Two-Way Alternating Automata and Finite Models

Tedious proofs of irrelevant results

Mikolaj Bojanczyk

Warsaw University

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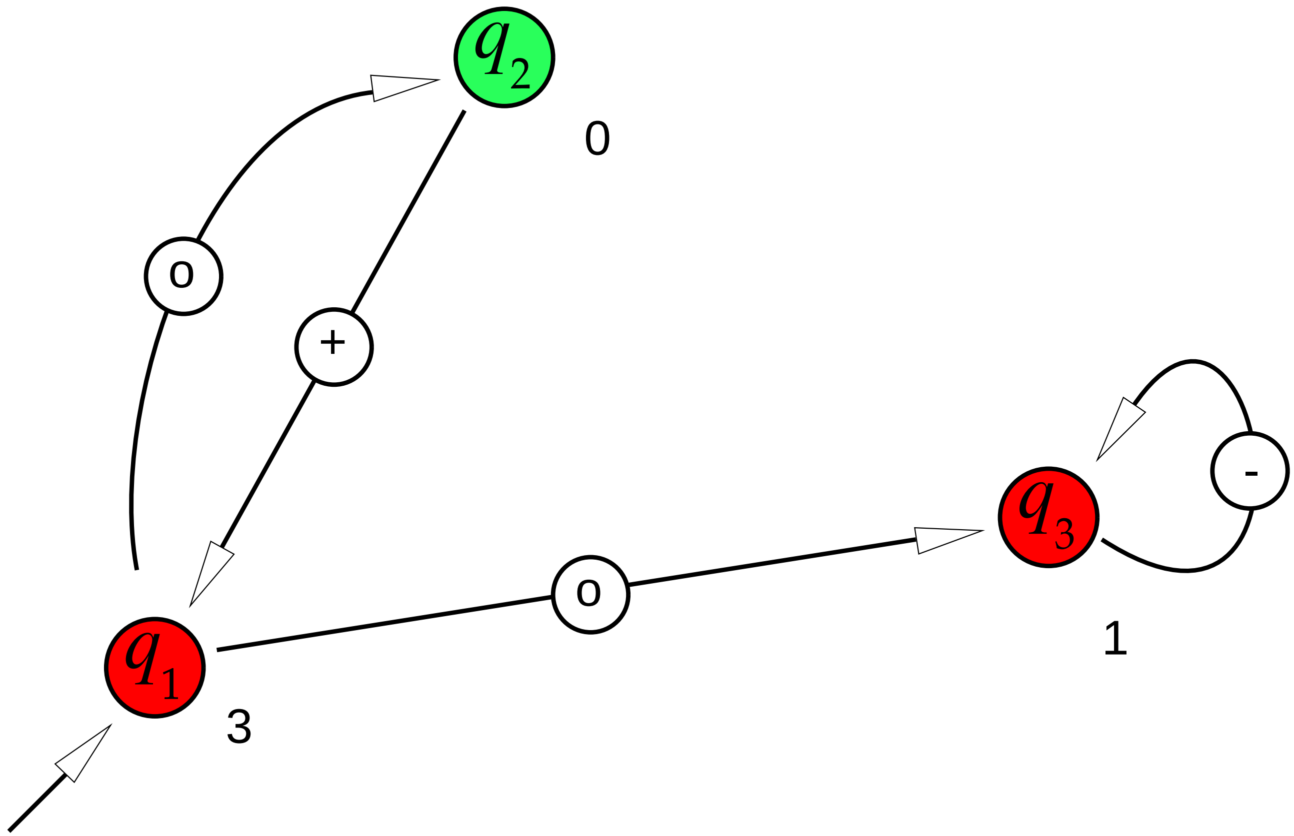
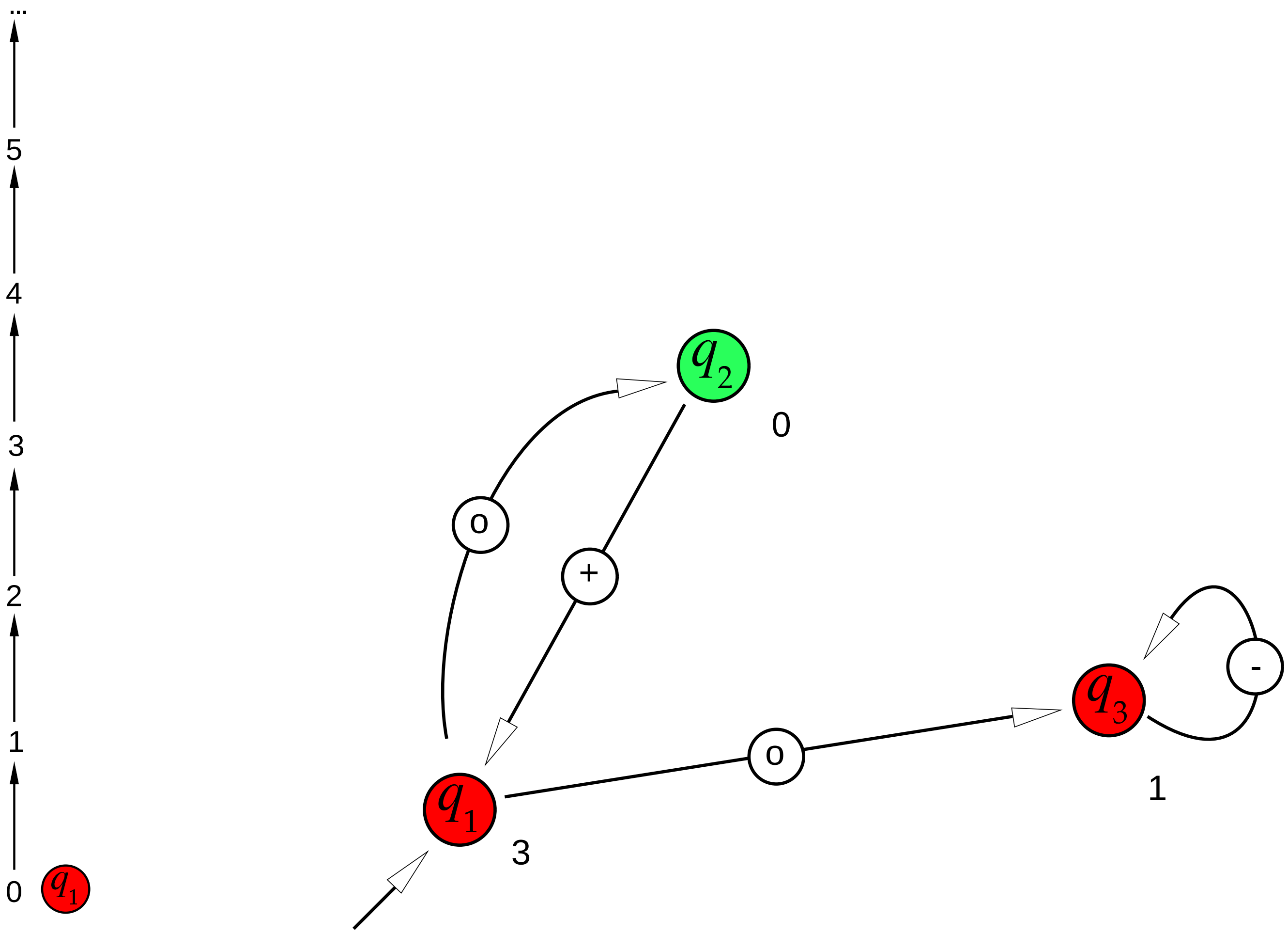
Some example properties recognized by alternating two-way automata:

- There is a vertex labelled by “a” in the graph
- There is an infinite path in the graph
- There is an infinite path in the graph and no vertex of this path is the starting point of some infinite backward path

The automaton $\mathcal{A}$



An example: $\mathbb{N}$



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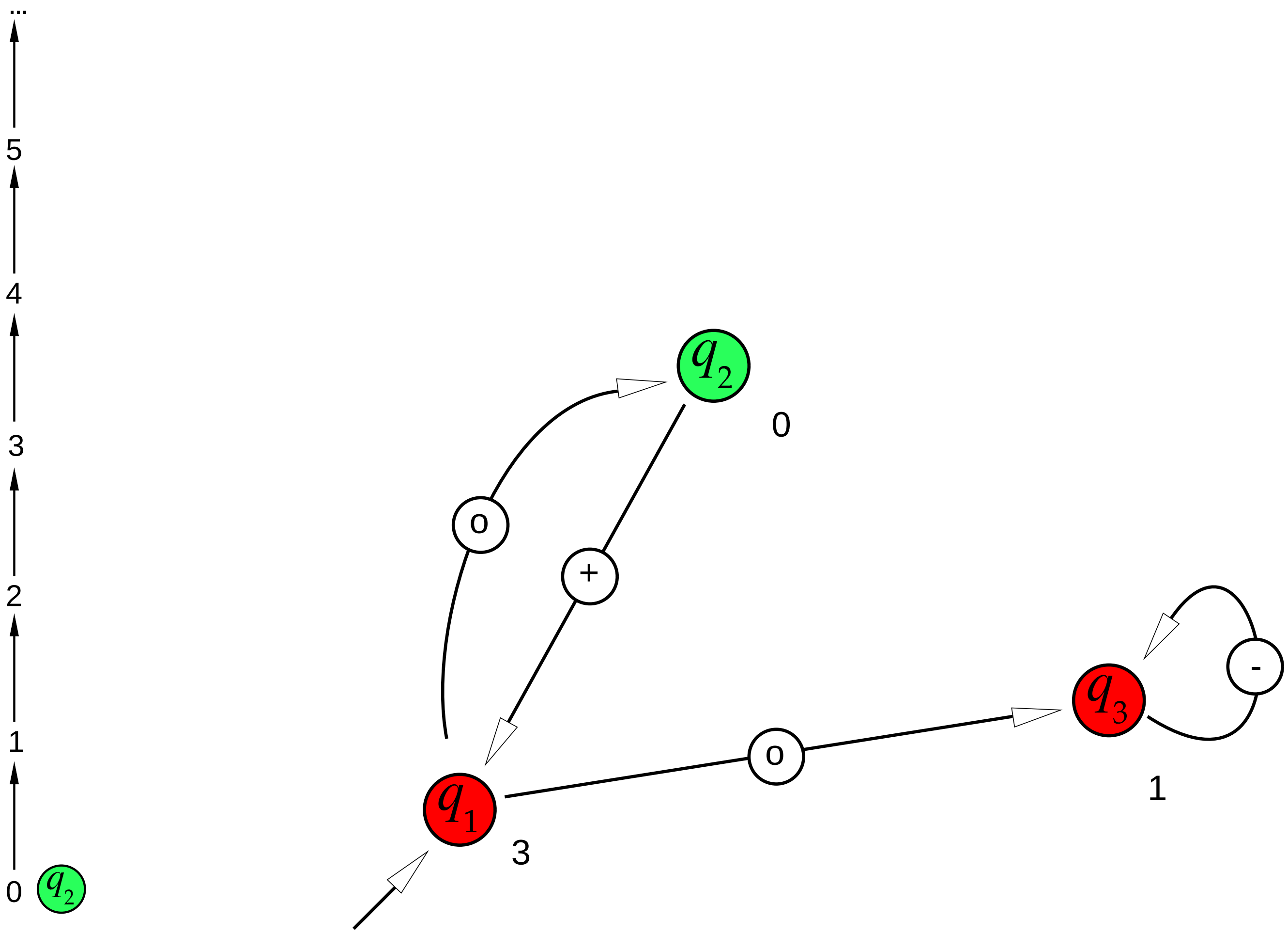

