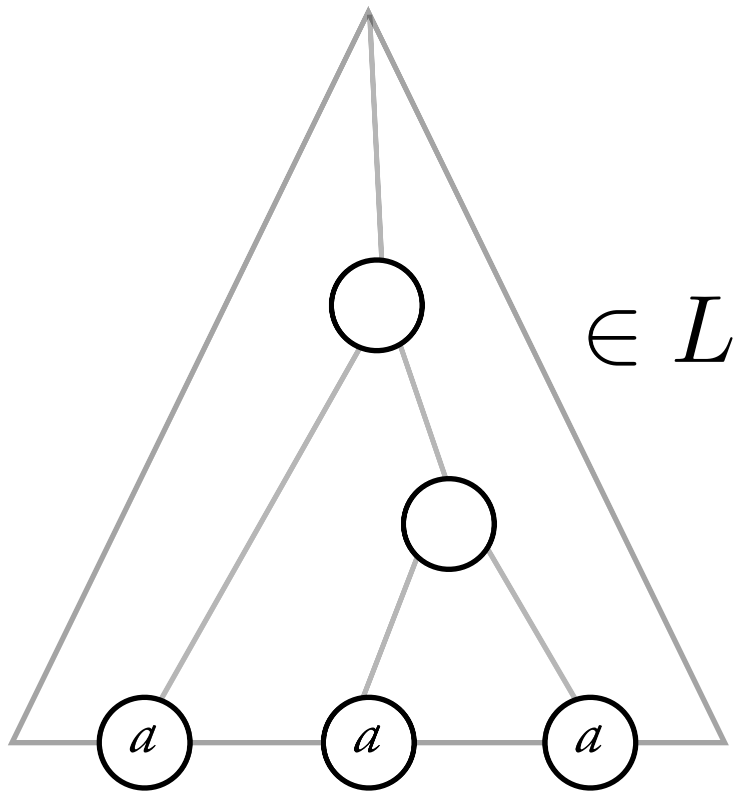

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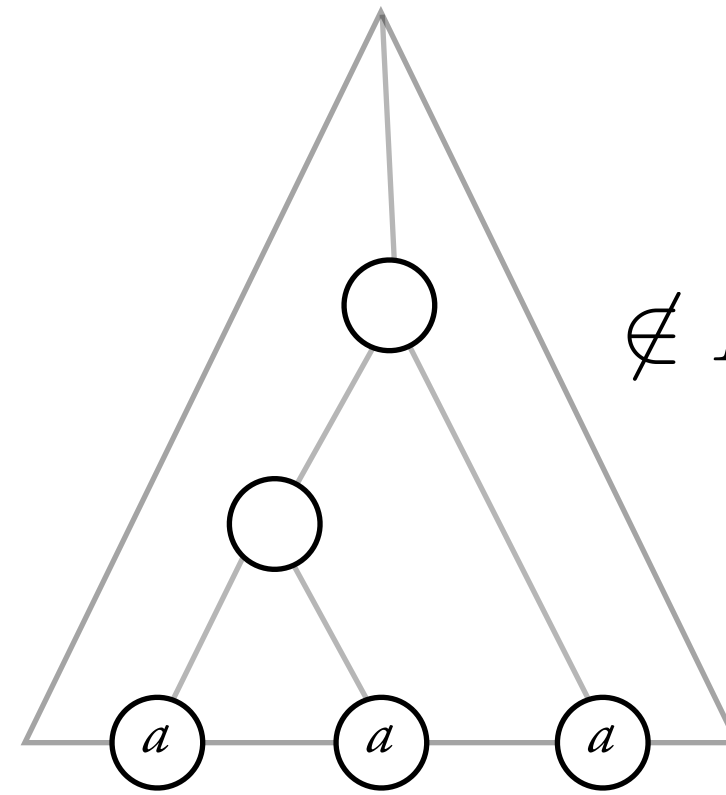

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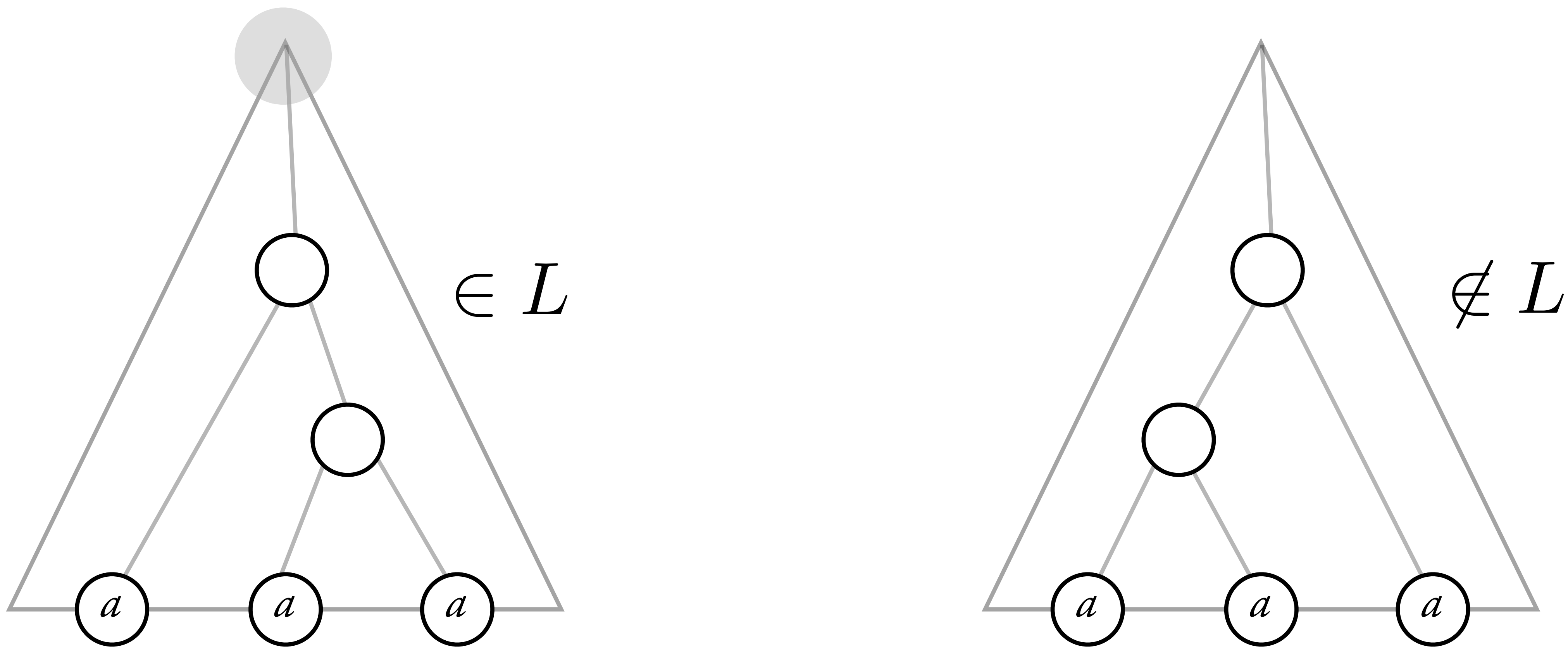

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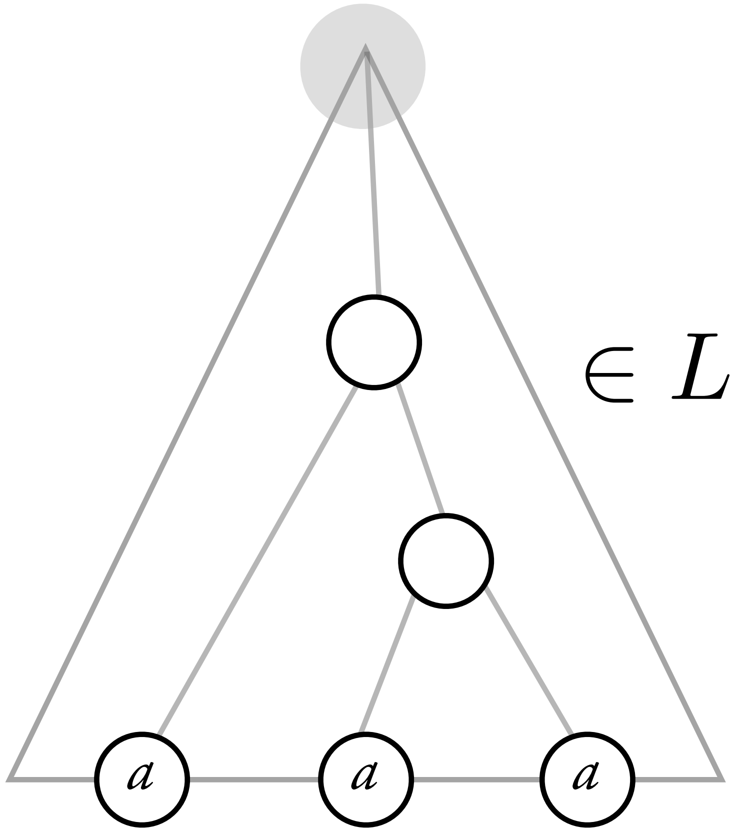

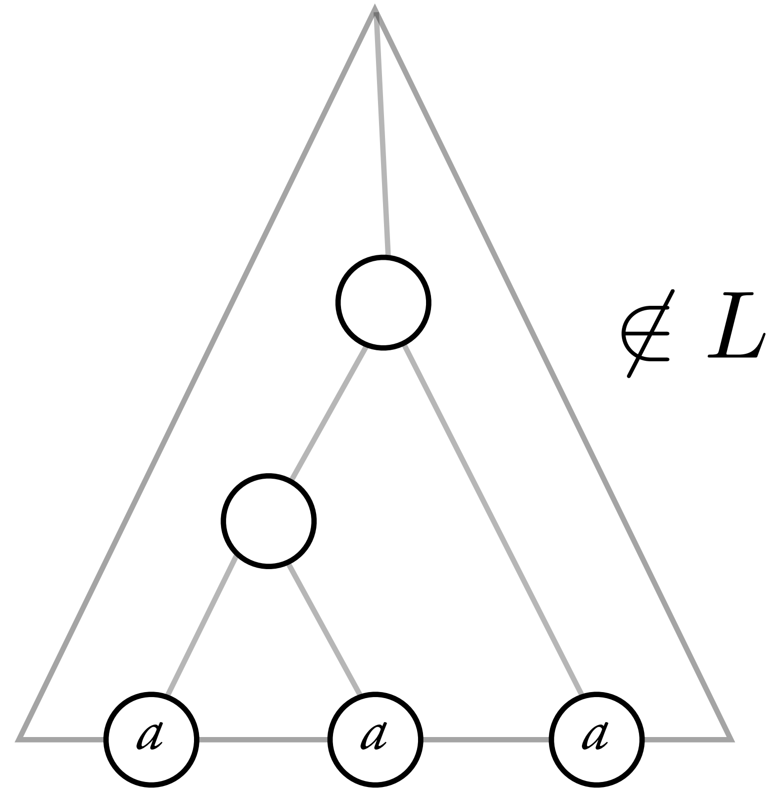

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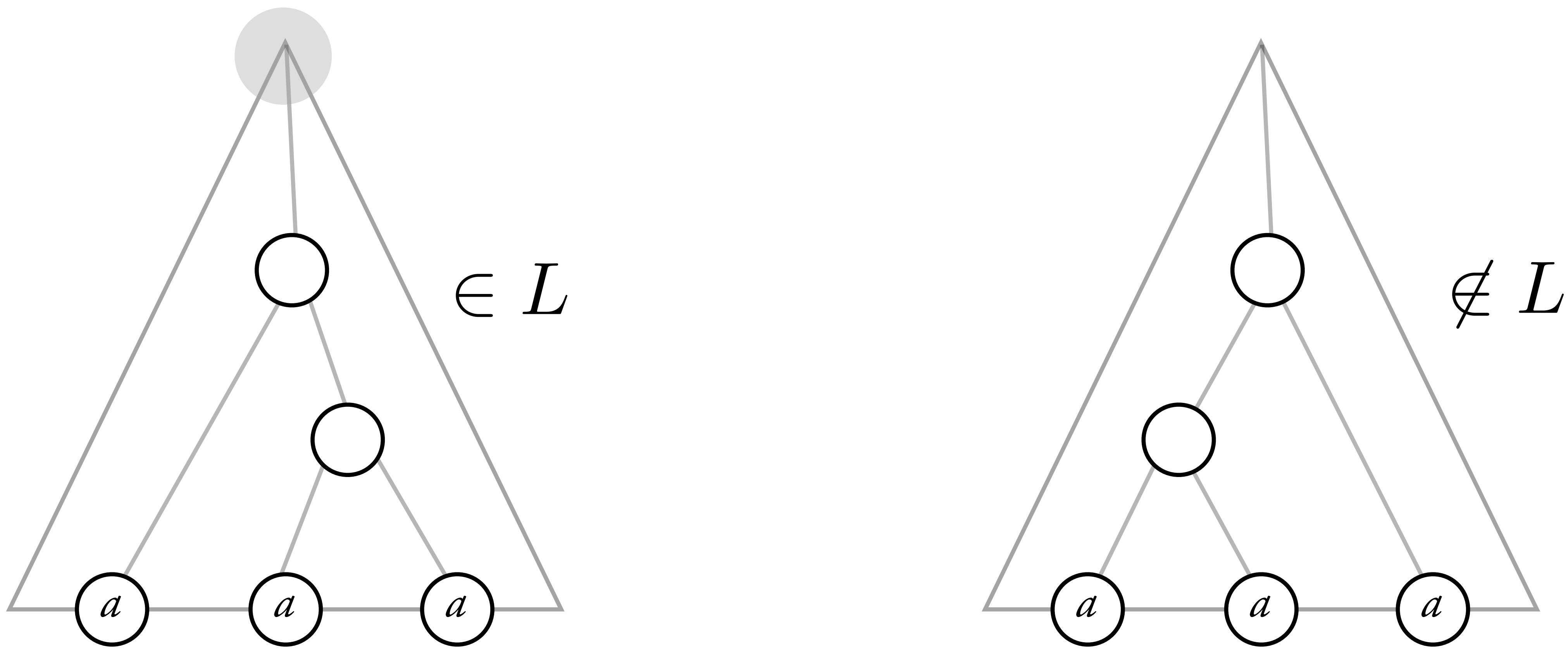

