1 Tree automata

What is a Tree Automaton?
Decision Problems

2 Logic

Logic for Words
Logic for Trees
Transitive Closure Logic

3 Temporal Logics

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XPath

4 Tree-Walking Automata, 1

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Expressive Power
Pebble Automata

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Tree-Walking Automata Cannot Be Determinized

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Some logics that describe tree properties

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monadic second-order logic

"There is a set of nodes that is closed under parents, has an *a* label, and has no *c* label"

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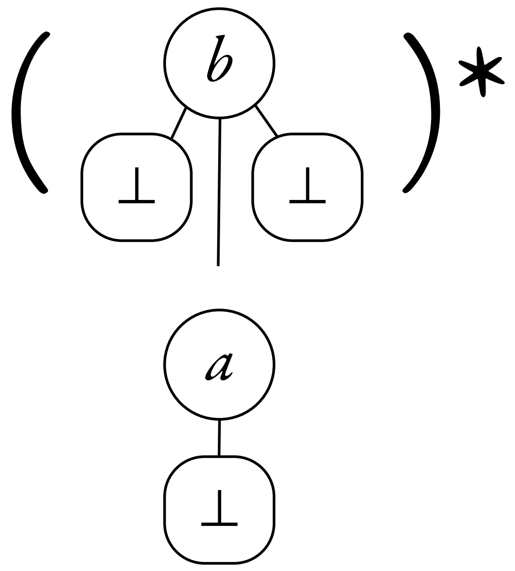

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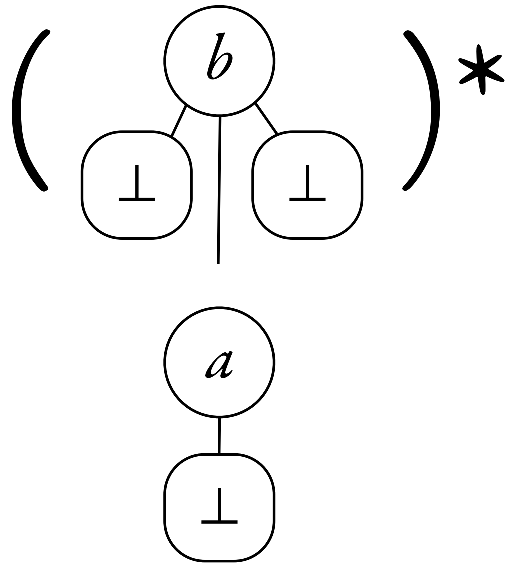

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Monadic- and First-Order Logic for Words

definition

weakness of first-order logic

MSO=regular

Monadic- and First-Order Logic for Trees

Transitive Closure Logic and Regular Expressions

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grandfather of logics for regular languages

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Thm. (Thatcher, Wright `68)
A tree language is regular if and only if it can be defined in monadic second-order logic.

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First-Order Logic for Words

Alphabet: $A=\{a,b,c\}$ $A^*ab^*aA^*$

first-order logic

$$\exists x\ \exists y\quad a(x) \land a(y) \land x<y \land (\forall z\quad x<z<y \Rightarrow b(x))$$

quantification
is over positions

label predicates

order on positions

Formal definition: a word $w{=}a_1\,a_2 \cdots a_n$ word is interpreted as
structure $\underline{w}= \langle \{1,..,n\}, <, a(x), b(x), c(x) \rangle$
A formula Ψ gives a language $L_\Psi=\{w : \Psi \text{ holds in } \underline{w}\}$

Thm. Every language definable in first-order logic is regular, but not conversely, eg. $(aa)^*$.

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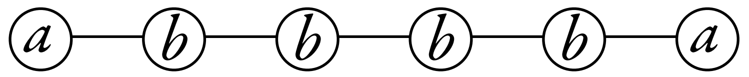

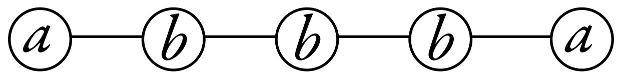

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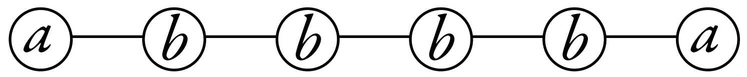

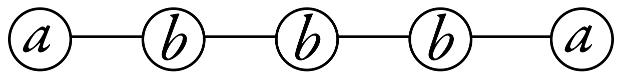

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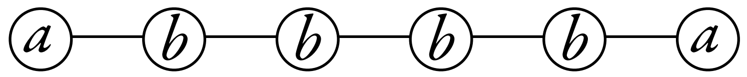

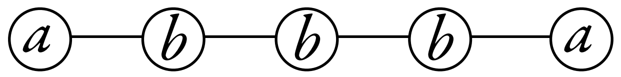

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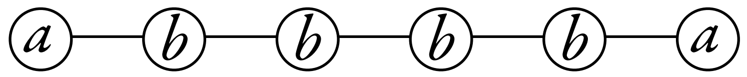