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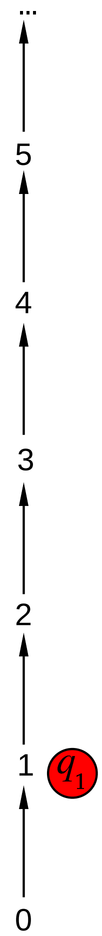

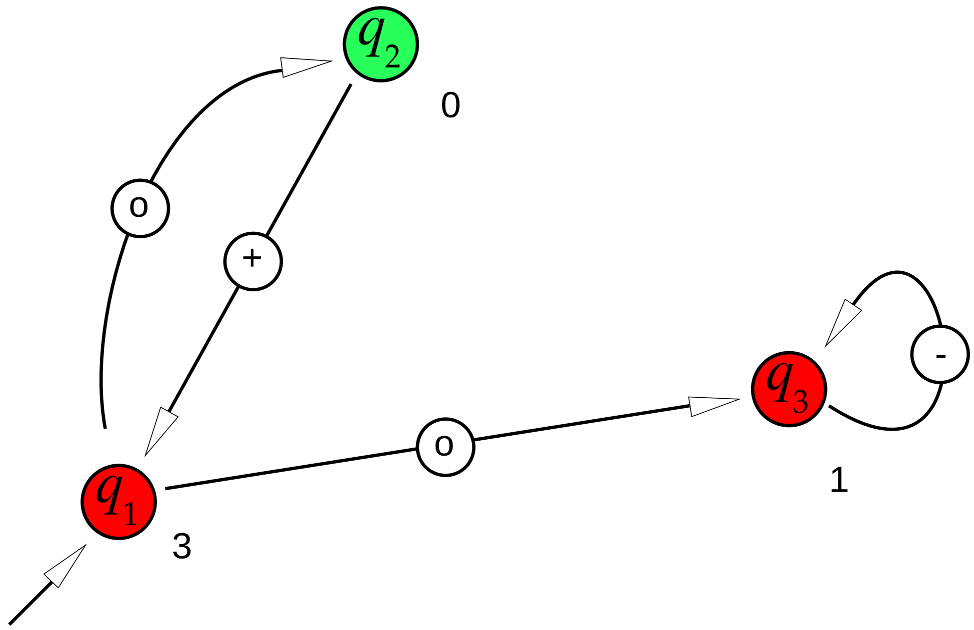

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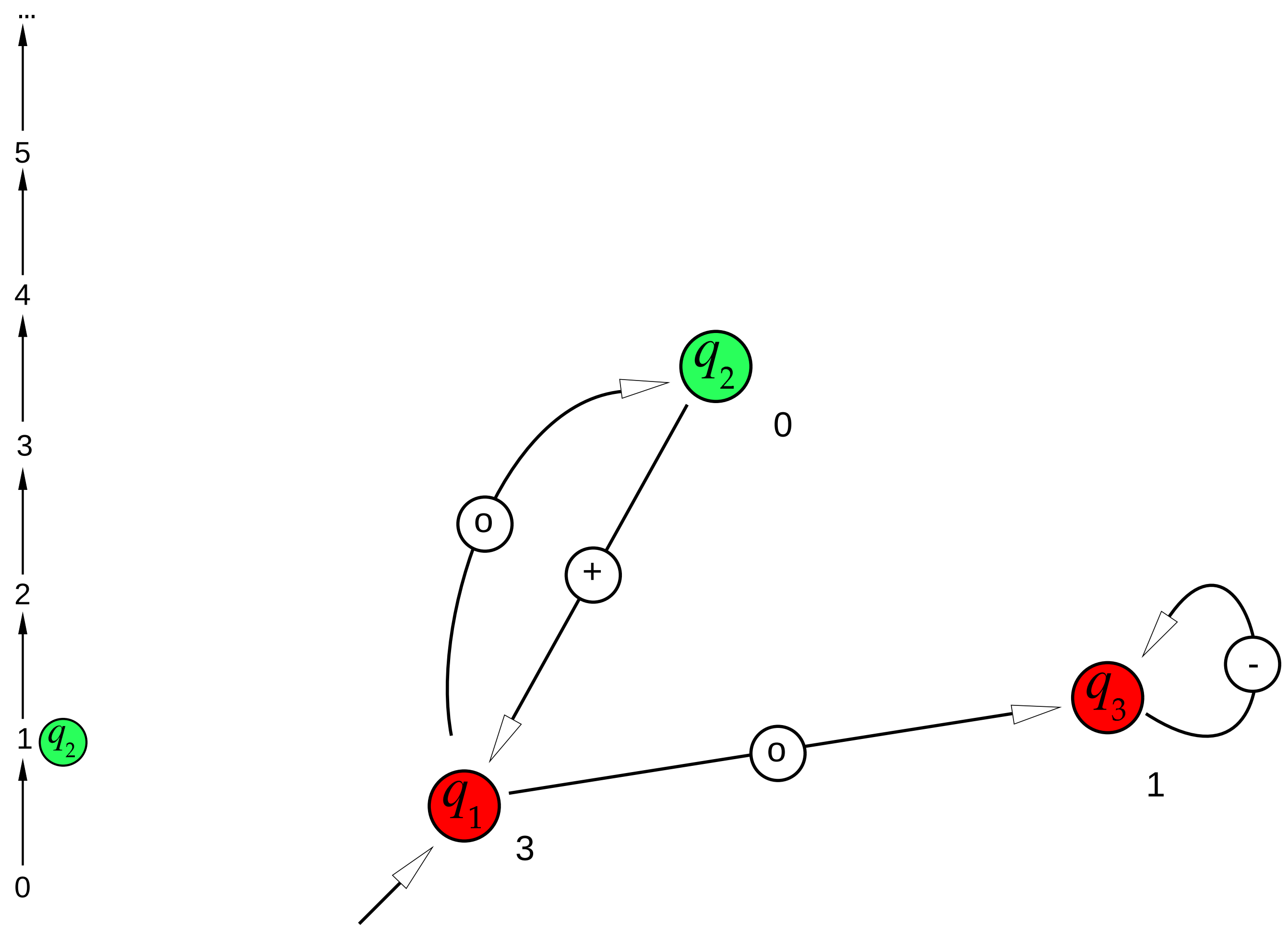

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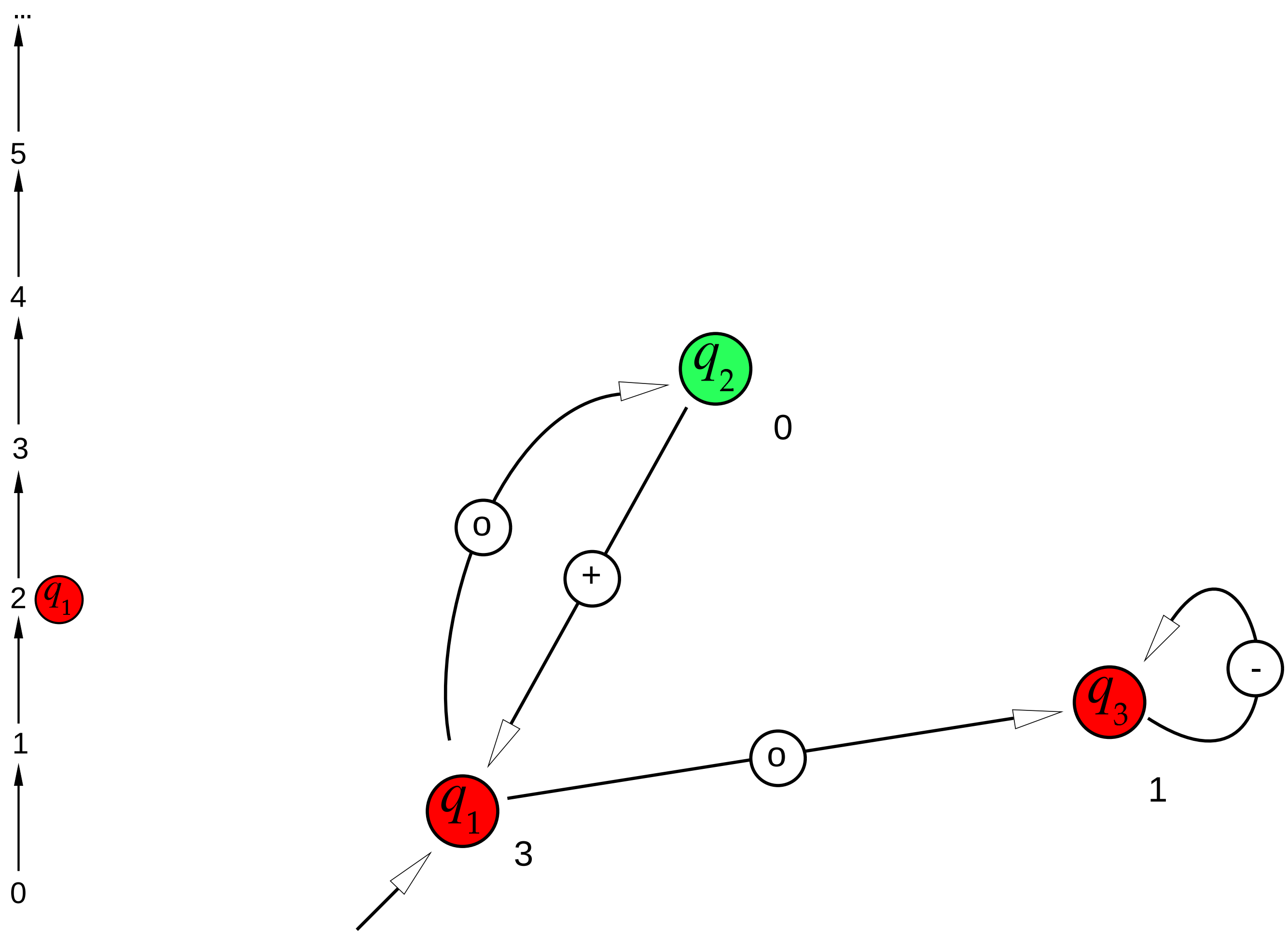