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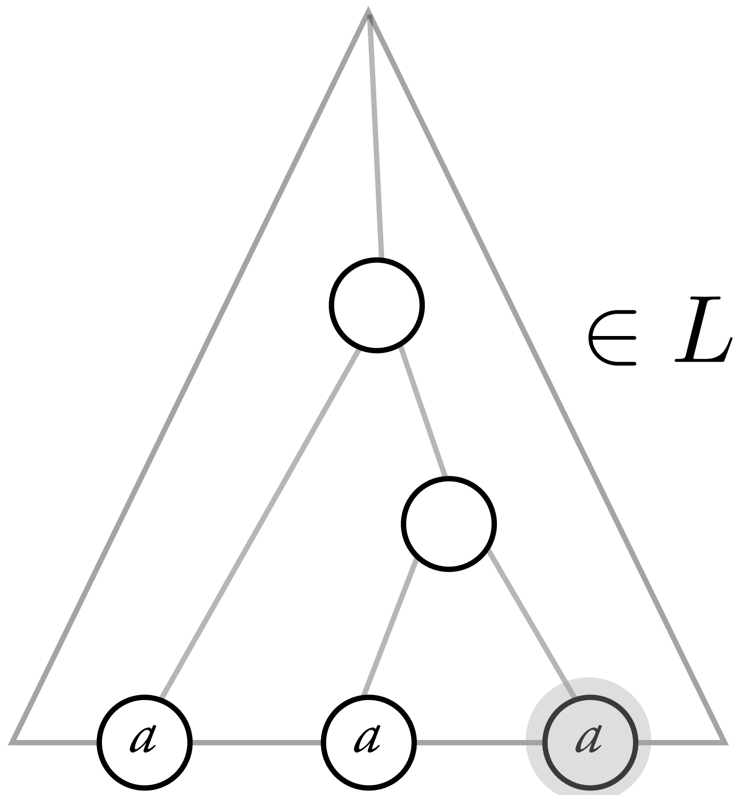

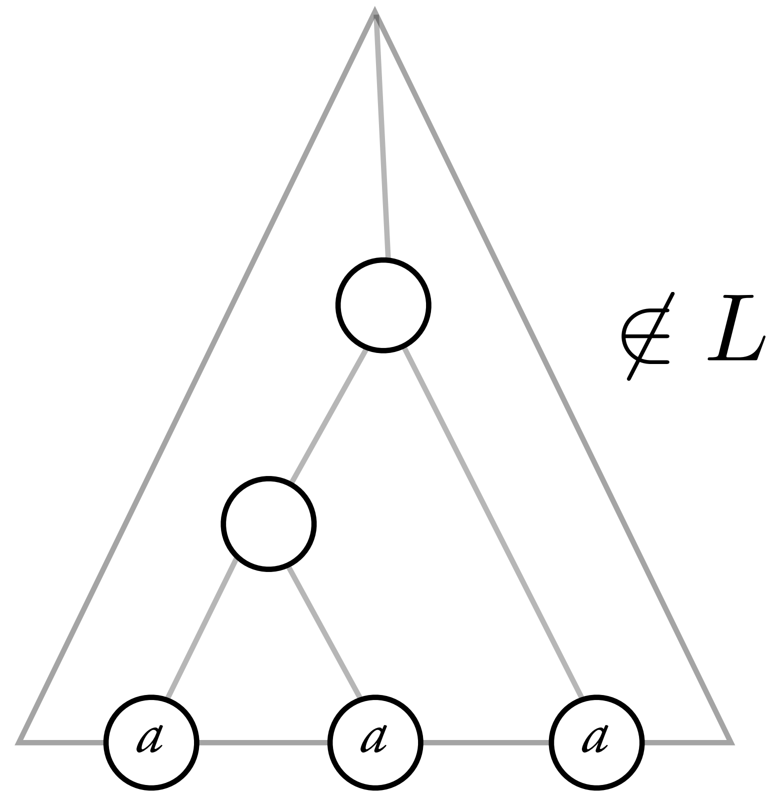

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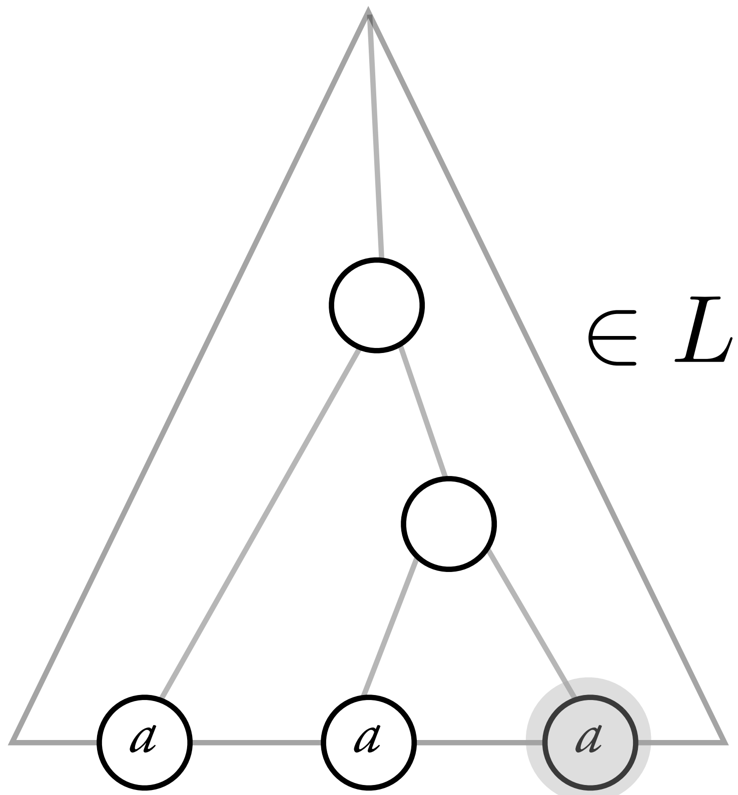

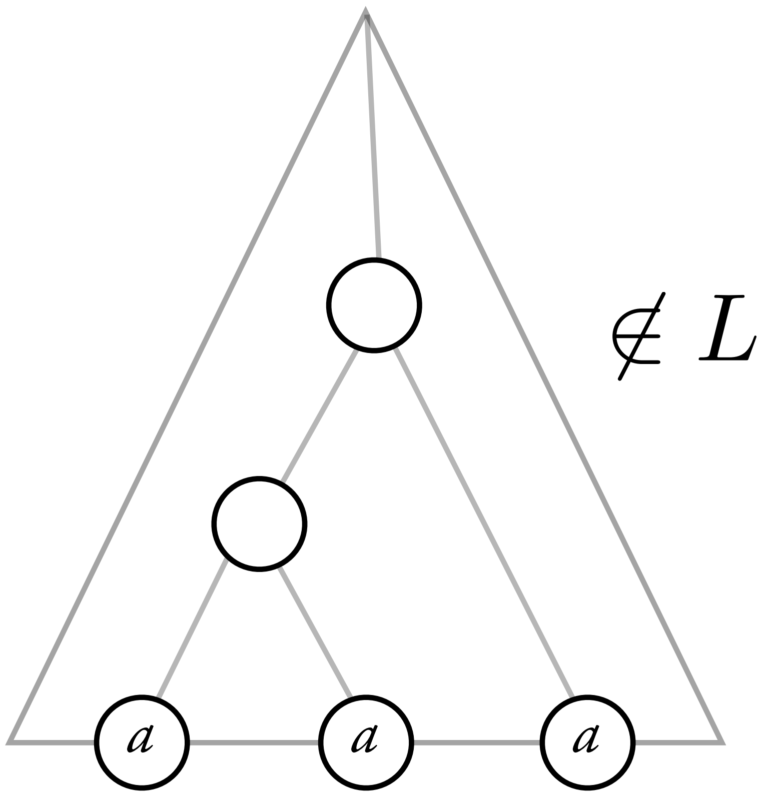

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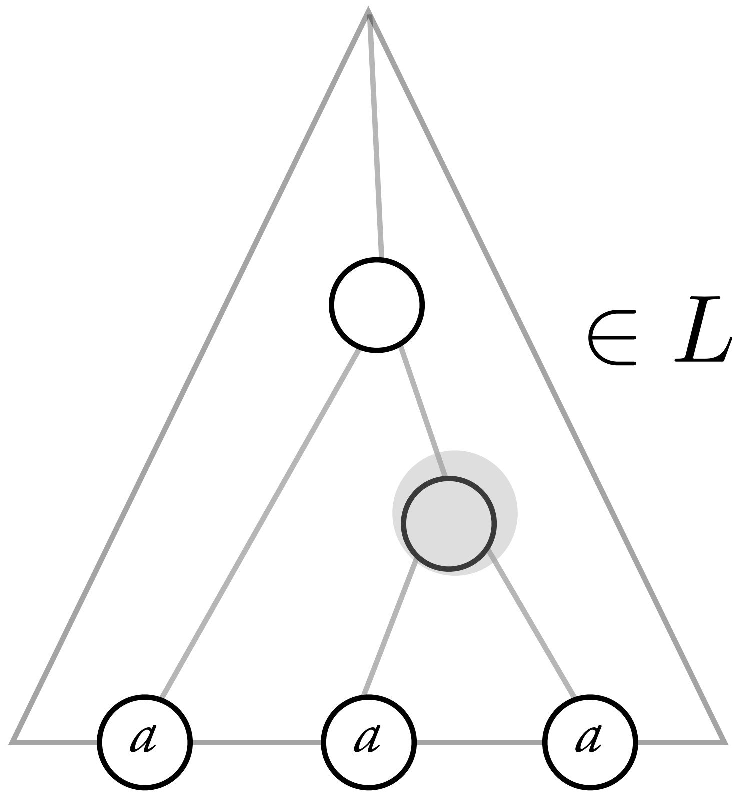

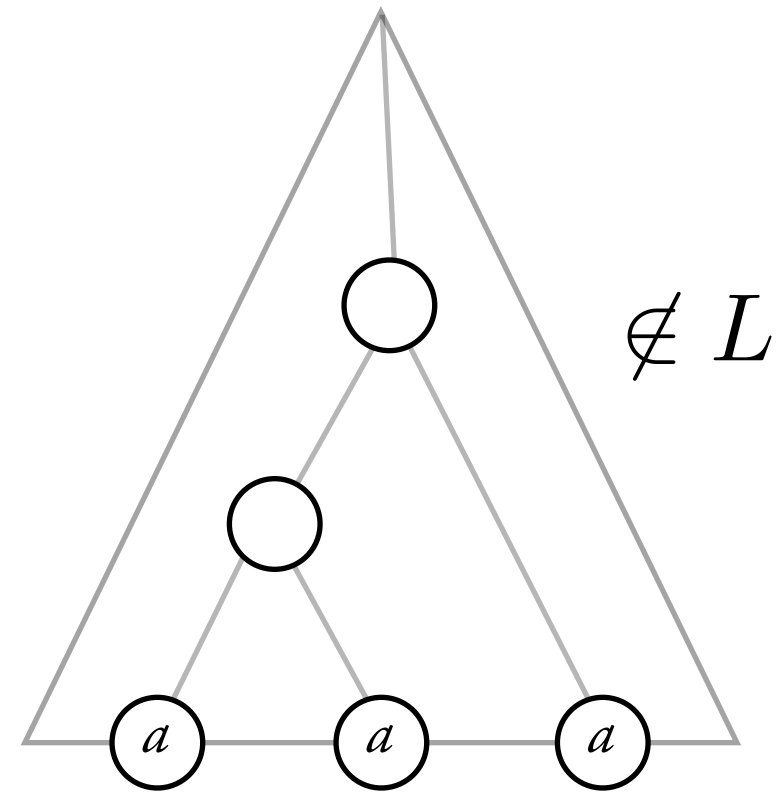

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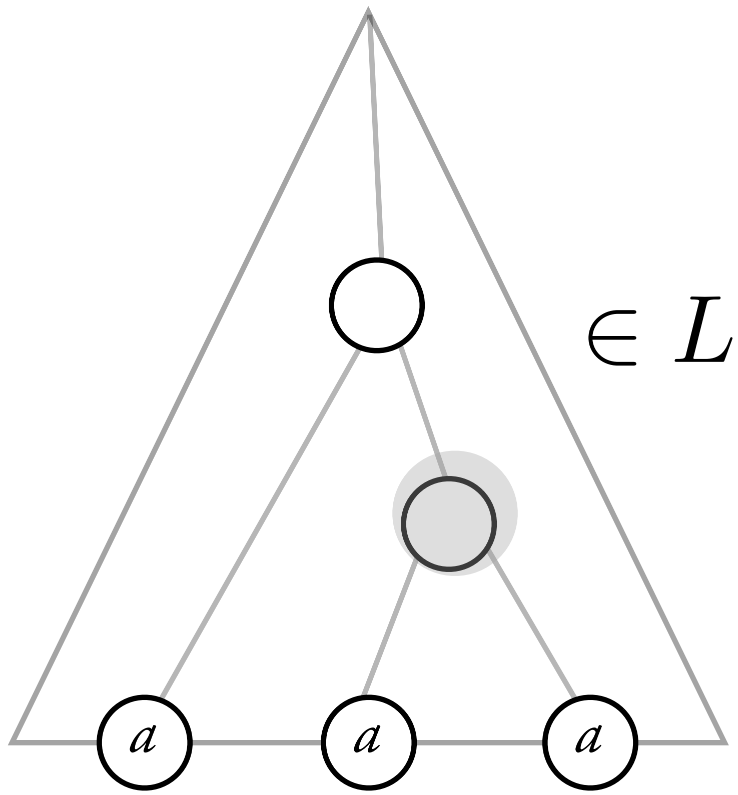