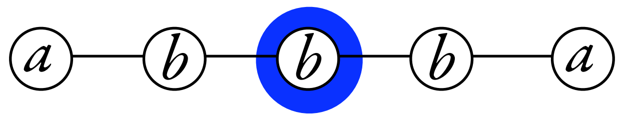

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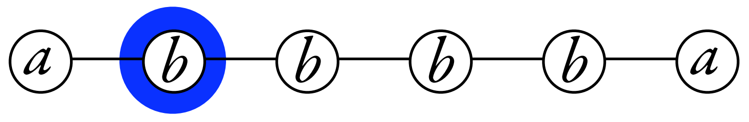

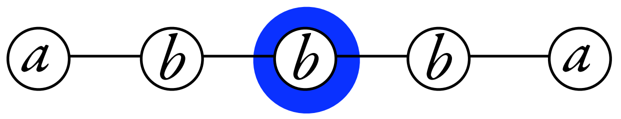

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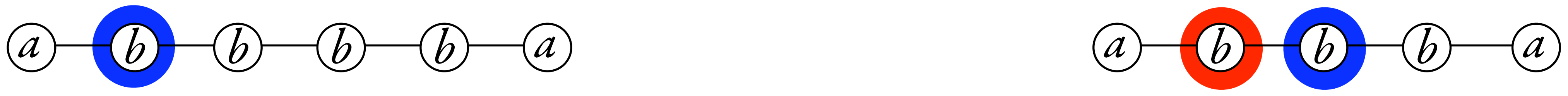

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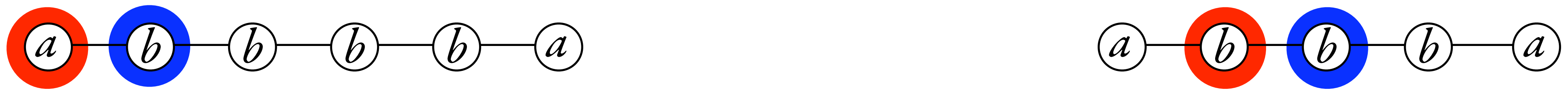

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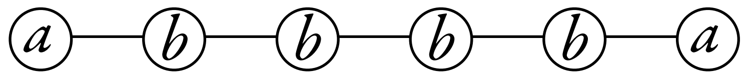

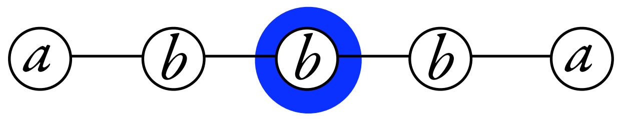

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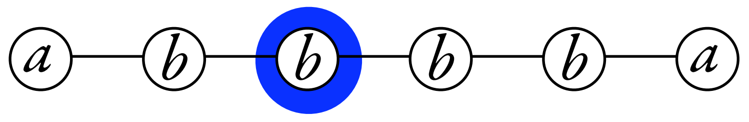

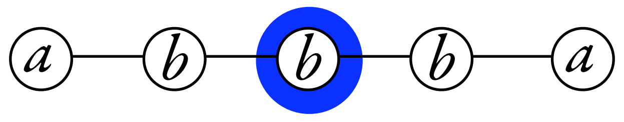

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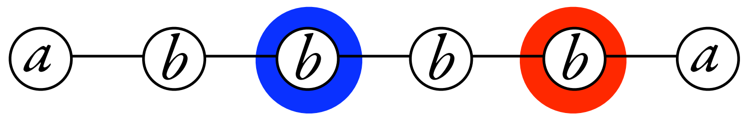

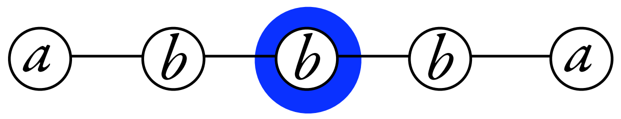

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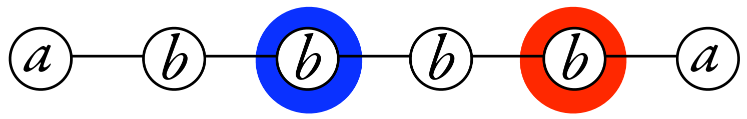

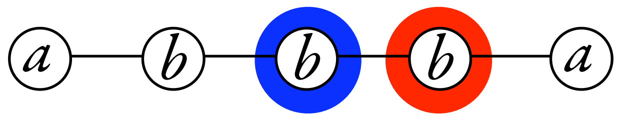

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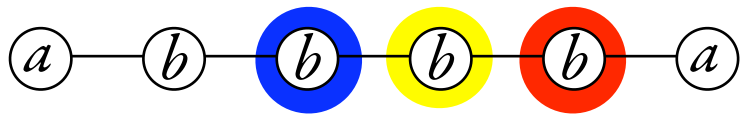