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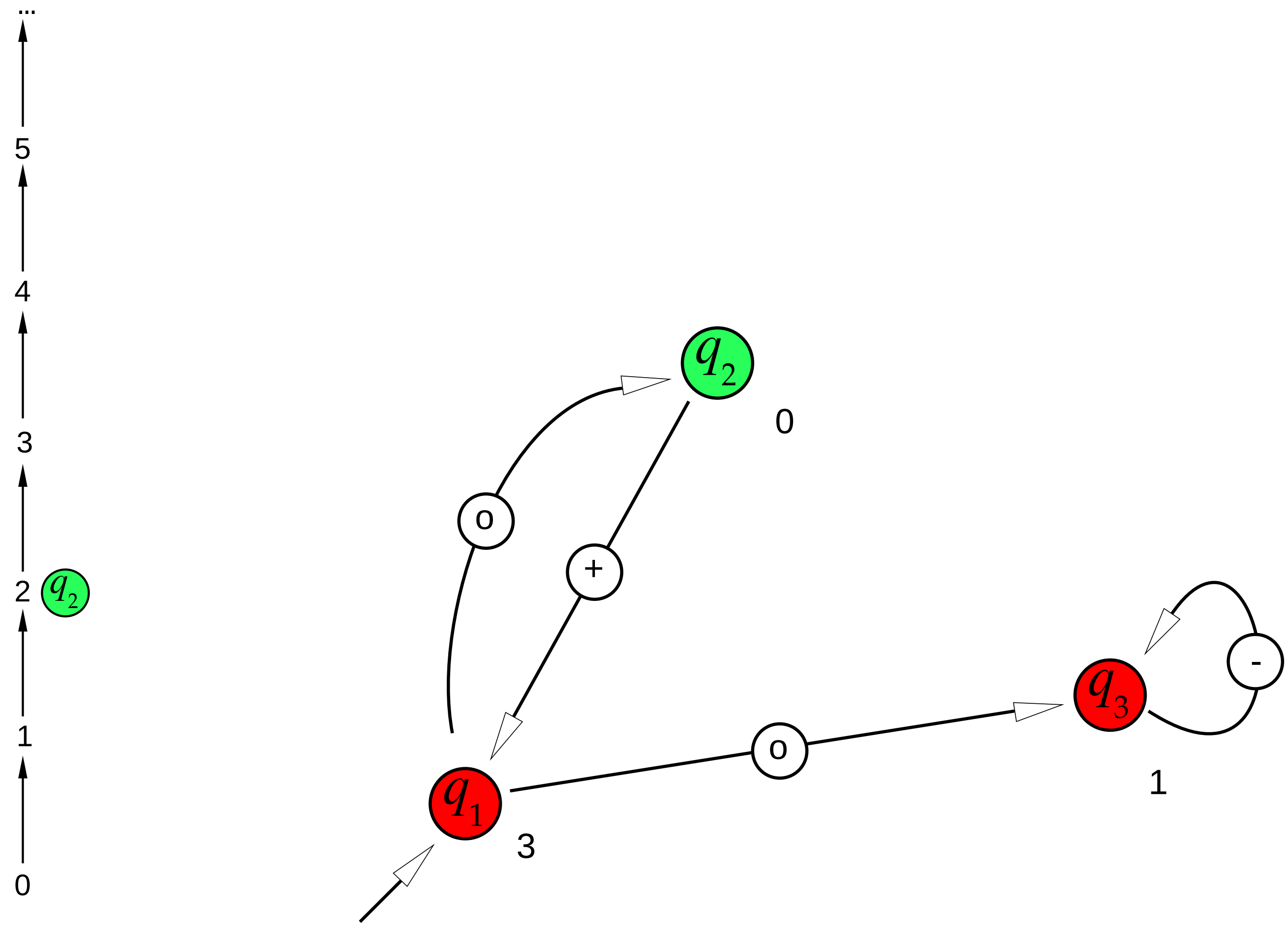

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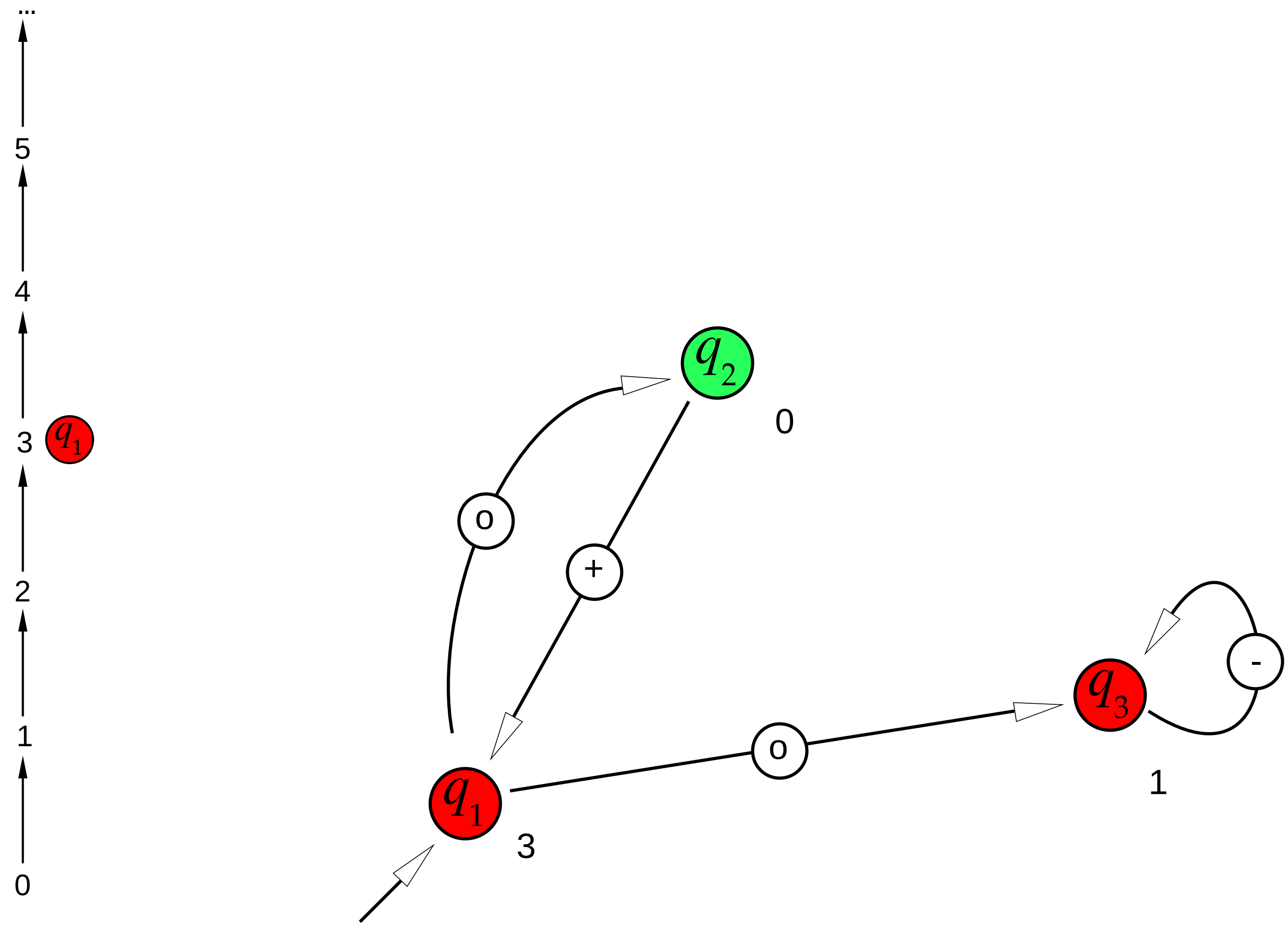

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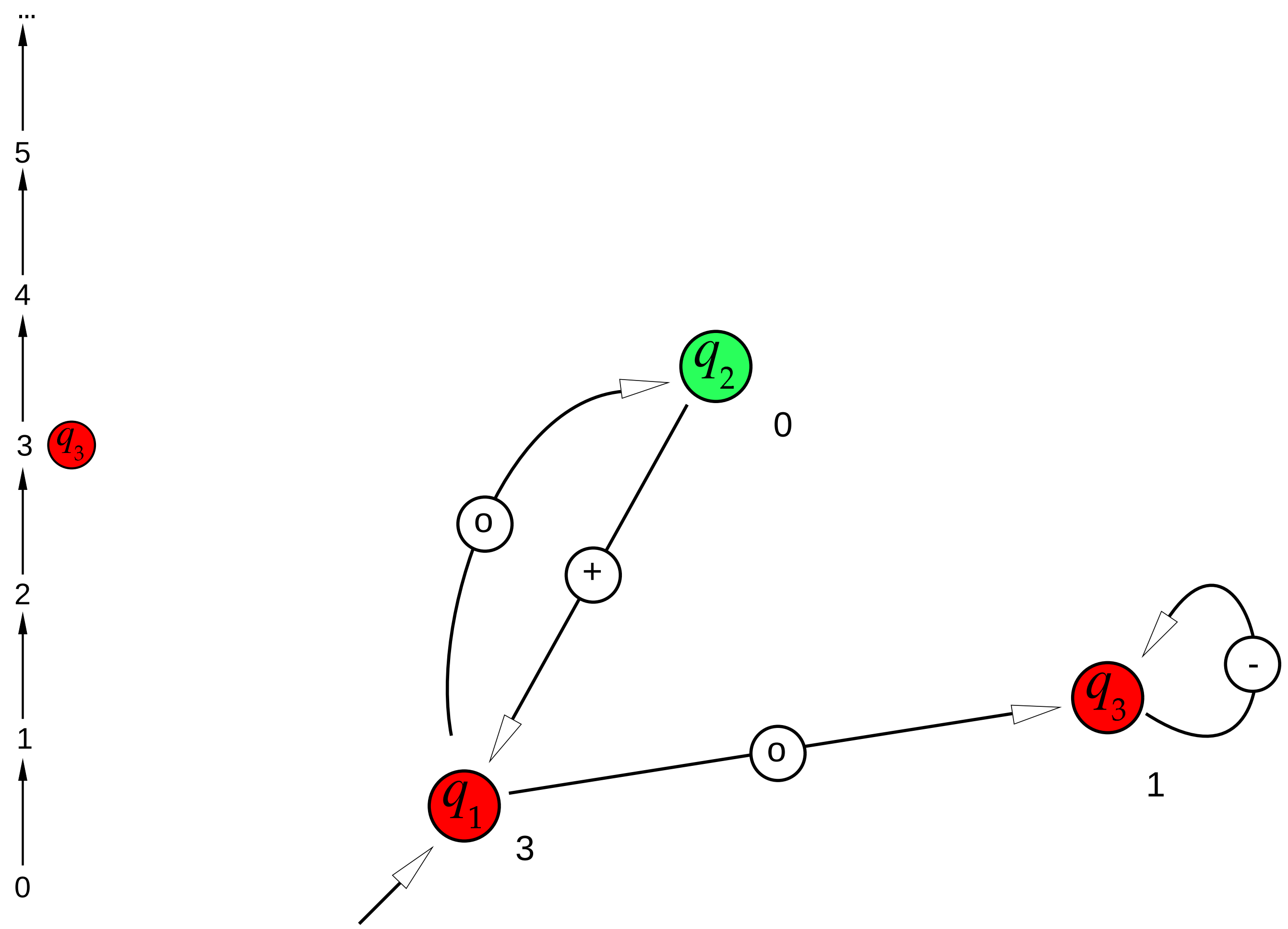