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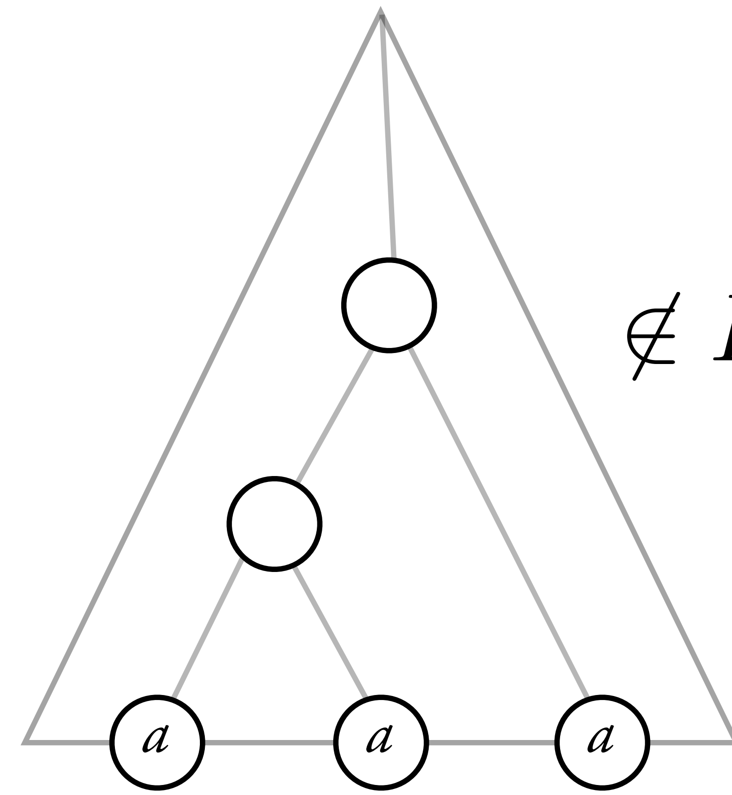

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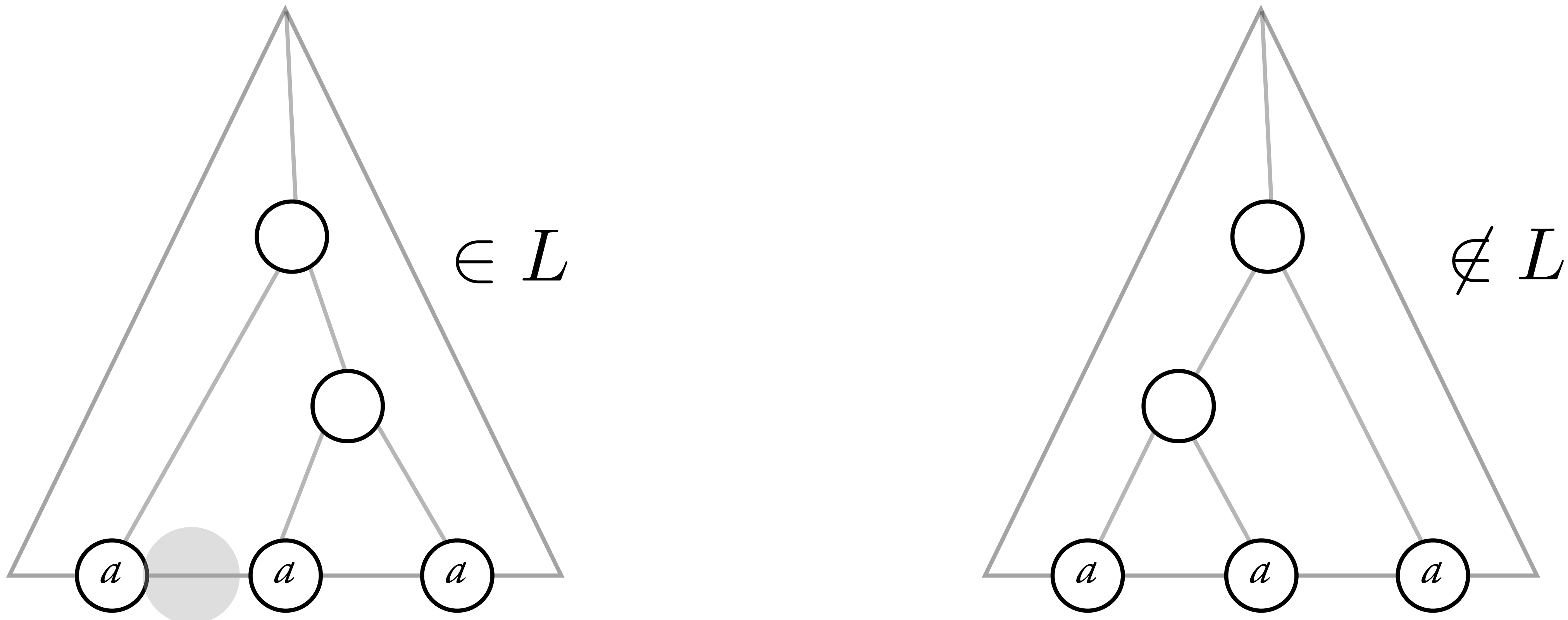

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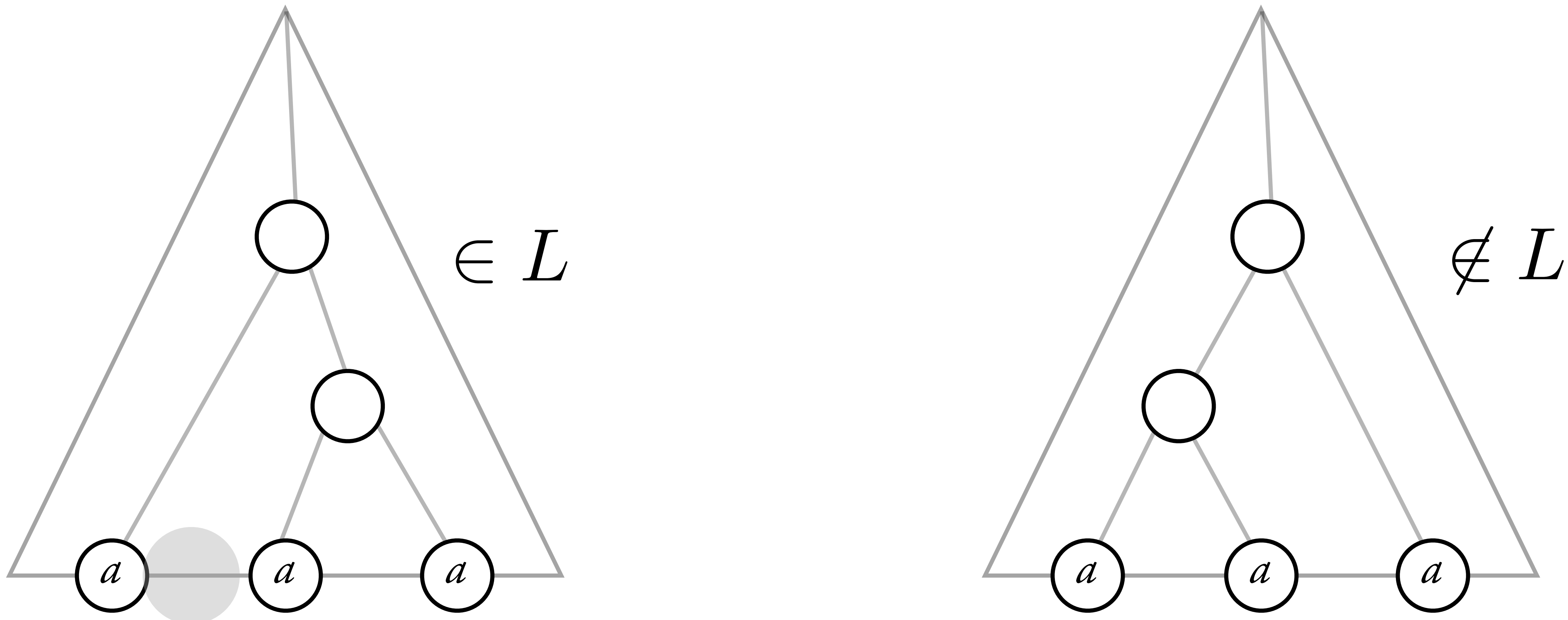

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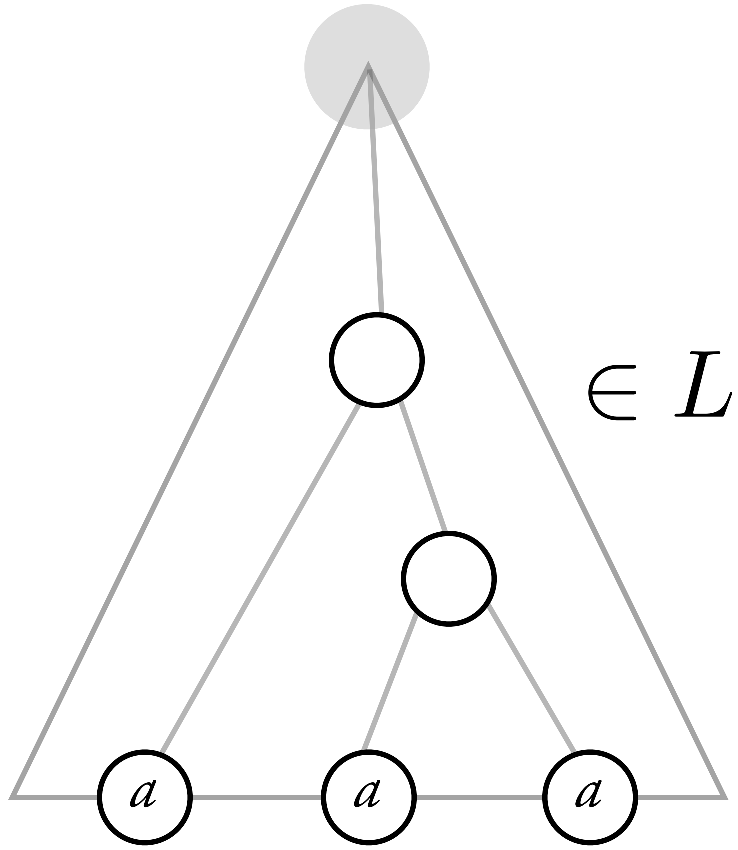