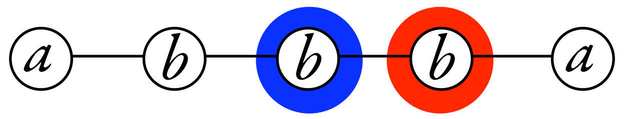

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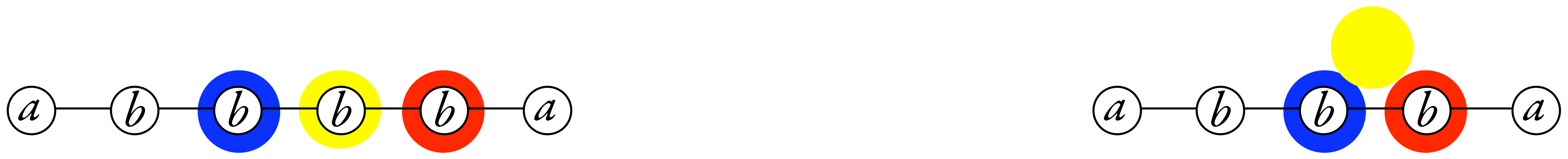

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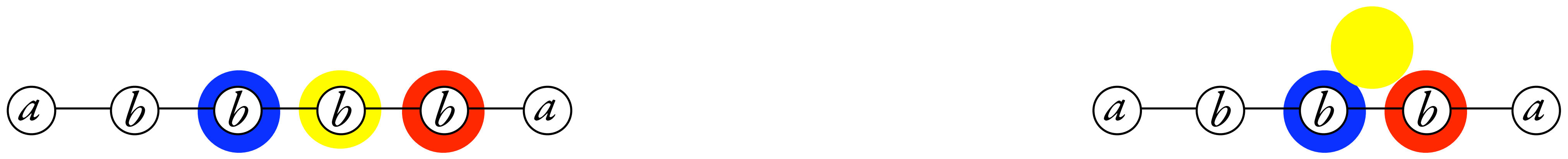

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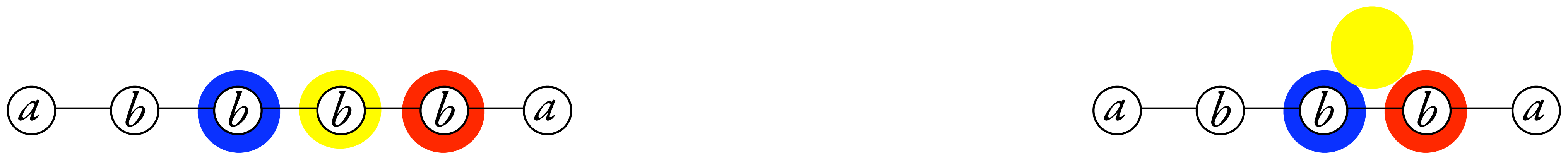

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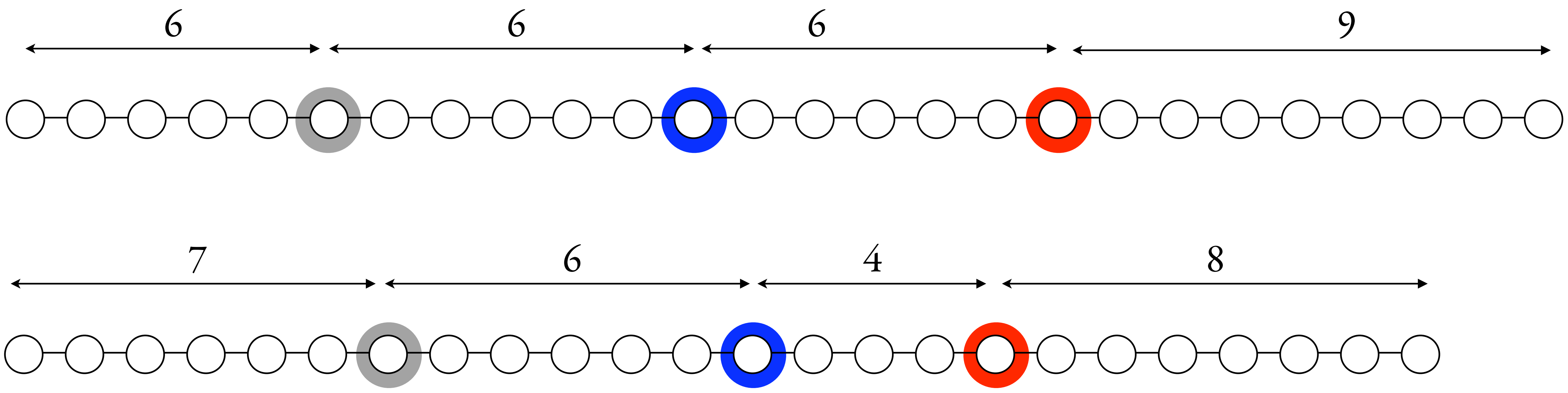