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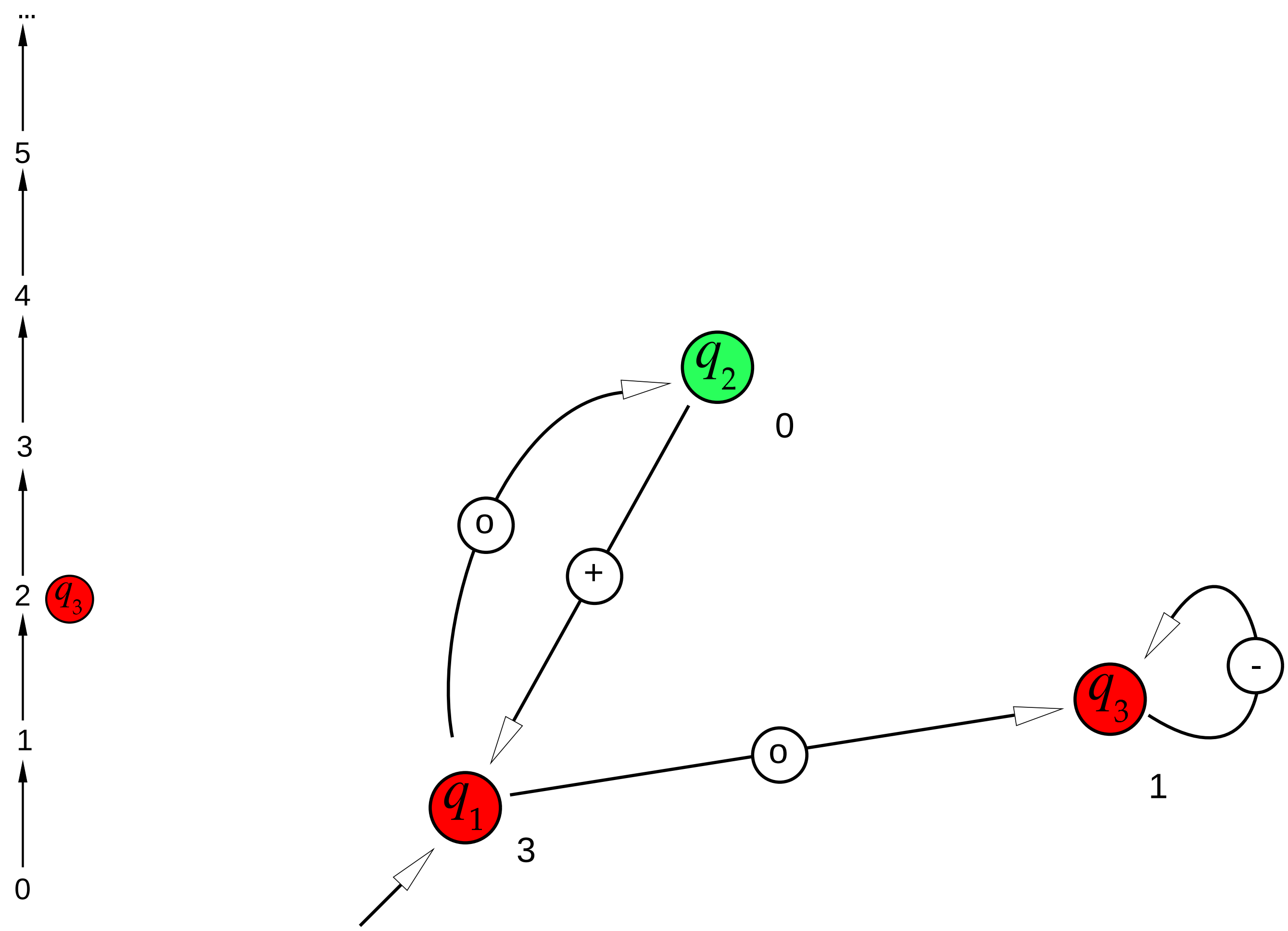

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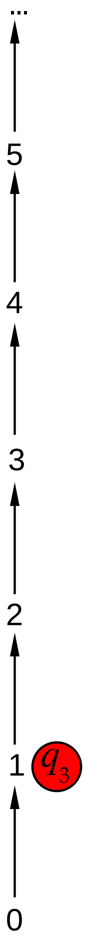

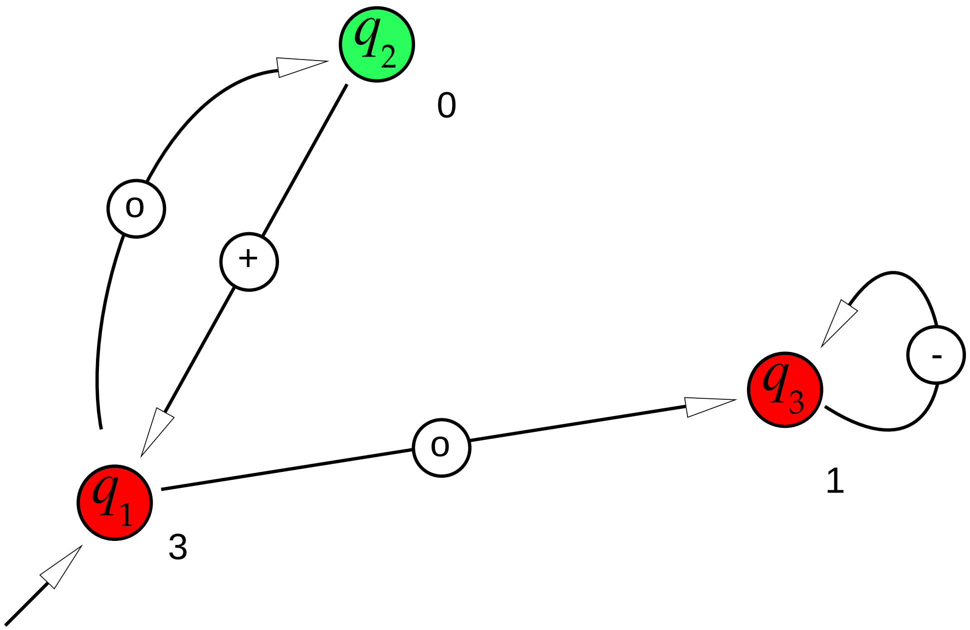

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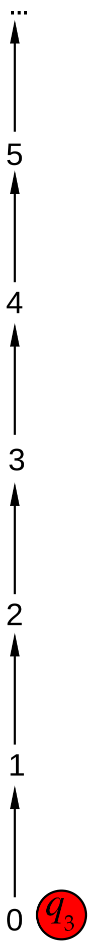

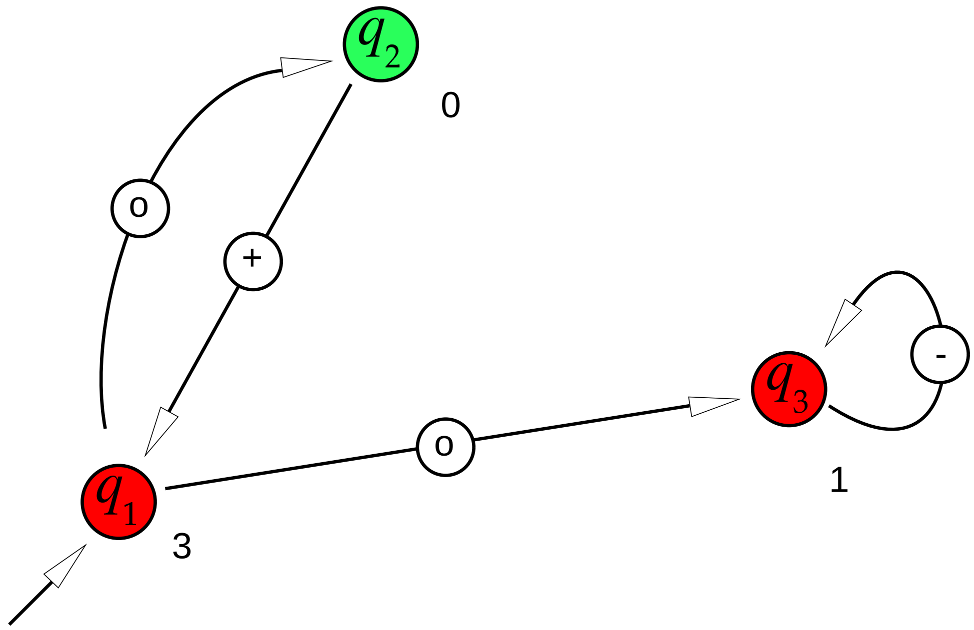

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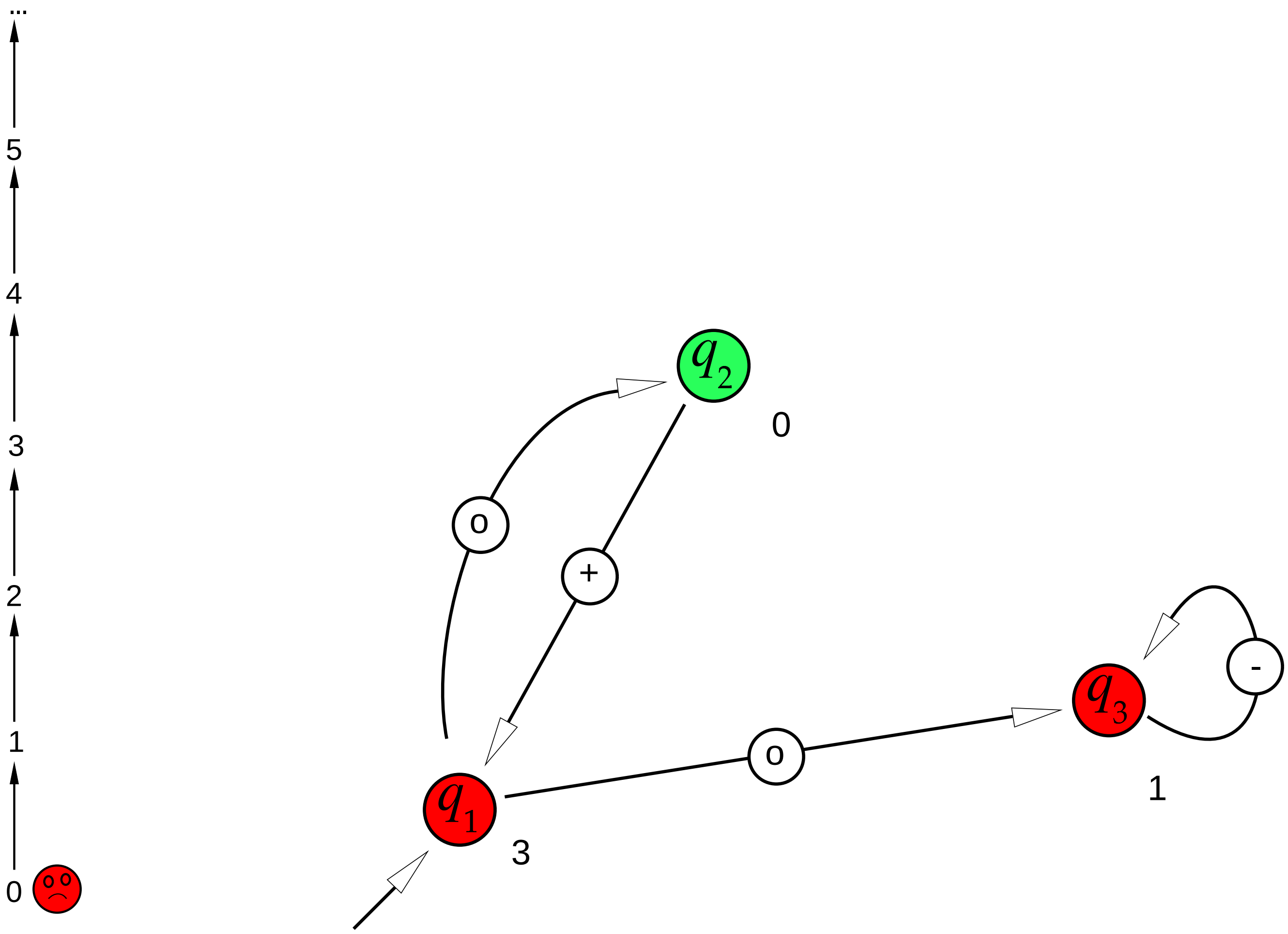

Parity condition

An infinite sequence $a_1, a_2, \ldots$ of elements from a finite set of natural numbers satisfies the *parity condition* if the lowest number occurring infinitely often is even.

$\mathcal{A}$ accepts only infinite graphs

Fact 0 For any graph G, the automaton $\mathcal{A}$ accpets in a vertex v_1 and state q_1 iff

1. No infinite backward path condition. v_1 is not the beginning of a sequence $v_1 v_2 \ldots$ where for all $i \in \{1, 2, \ldots\}$, (v_{i+1}, v_i) is an edge in G.
2. Infinite forward path condition. v_1 is the beginning of a sequence $v_1 v_2 \ldots$ where for all $i \in \{1, 2, \ldots\}$, (v_i, v_{i+1}) is an edge in G and $\mathcal{A}$ accepts in v_i and q_1.