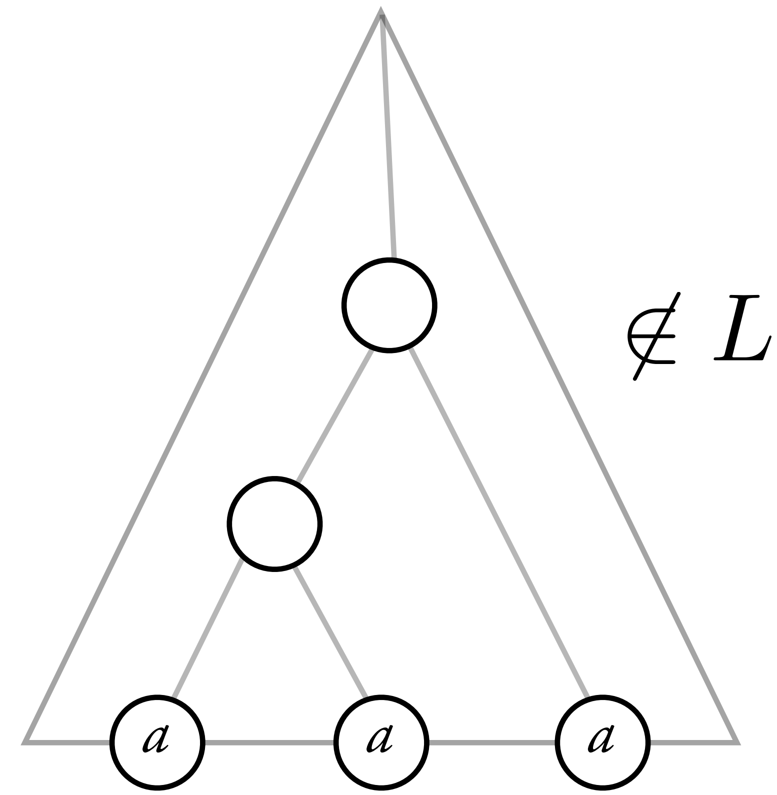

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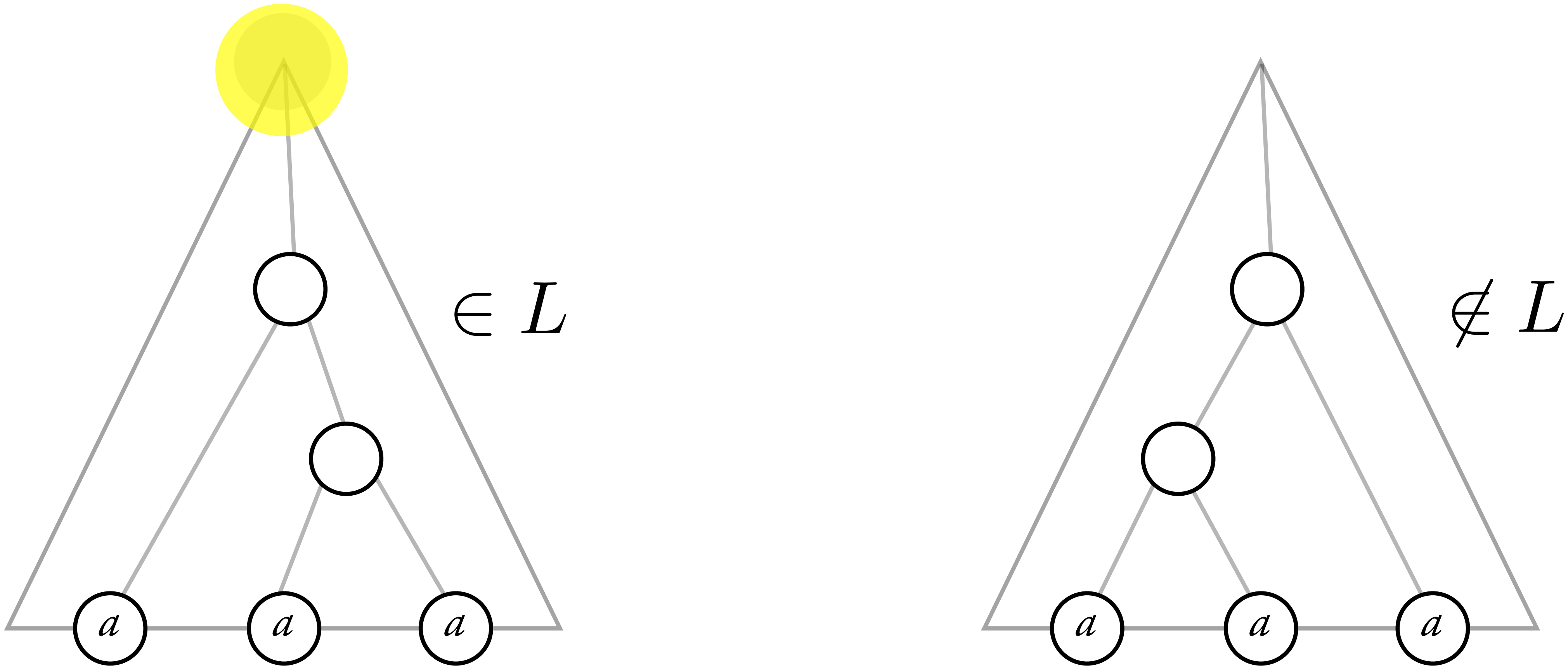

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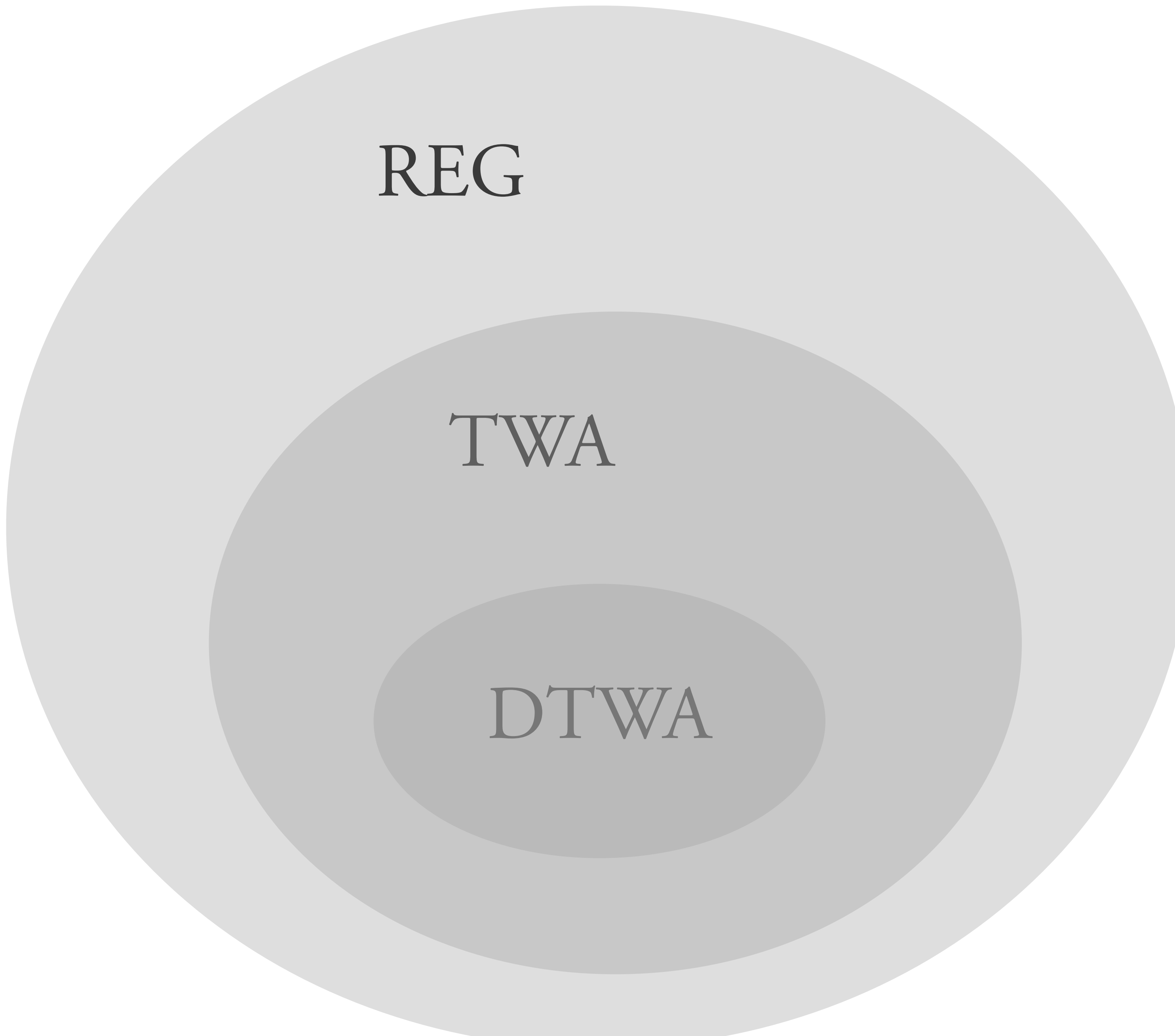
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Plan

-A tree-walking automaton

-Expressive power

-Pebble automata and logic

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What if the automaton could mark nodes with pebbles?

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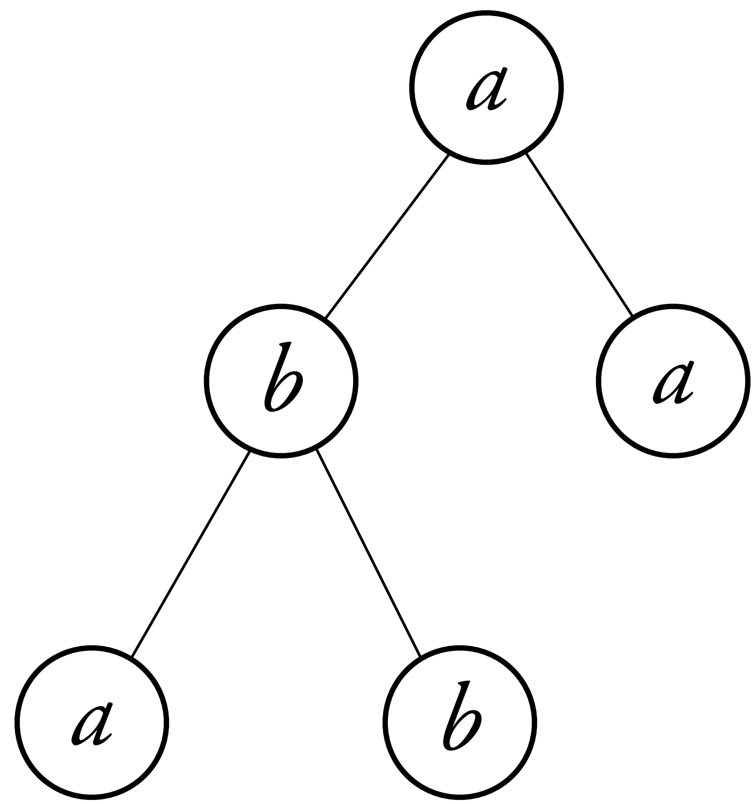

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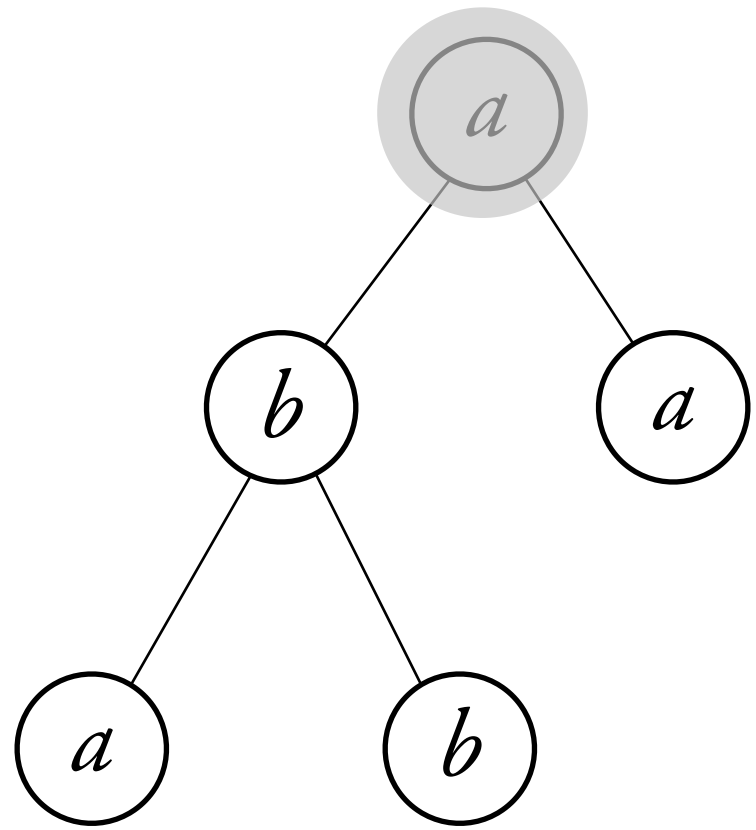

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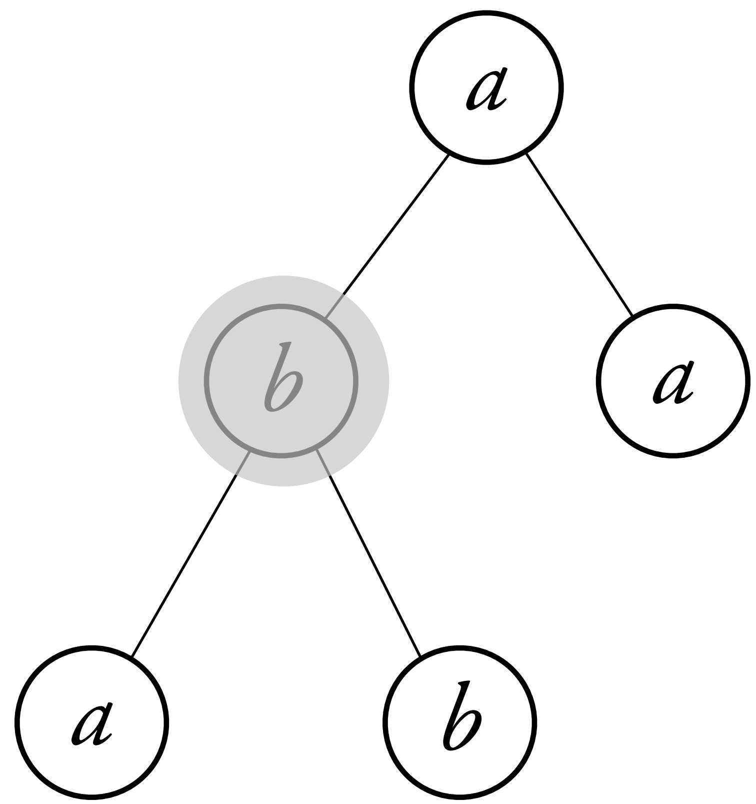

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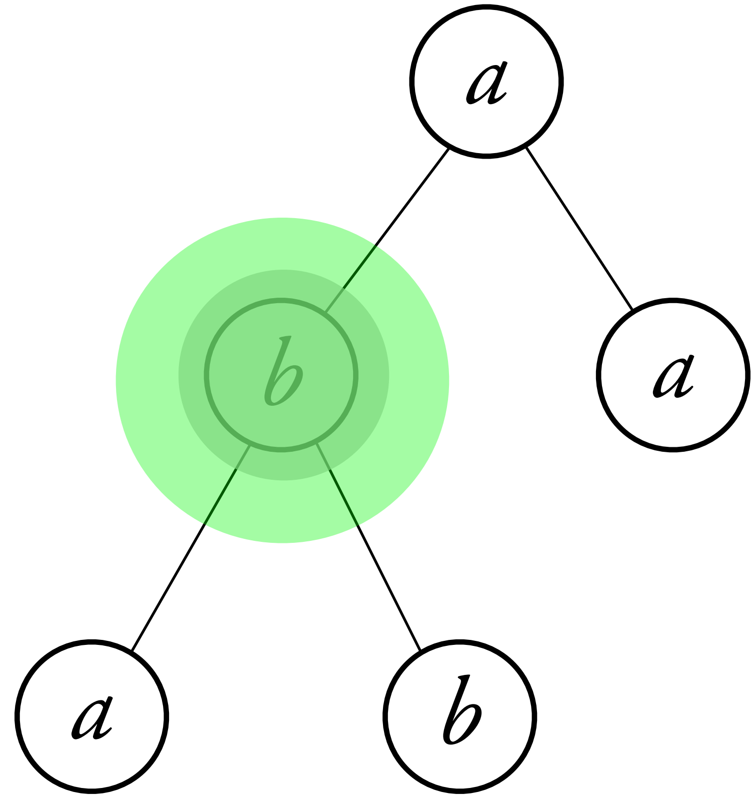