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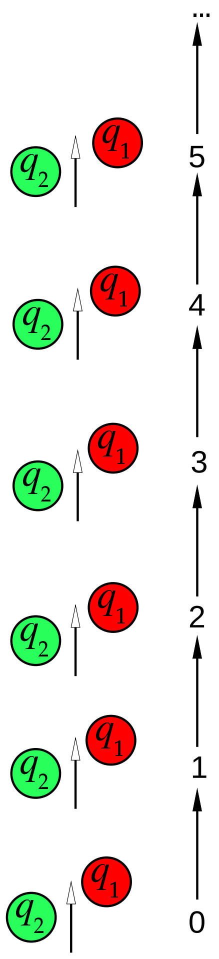
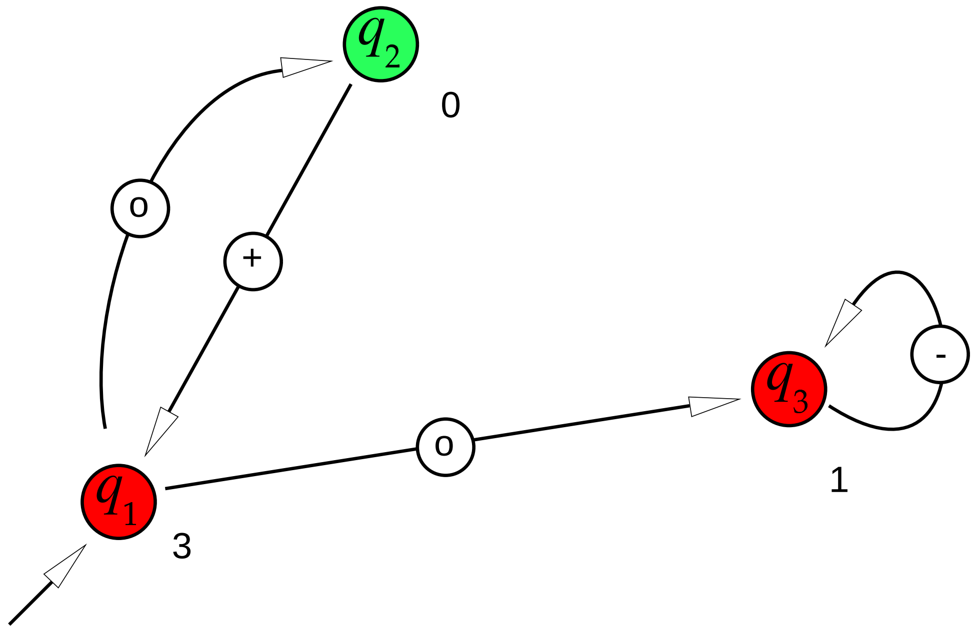
All three are equivalent

A strategy for the good player

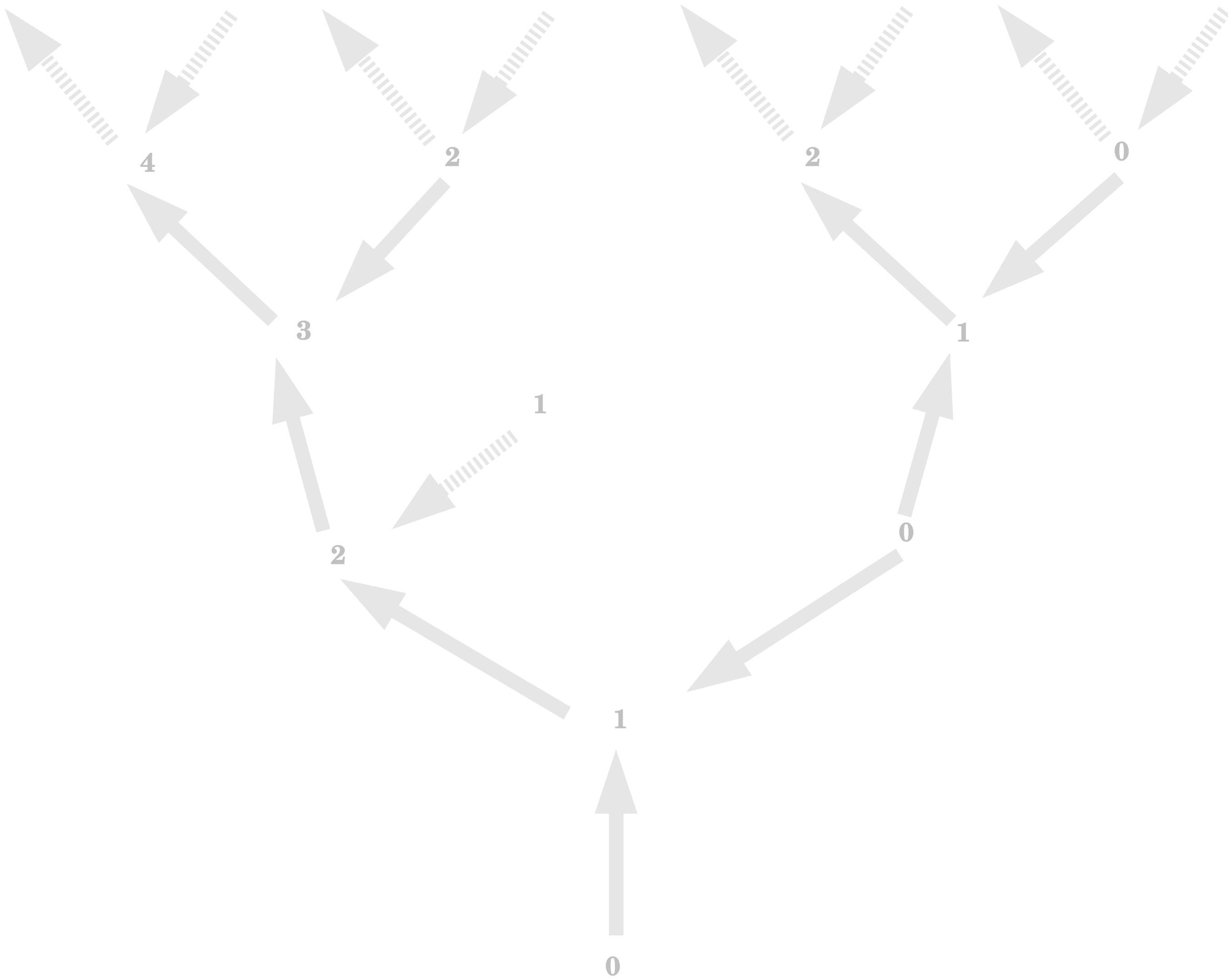




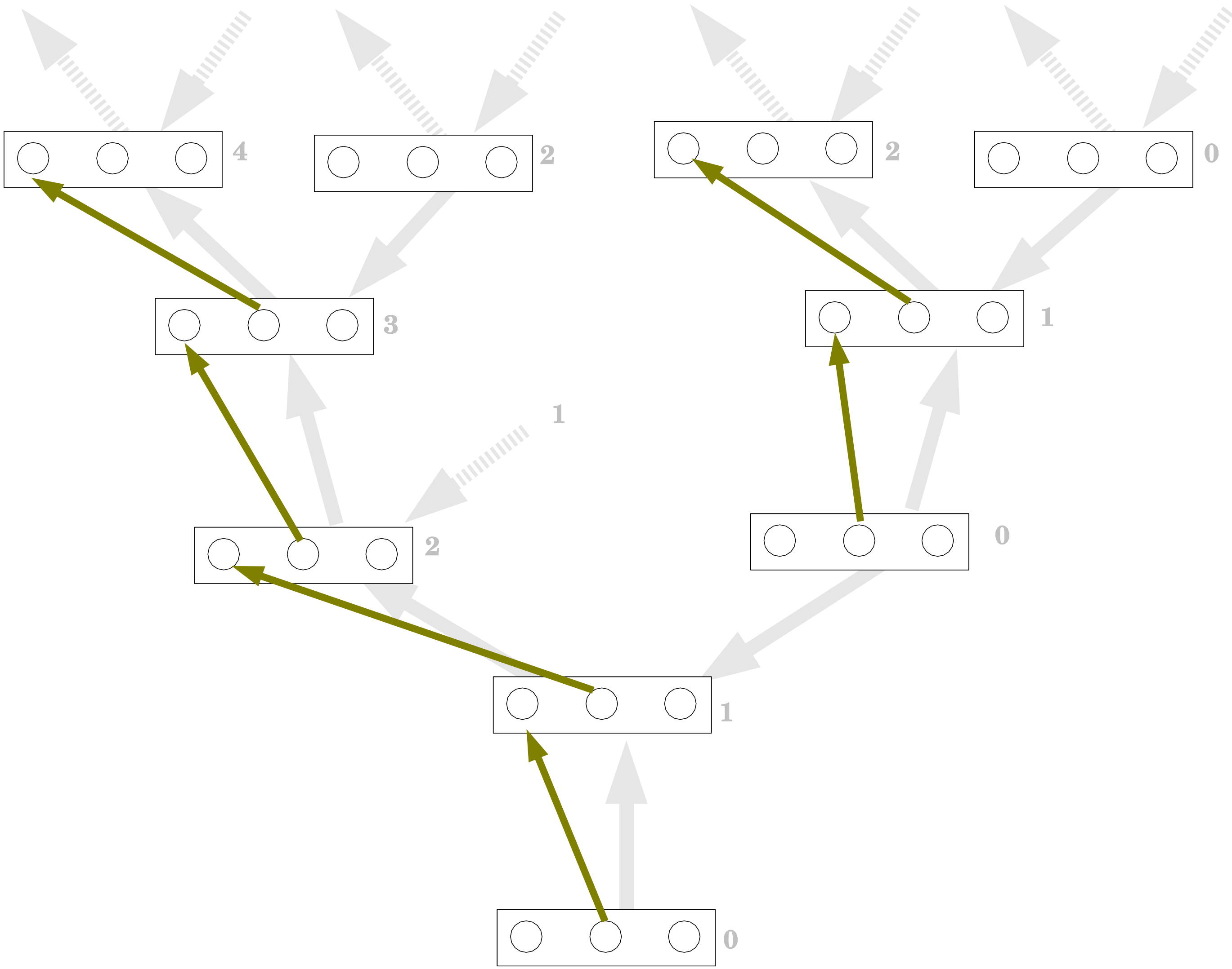
Memoryless strategies

Thm:[Emmerson-Jutla/Mostowski] One of the players has a winning strategy and, moreover, it is a memoryless strategy

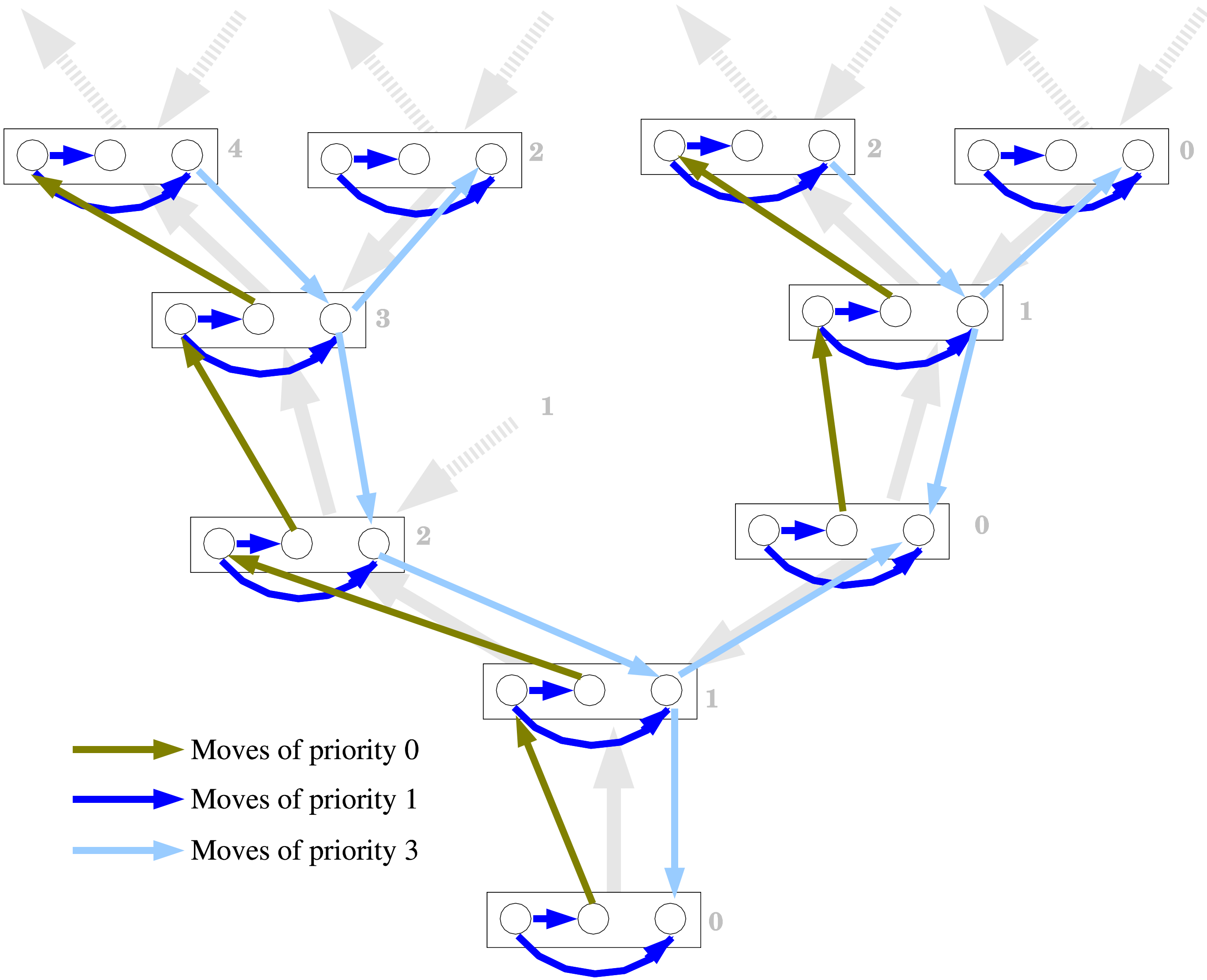
The graph $\mathcal{N}$



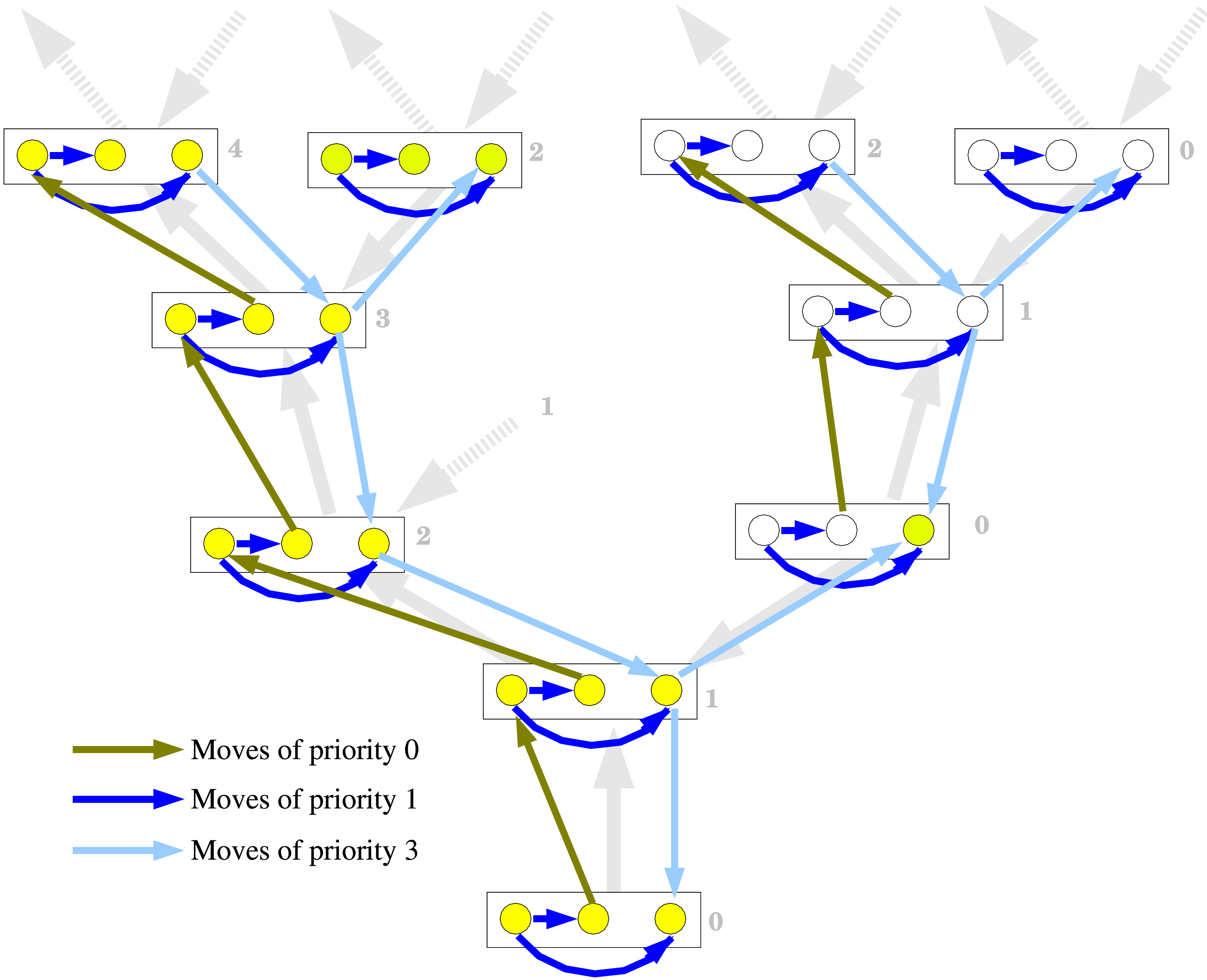
Its unwinding



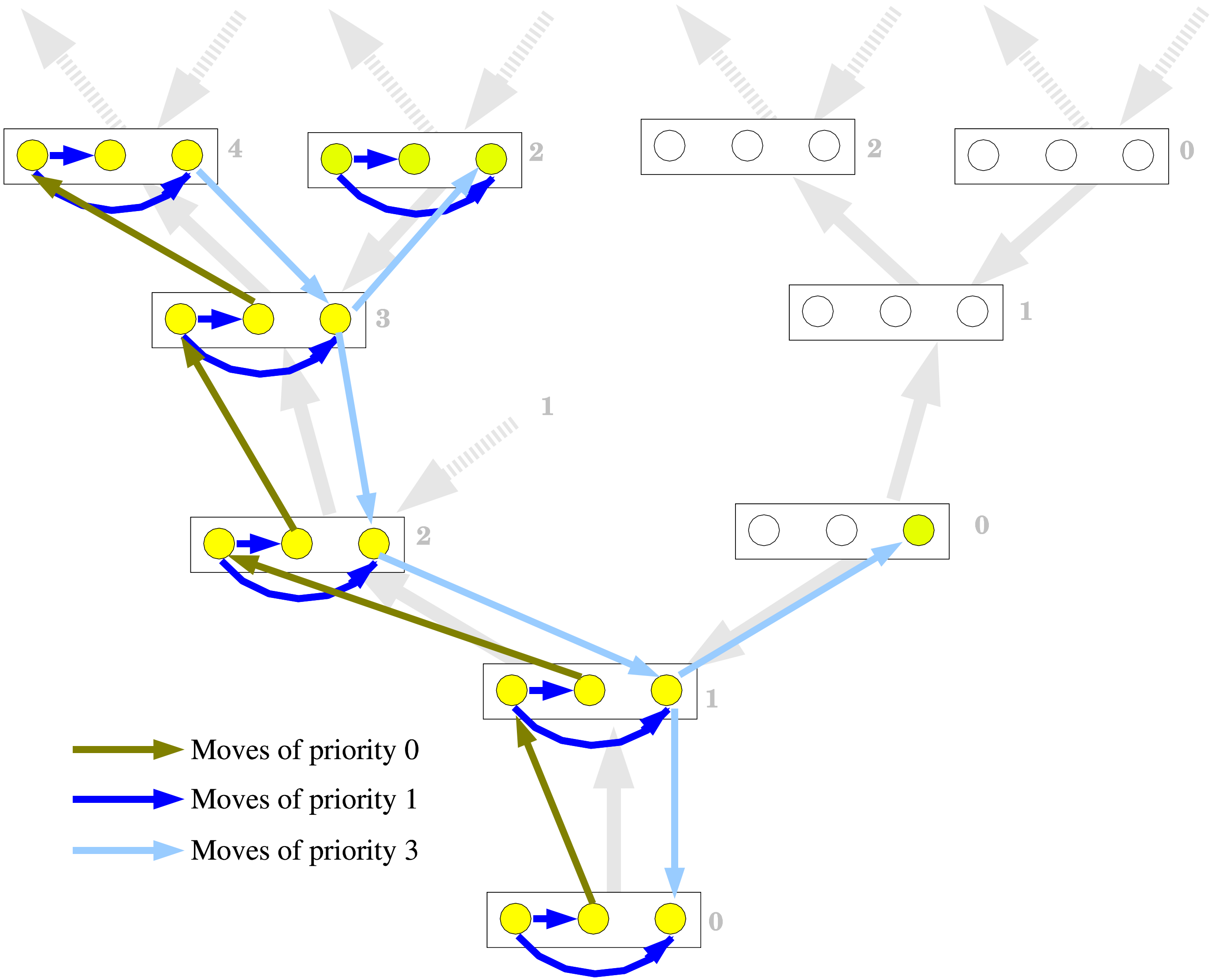
A strategy s for the green player



Locally possible moves under s



Locally possible moves under s with accessible positions



The graph $\mathrm{Gr}(t, s)$

Parity length

- The i-*length* of a sequence of numbers $a = a_1 a_2 \dots a_n$ is the length of the longest sequence of i-s in the sequence a' resulting from a by taking out all numbers greater than i.
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- The *parity length* of a sequence of numbers maximal i-length of the sequence for odd i.
- The *parity length* of a path labelled by priorities is the parity length of the corresponding sequence of priorities.

Properties of $\mathrm{Gr}(t, s)$

- s is a winning strategy for the green player iff no infnite path in $\mathrm{Gr}(t, s)$ violates the parity condition (the parity length of paths in $\mathrm{Gr}(t, s)$ is finite).

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- t can be wound back into a finite graph iff for some s, the parity length of paths in $\mathrm{Gr}(t, s)$ is bounded, i. e. there is some $M \in \mathcal{N}$ such that all paths in $\mathrm{Gr}(t, s)$ have parity length not greater than M.

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- The finite graph question thus becomes: is there some tree t and strategy s such that the parity length of paths in $\mathrm{Gr}(t, s)$ is bounded.

Regular trees and languages

- A tree language is *regular* iff it is recognized by some finite automaton.
- A tree is *regular* iff it contains a only finitely many non-isomorphic subtrees.

Thm:[Rabin]Every regular tree language contains some regular tree.