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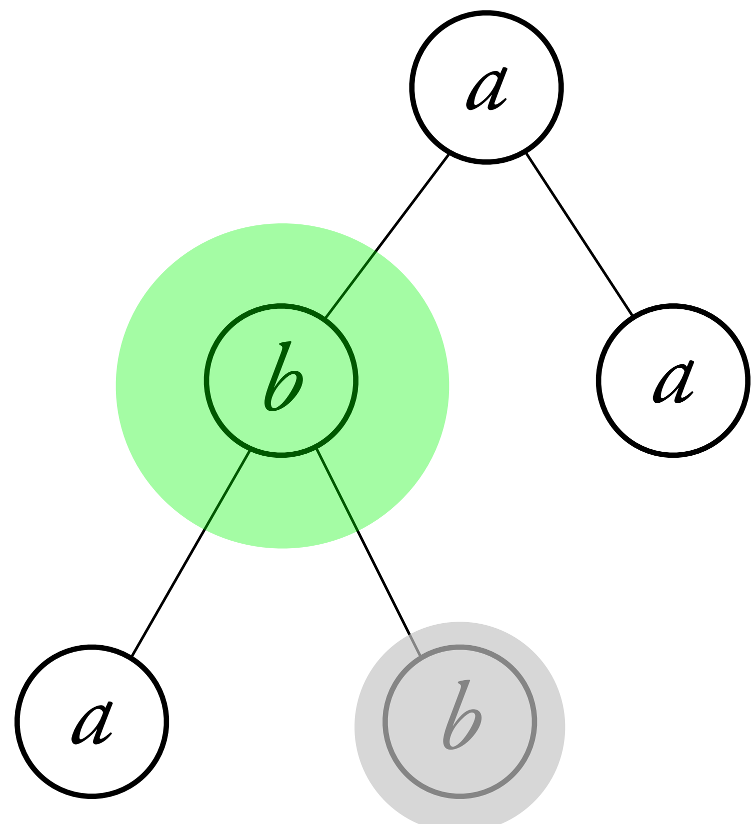

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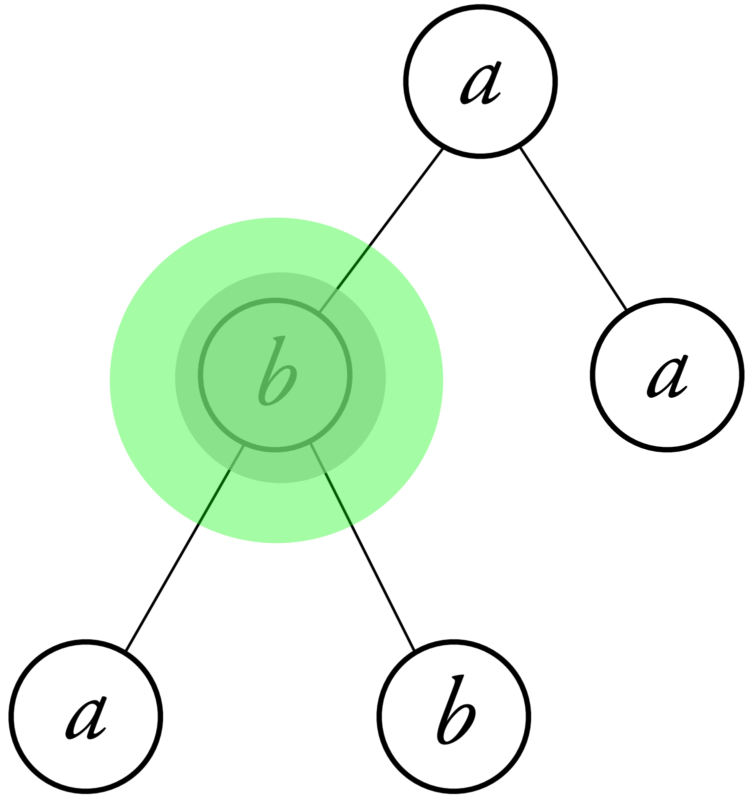

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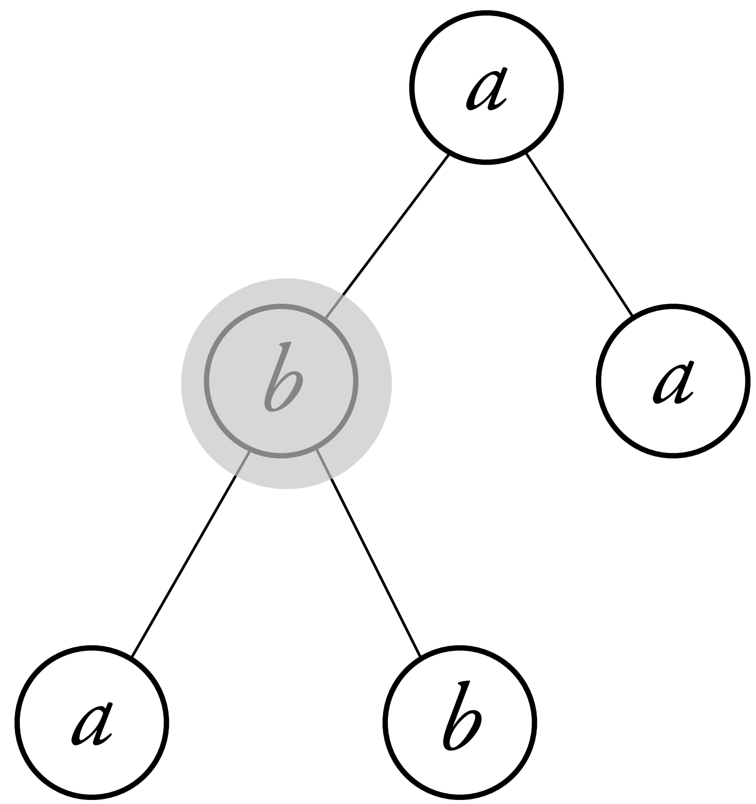

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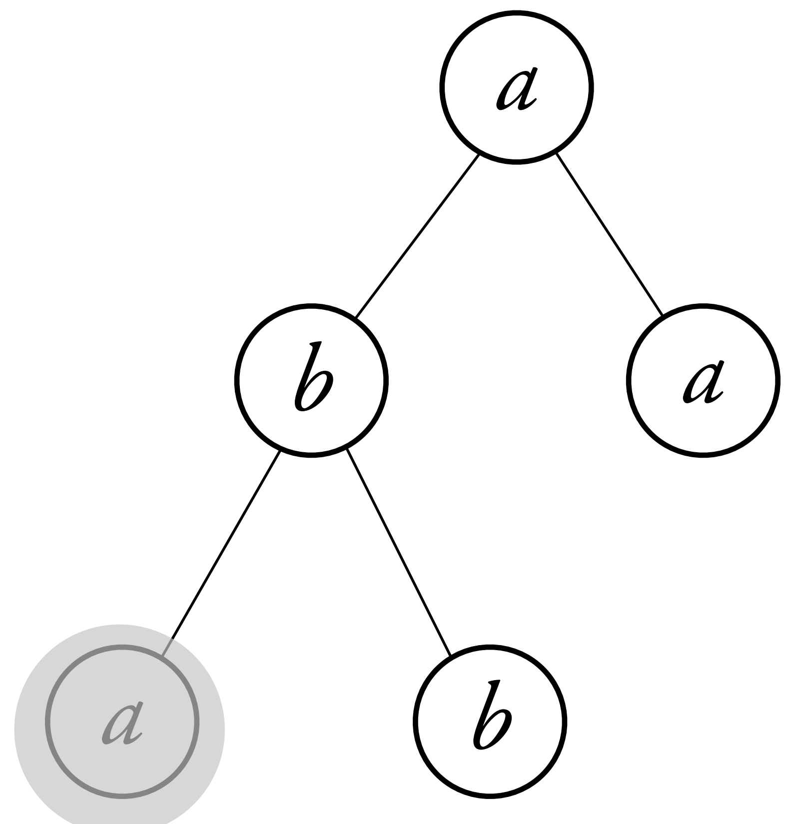

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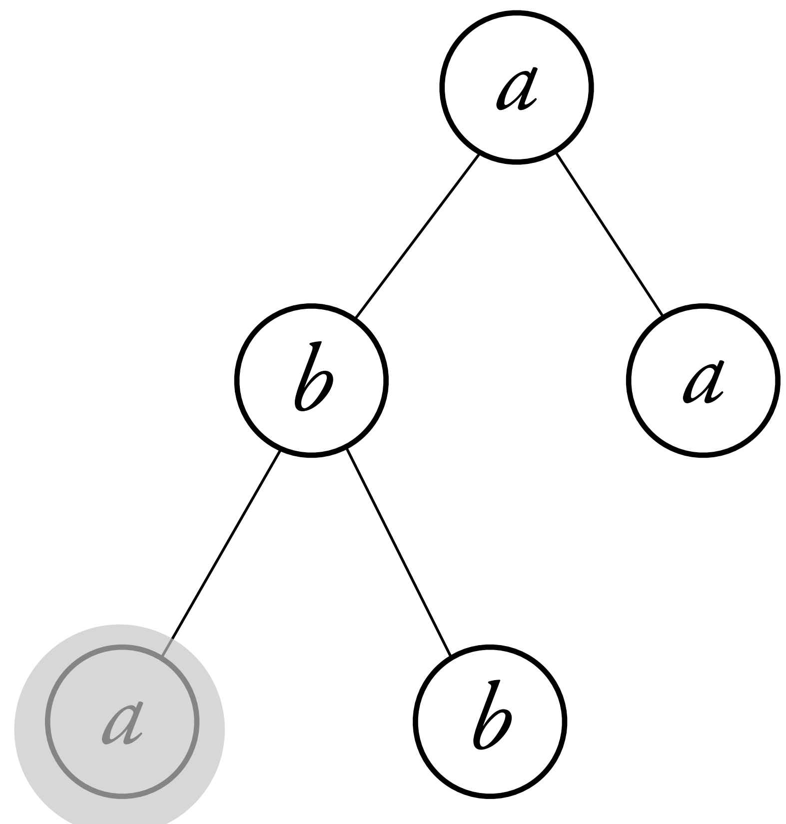

An *n-pebble automaton* has pebbles *1,…,n*.
New tests: “is pebble *i* on the current node?”
New commands: “place pebble *i* on the current node”
“lift pebble *i* from the current node”.

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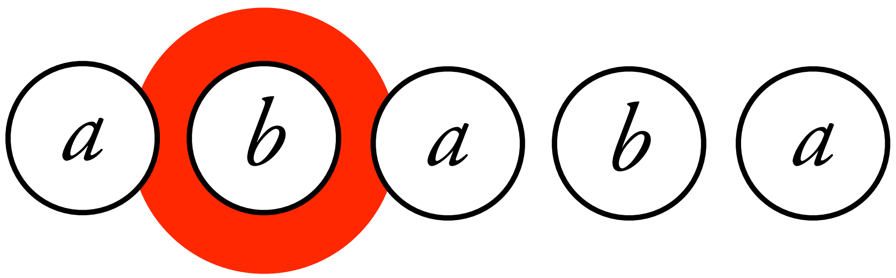

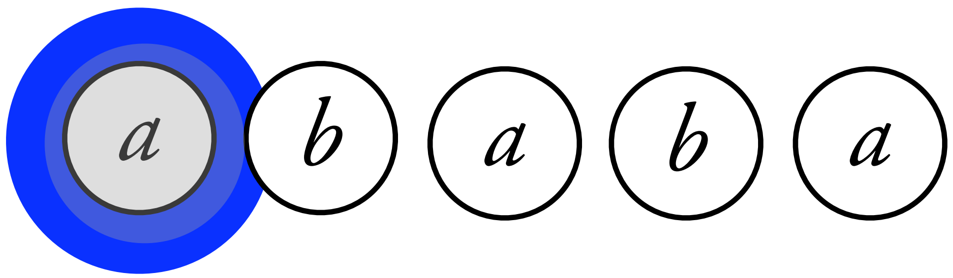

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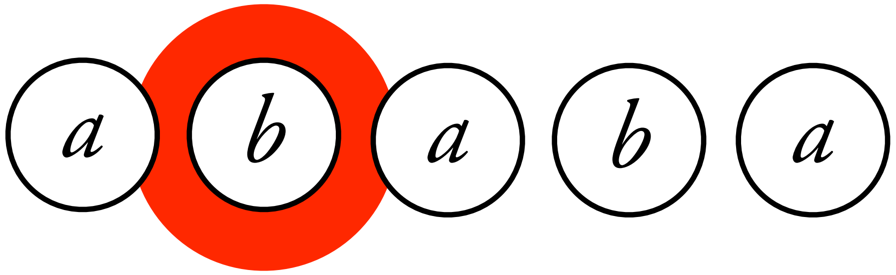