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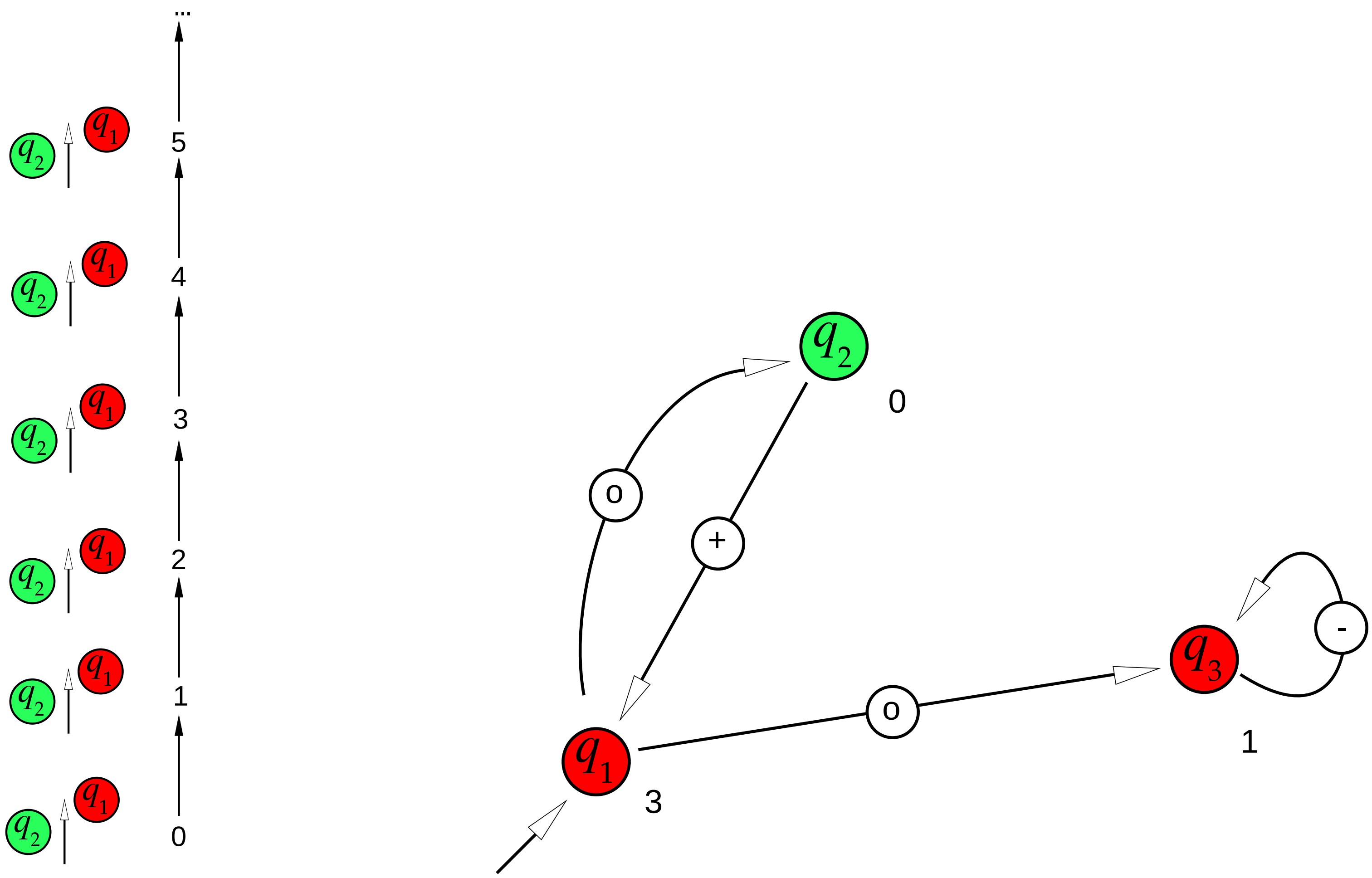
Thm: LF is nonempty iff LB is nonempty.

Thm: The finite graph problem is decidable

Signature



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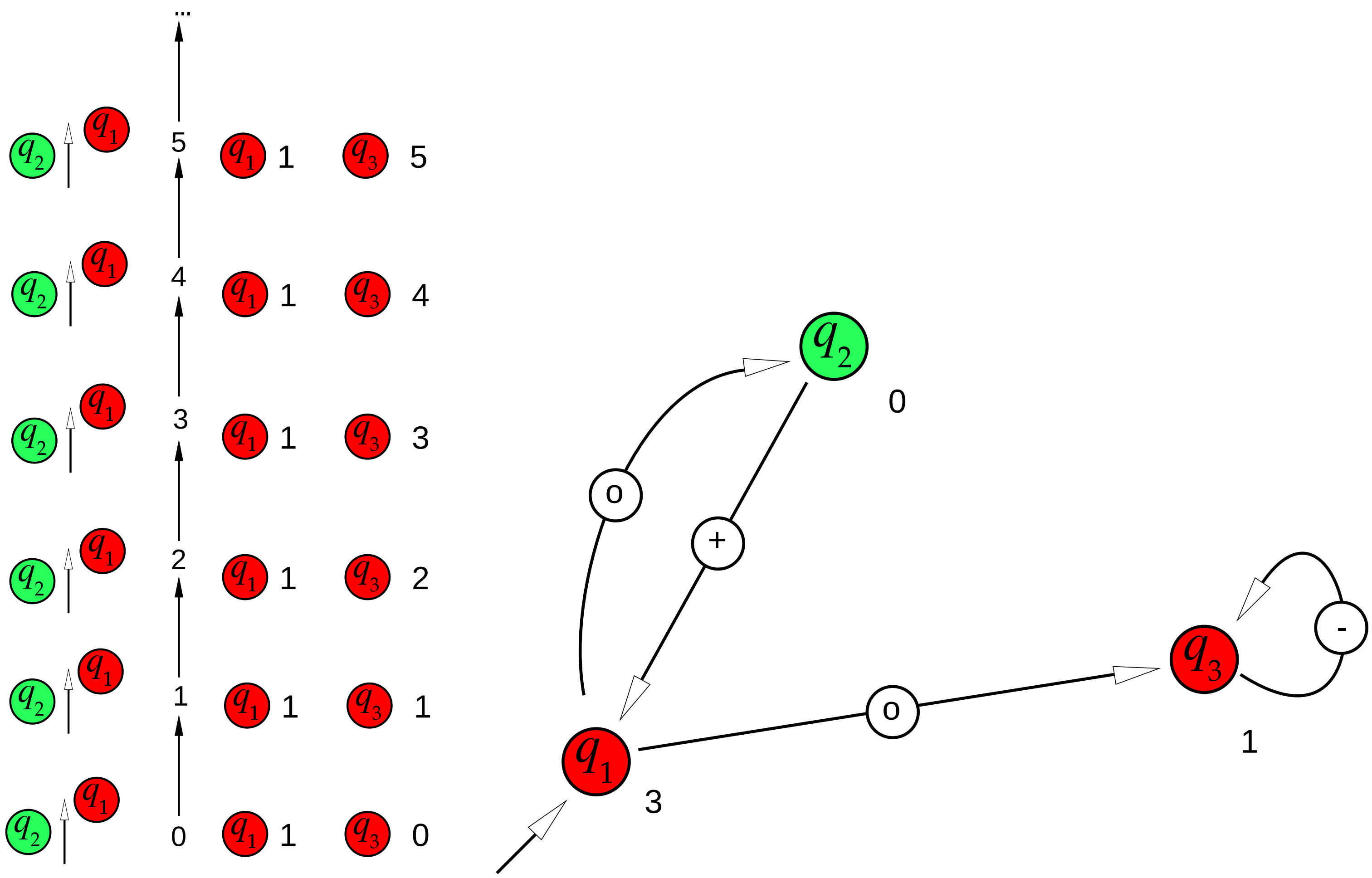
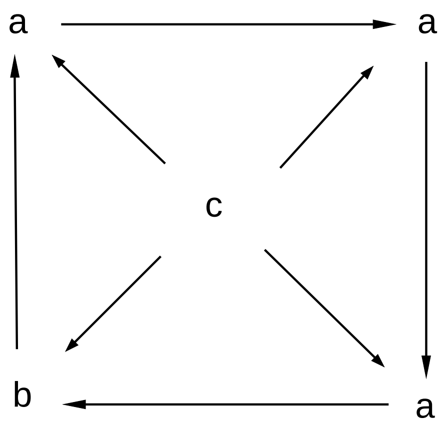
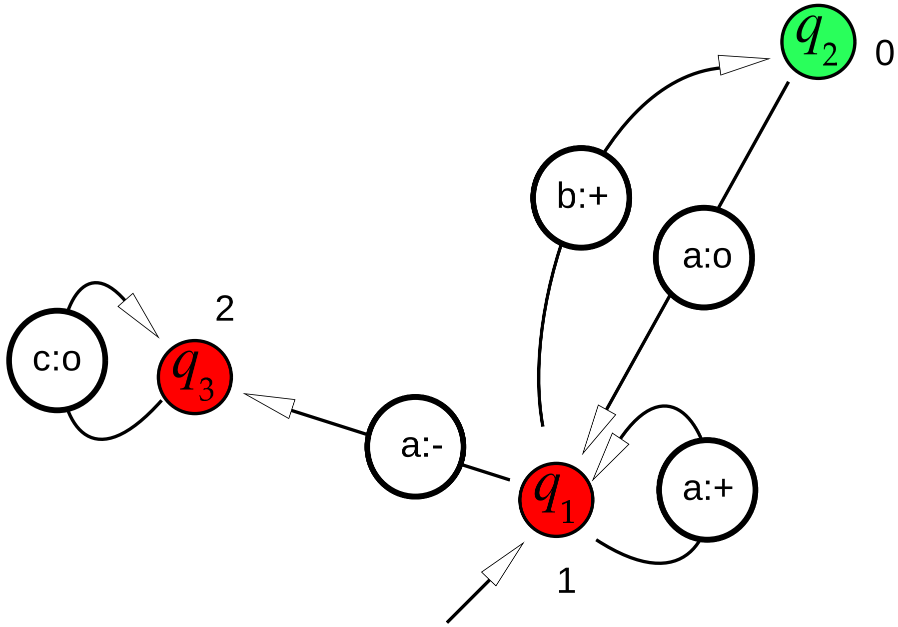


Another graph





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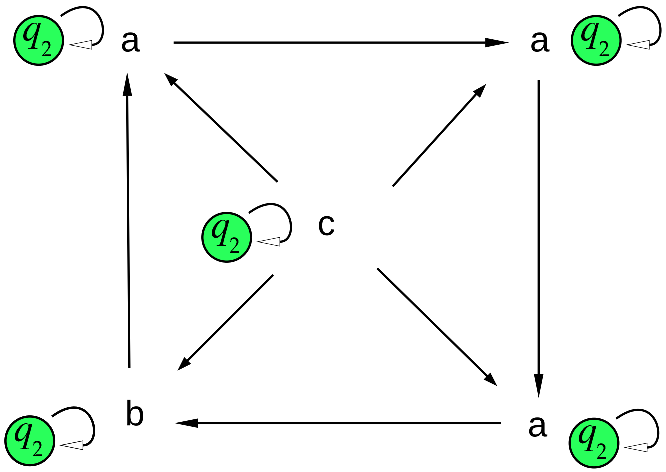