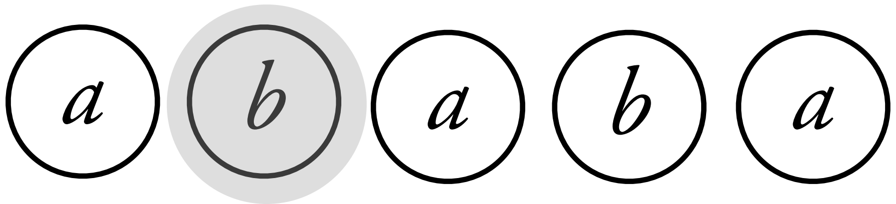

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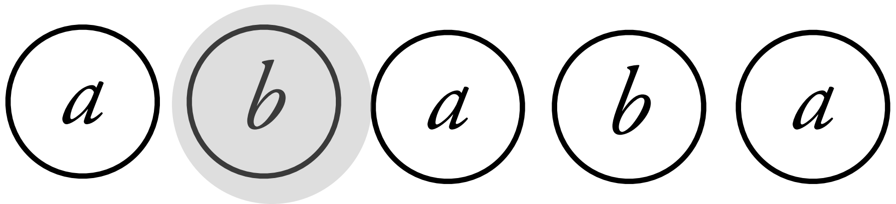

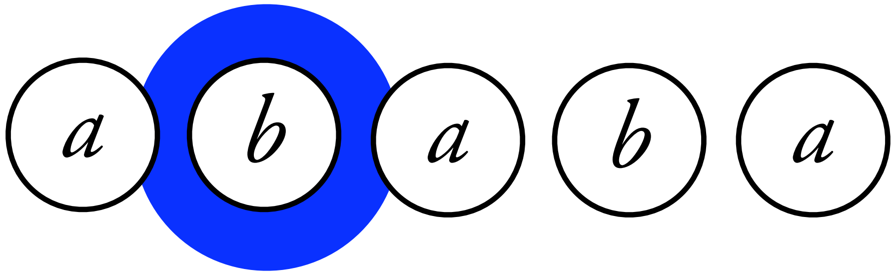

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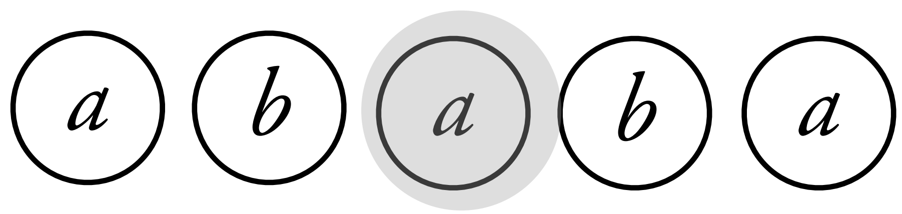

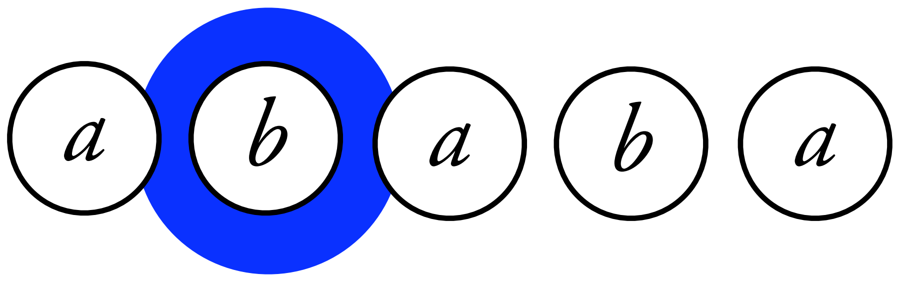

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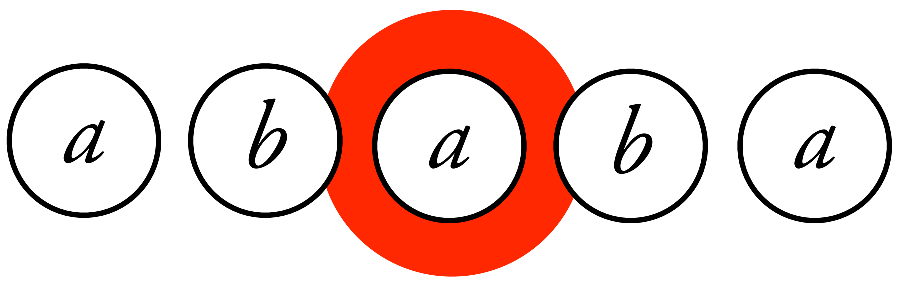

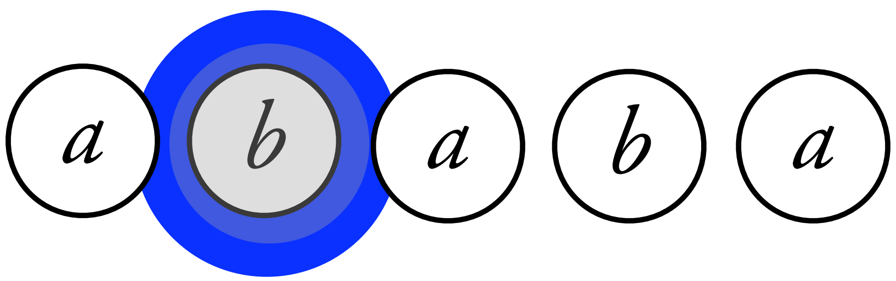

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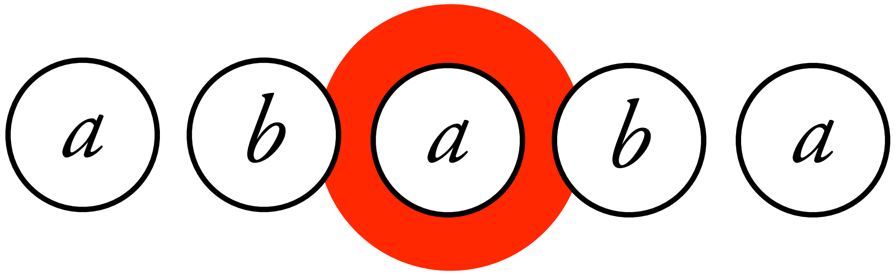

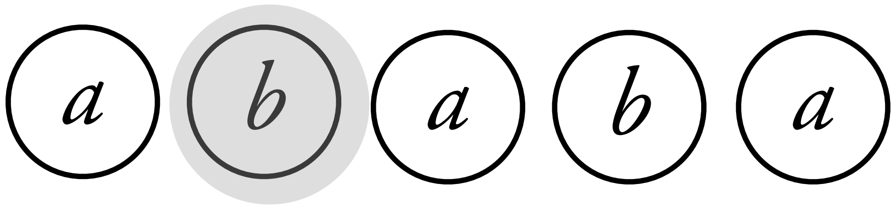

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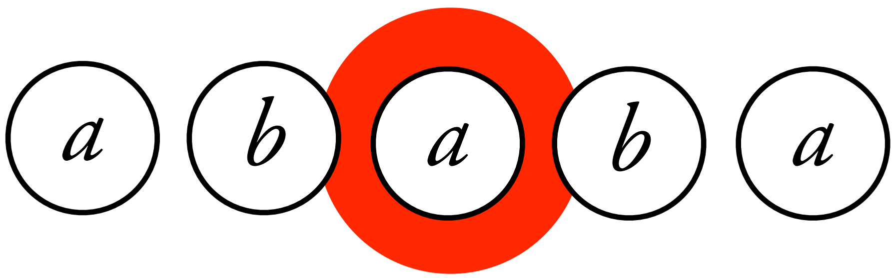

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initial				accepting

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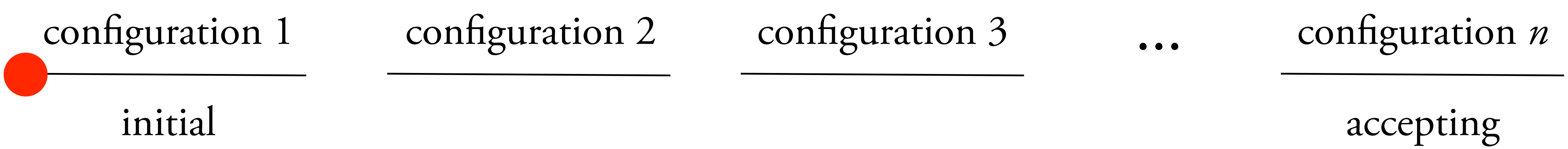

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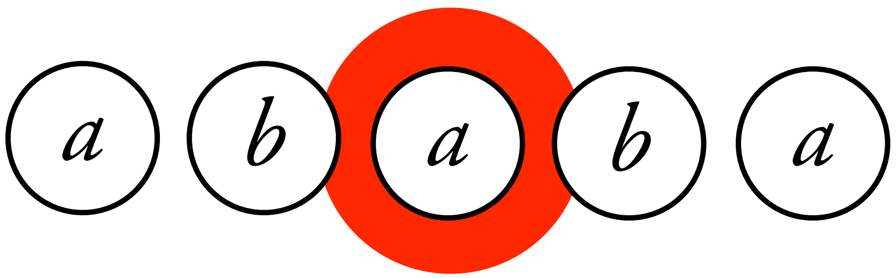

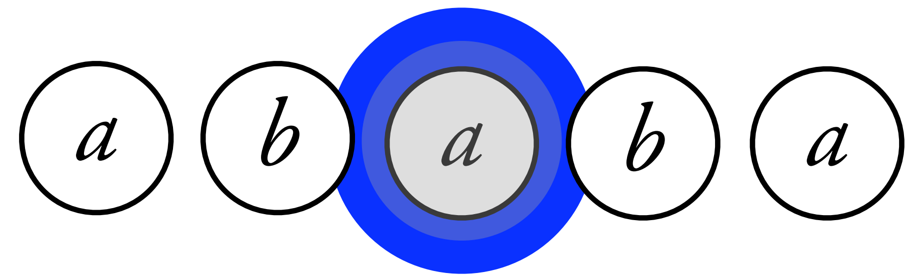

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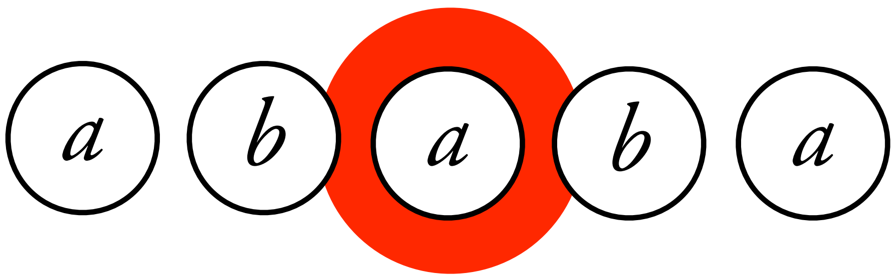

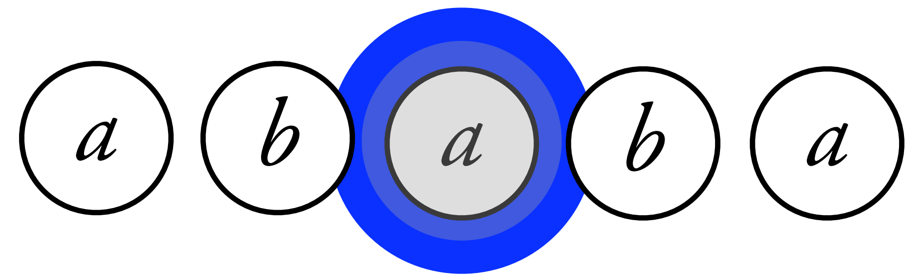

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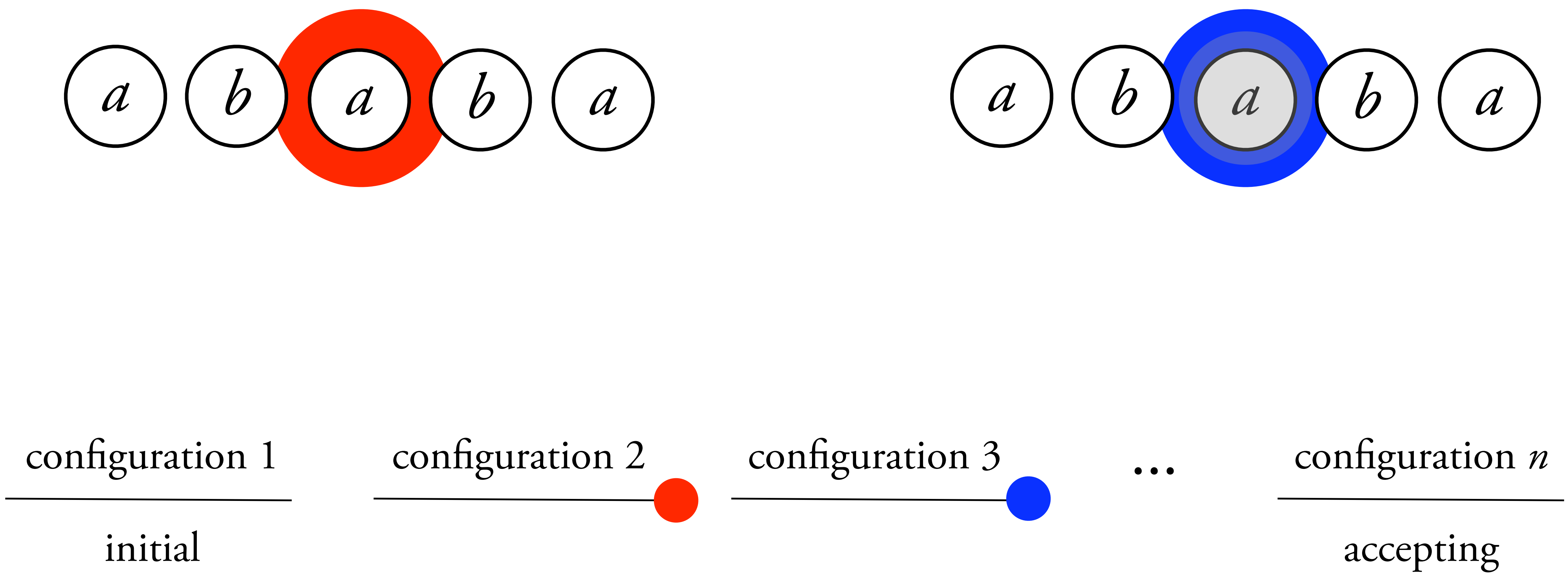

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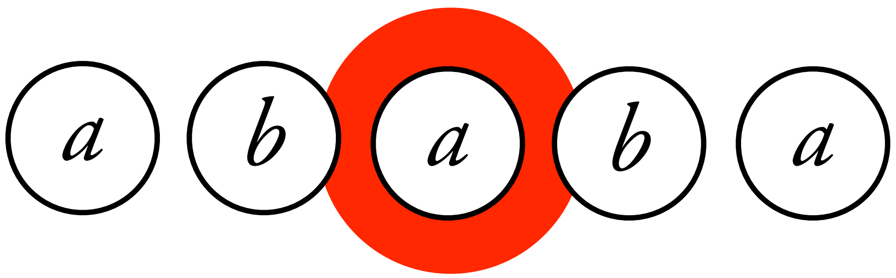

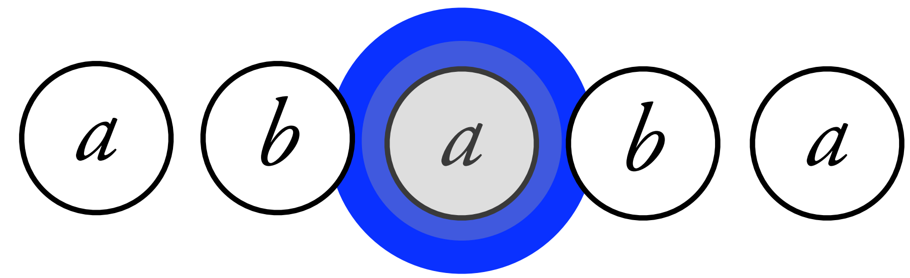

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Good news: with stack discipline, pebble automata have decidable emptiness.

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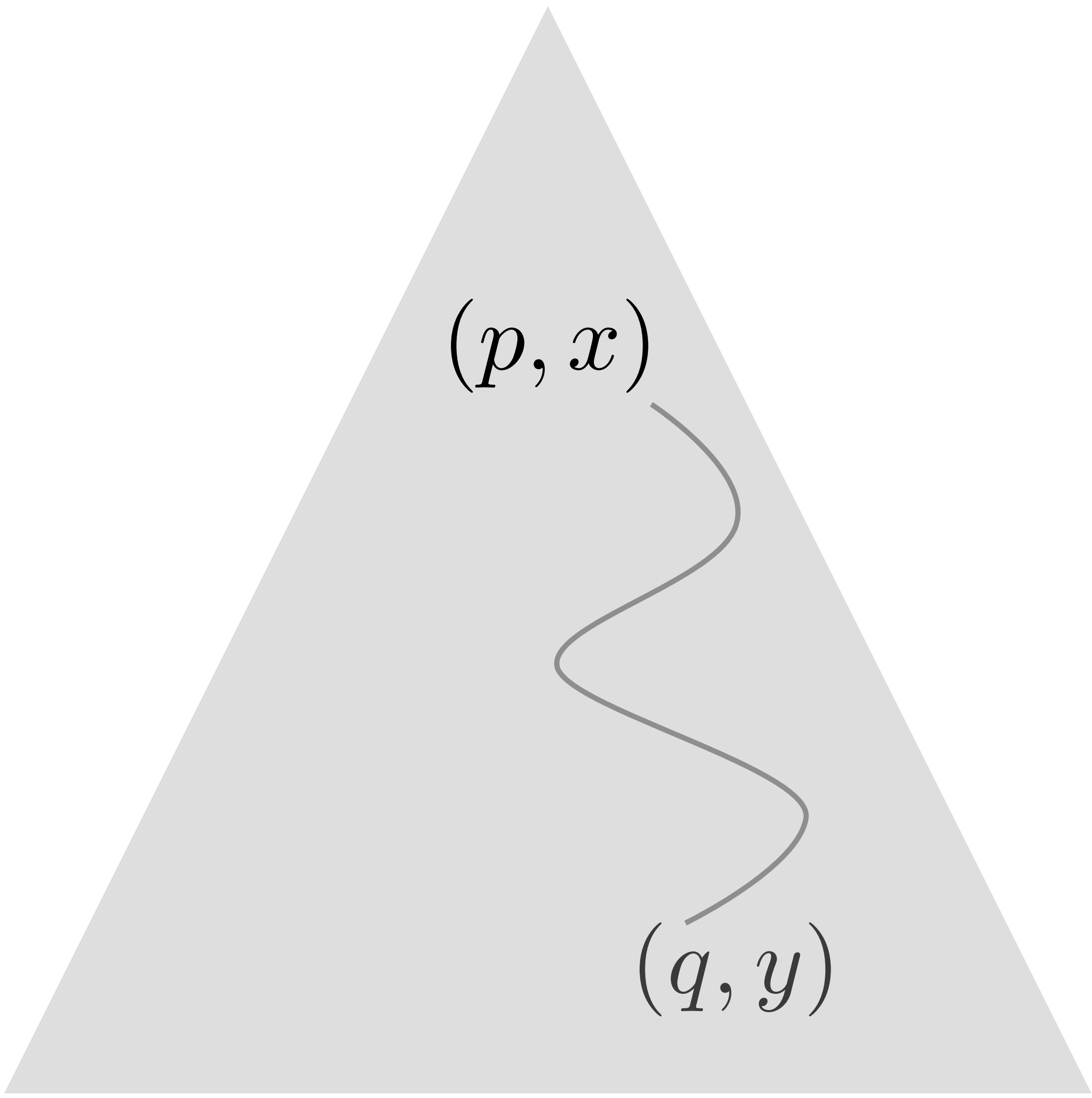
If the pebble automaton has *n* pebbles, the tree automaton may have

$$\overbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}^{n \text{ times}}$$

states. Likewise, emptiness is non-elementary.

$$\varphi_{p,q}(x, y, x_1, \ldots, x_i)$$

There is a run that begins in (p, x) and ends in (q, y).
The pebbles at the beginning and end are in $x_1, \ldots, x_i$.
During the run, pebble x_i is not lifted, but pebbles can be added.



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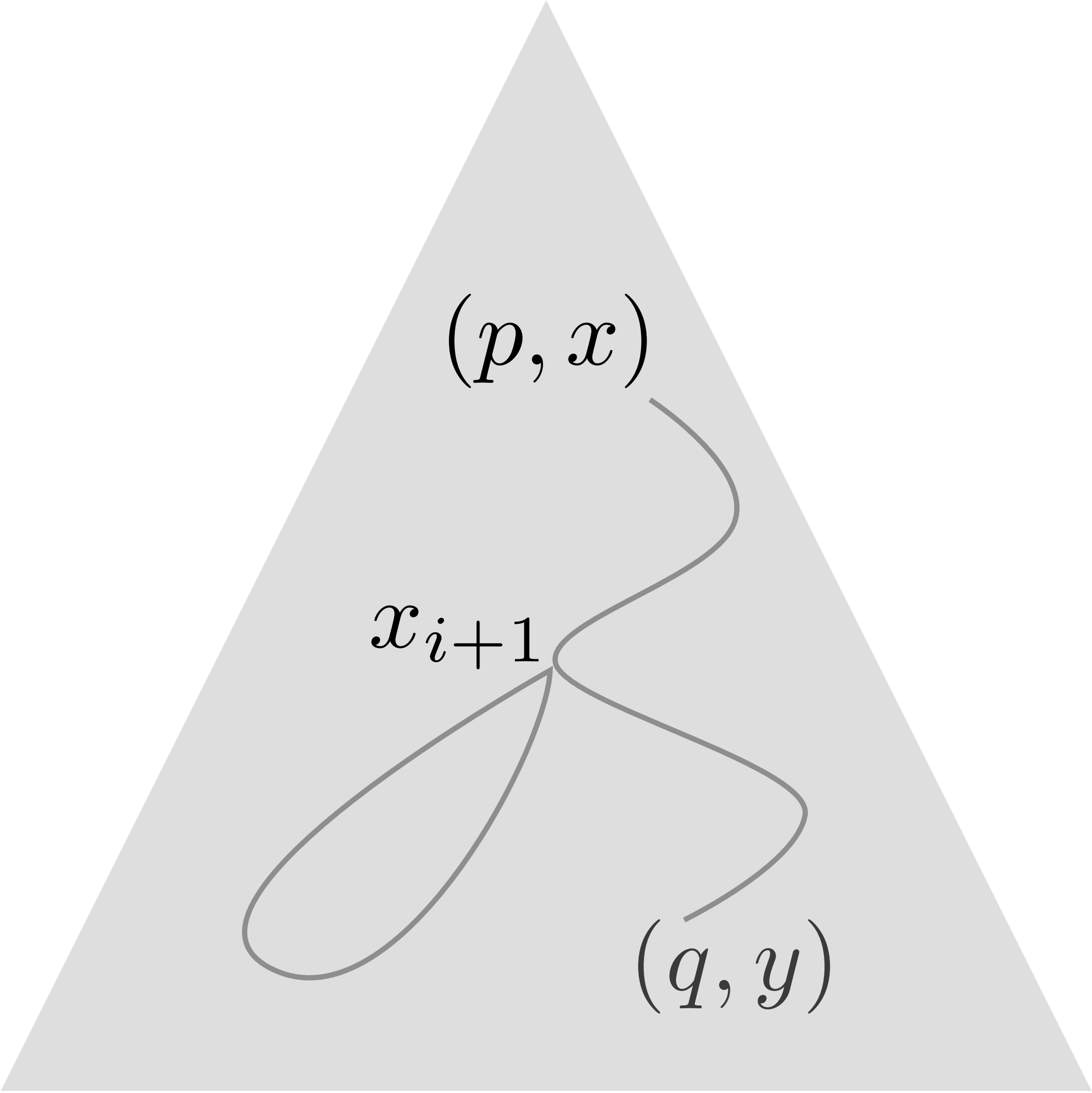
There is a run that begins in (p, x) and ends in (q, y).
The pebbles at the beginning and end are in $x_1, \ldots, x_i$.
During the run, pebble x_i is not lifted, but pebbles can be added.