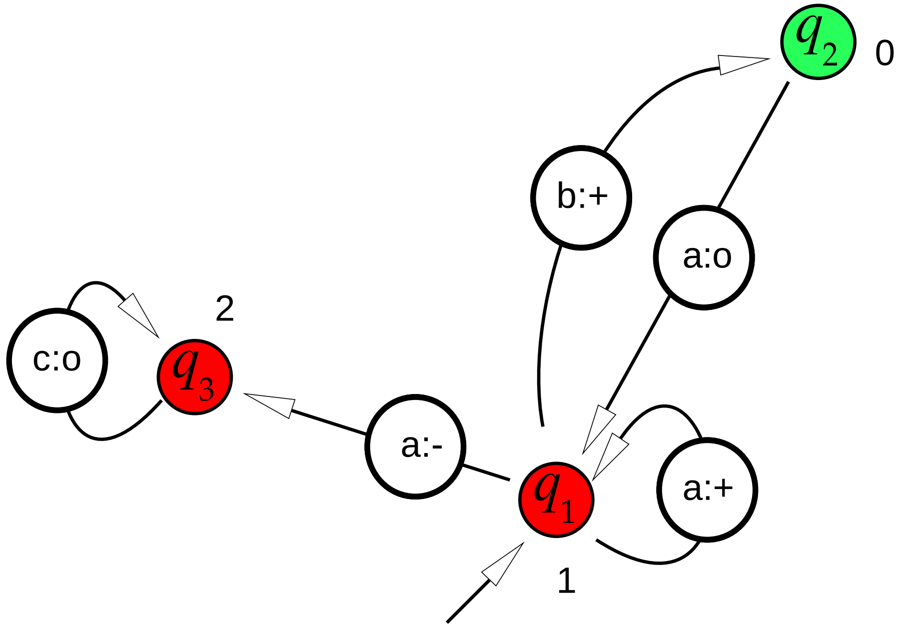

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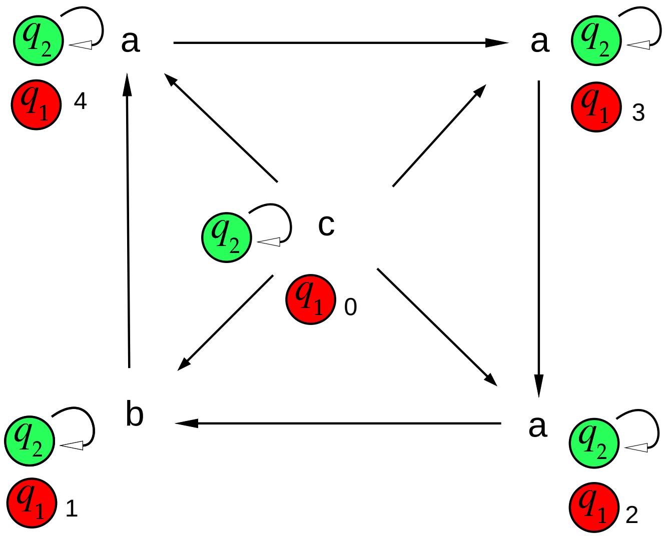
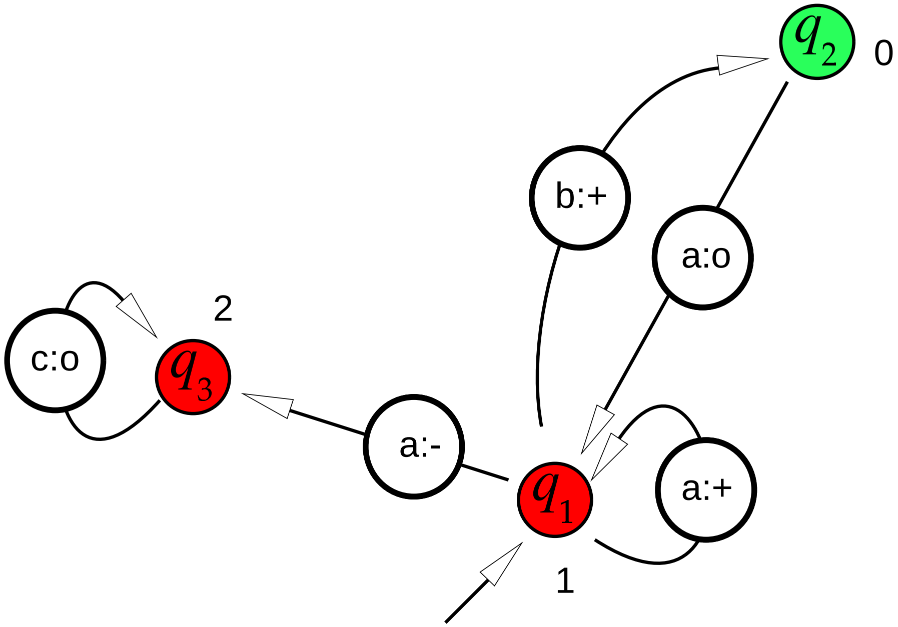
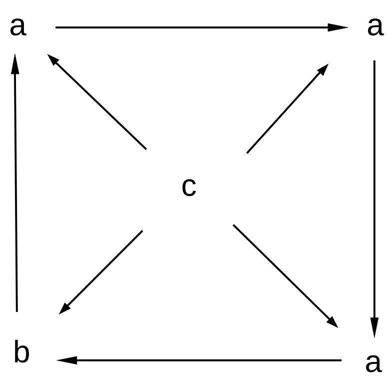




Tree unwinding



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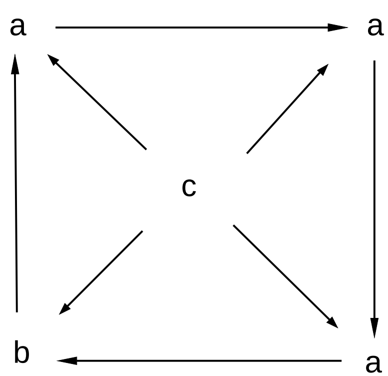

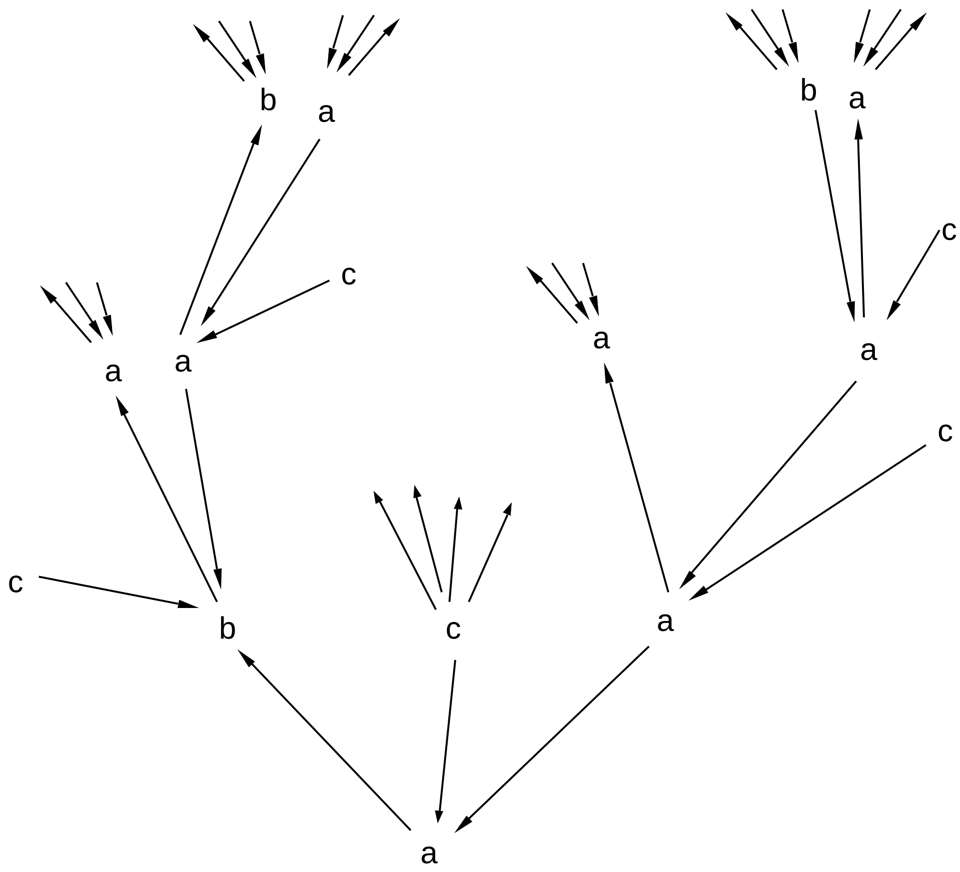

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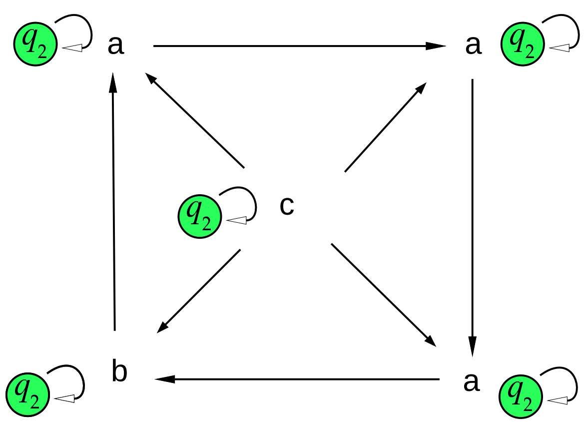

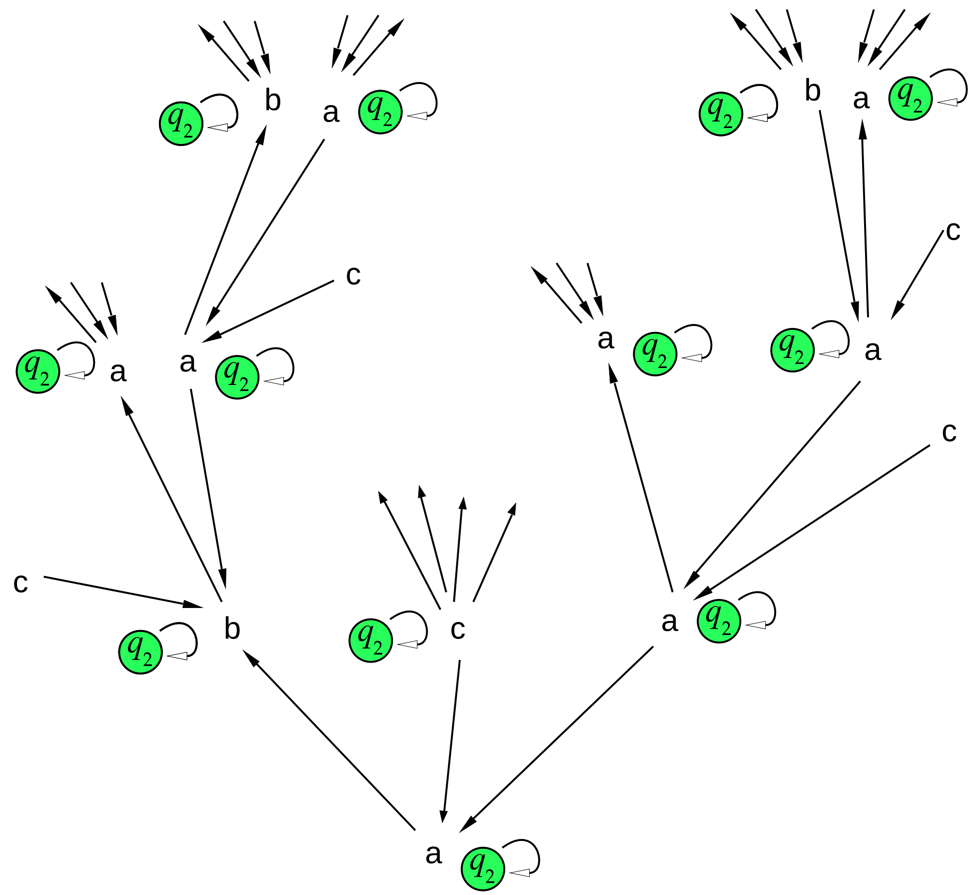

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