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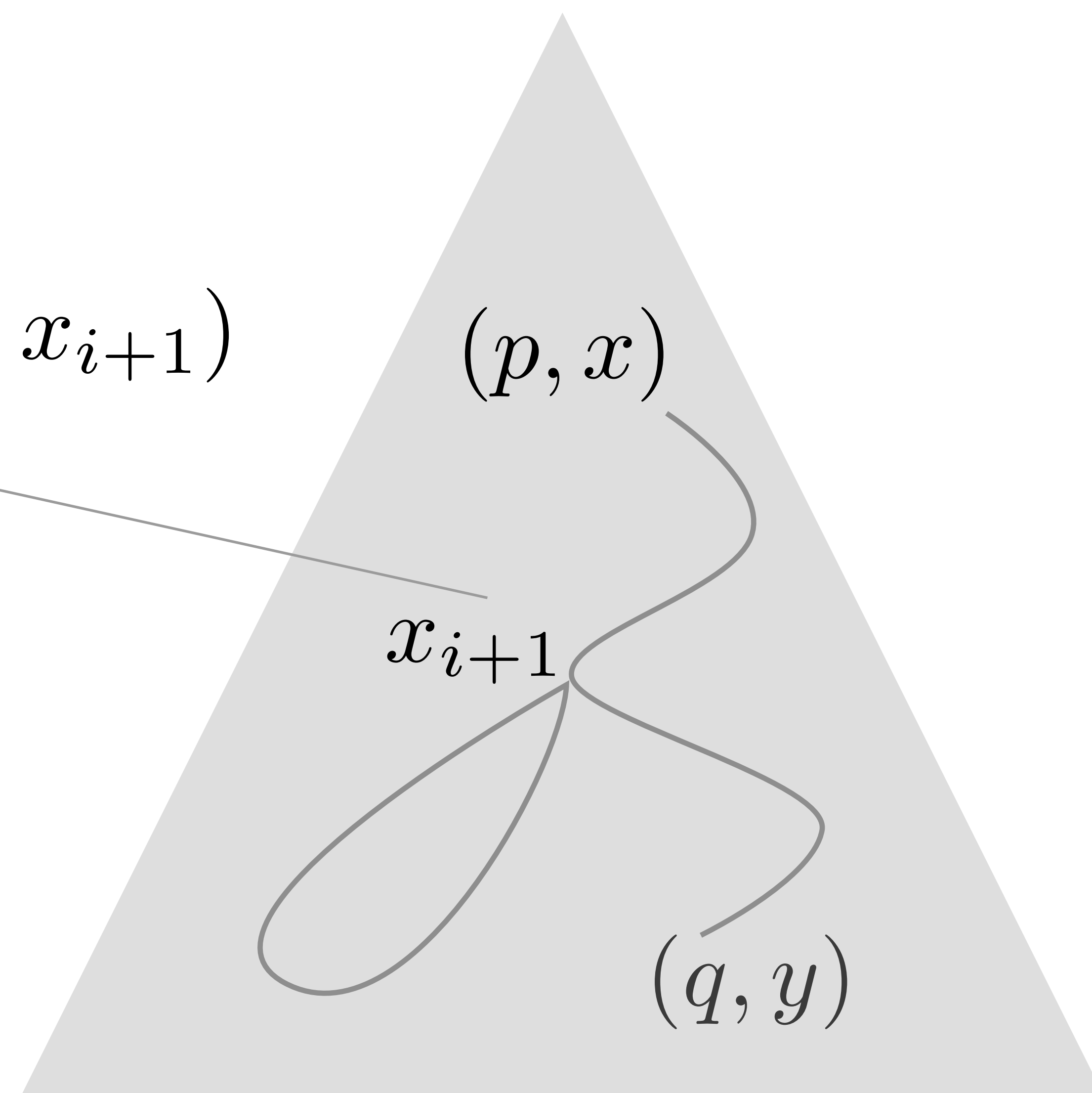
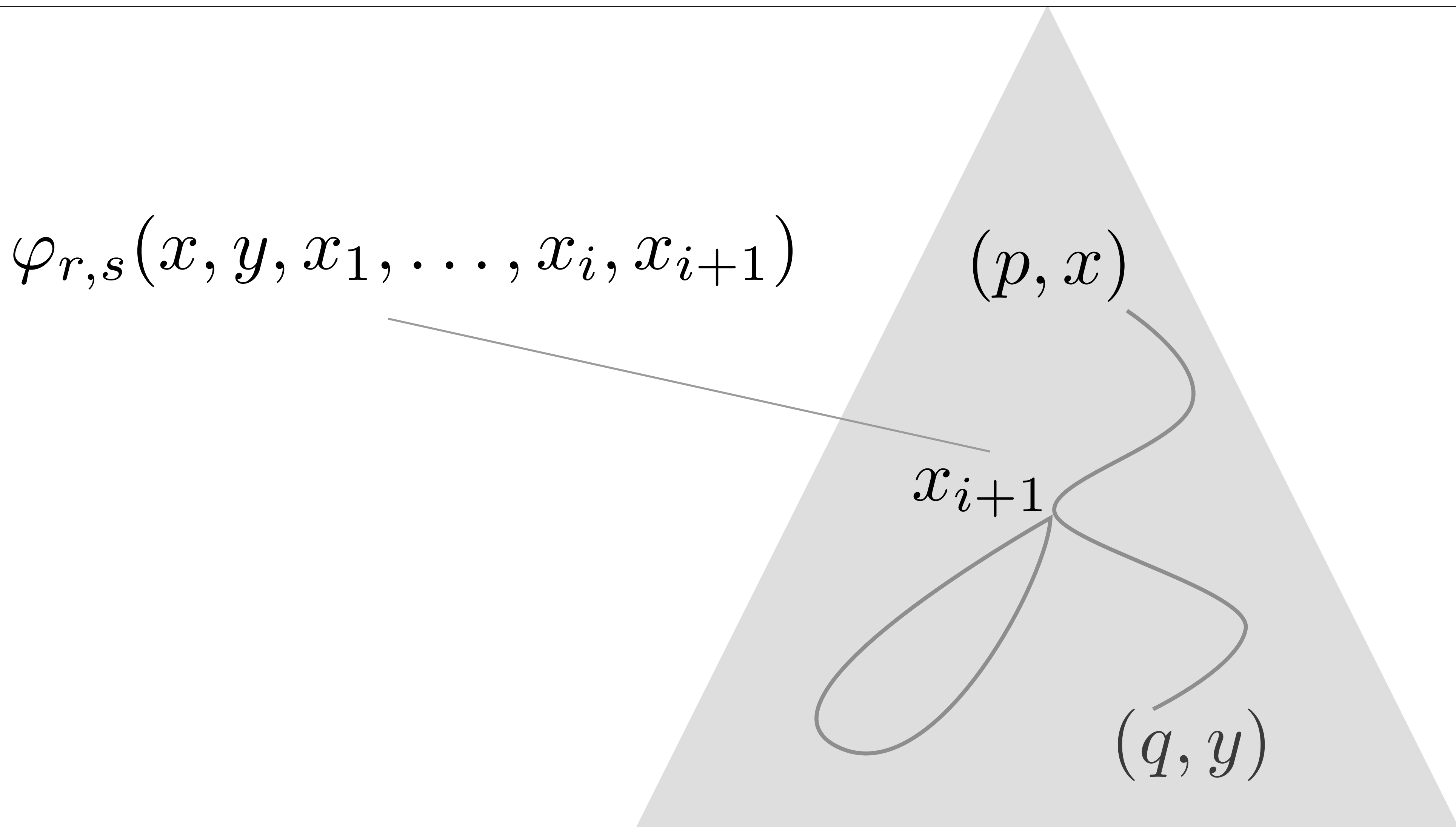
$$\varphi_{r,s}(x, y, x_1, \ldots, x_i, x_{i+1})$$



$$\varphi_{p,q}(x, y, x_1, \ldots, x_i)$$

What logic? Monadic second-order logic is good enough.

Pebble automata = first-order logic with positive transitive closure.

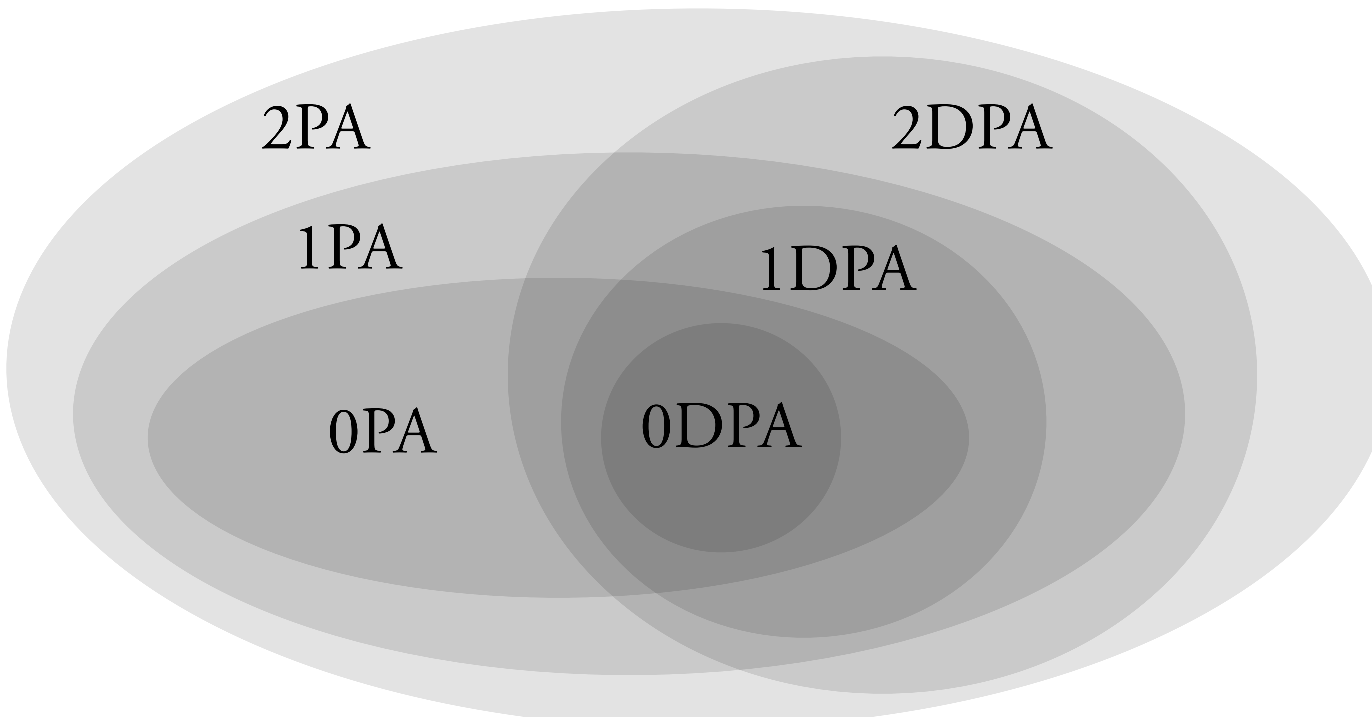


Theorem. [B., Samuelides, Schwentick, Segoufin 06]

-Pebble automata do not recognize all regular languages.

-Deterministic *n* pebbles are weaker than nondeterminstic *n* pebbles.

-*n* pebbles are weaker than *n+1* pebbles, both in det and nondet.



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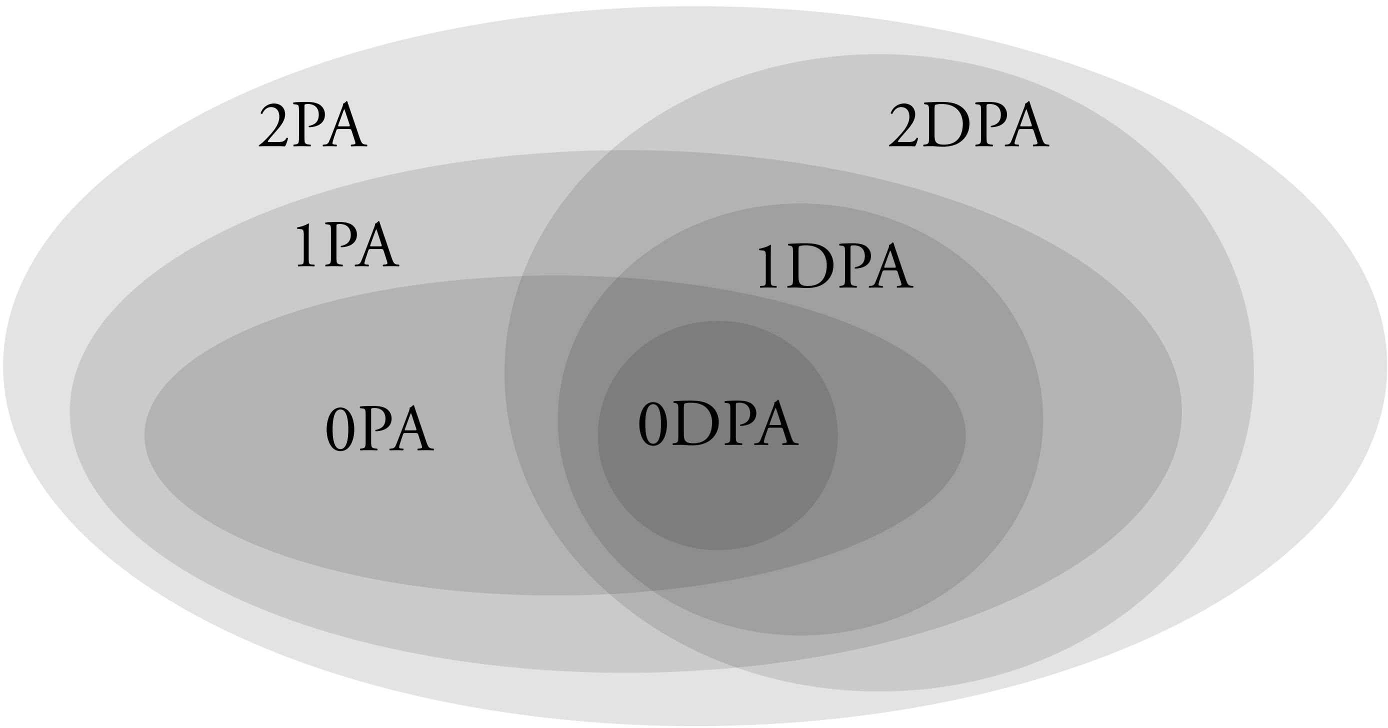

Open question:

$$\bigcup_i i\mathrm{PA} = \bigcup_i i\mathrm{DPA}$$

Known:

$$\forall i \quad 0\mathrm{PA} \not\subseteq i\mathrm{DPA}$$

Pebble automata = first-order logic with positive transitive closure.

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First-order logic.

$$\forall x \forall y \; a(x) \wedge child(x,y) \Rightarrow b(y)$$

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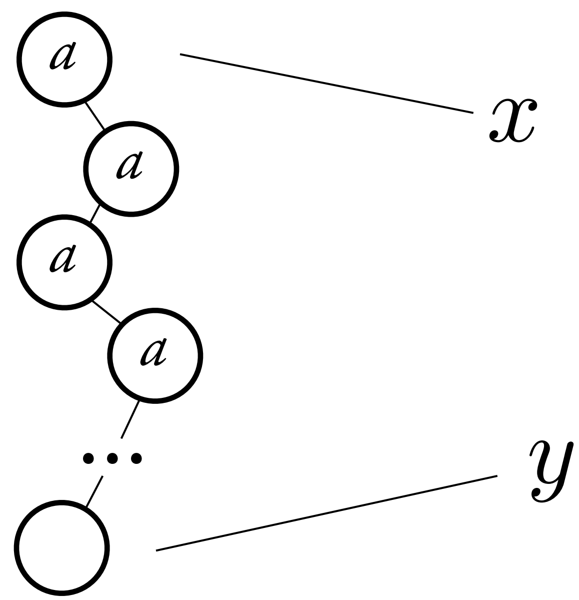
First-order logic with transitive closure.

$$TC\big(child(x,y) \wedge a(x)\big)(x,y)$$



For words,
first-order logic with transitive closure = regular languages.

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$$x = y \;\wedge\; a(x)$$

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What about trees?
first-order logic with positive transitive closure = pebble automata

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What about trees?
first-order logic with positive transitive closure = pebble automata

Theorem. [ten Cate, Segoufin '08]
For trees, not all regular languages are captured by first-order logic with transitive closure.

Conclusion

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What did we miss?

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-caterpillar expressions

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Open questions:
-complementation

Conclusion

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Open questions:
-complementation
-deteremінization of pebble automata

Conclusion

What did we miss?
-caterpillar expressions
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Open questions:
-complementation
-deteremінization of pebble automata
-better understanding