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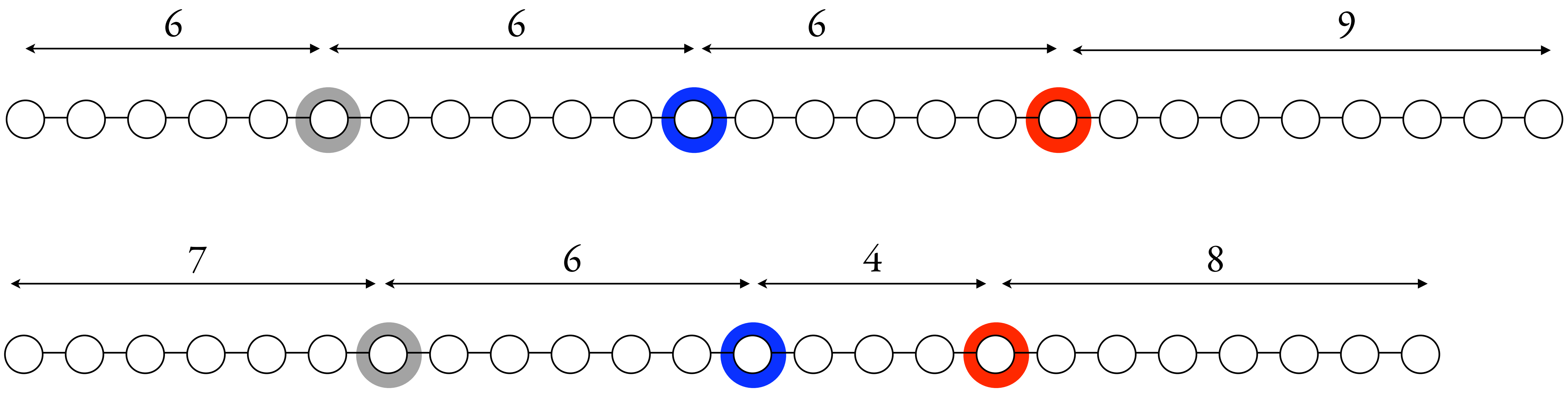

if there are $i=2$ rounds left, Duplicator will survive.

Fact. The language $(aa)^*$ cannot be defined in first-order logic (with order $<$ and labels).

Proof. For any number of rounds k, Duplicator has a strategy to survive the game played on words of length 2^k and 2^k+1

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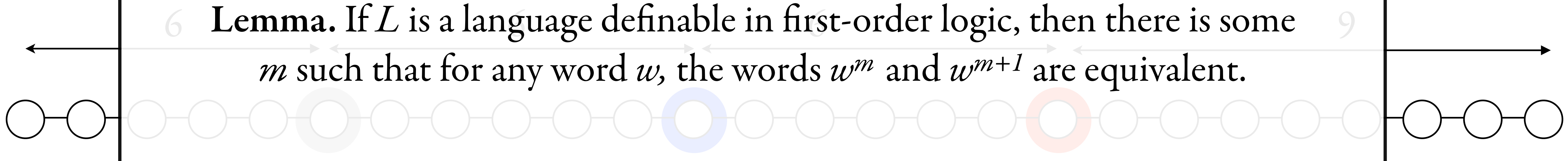
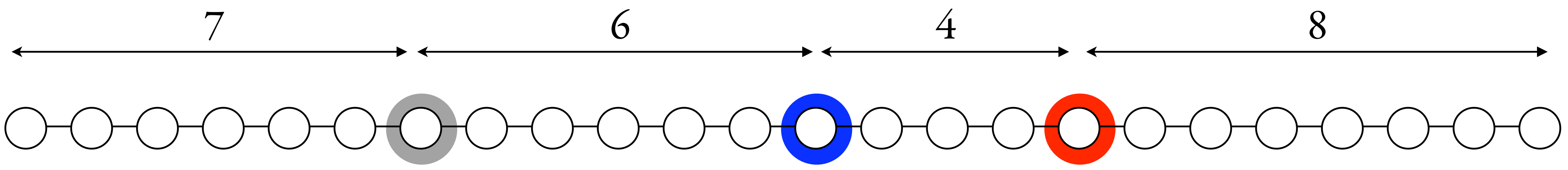
Lemma. If L is a language definable in first-order logic, then there is some m such that for any word w, the words w^m and w^{m+1} are equivalent.





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Contrary to what the above suggests, MSO is more succint than regular expressions.

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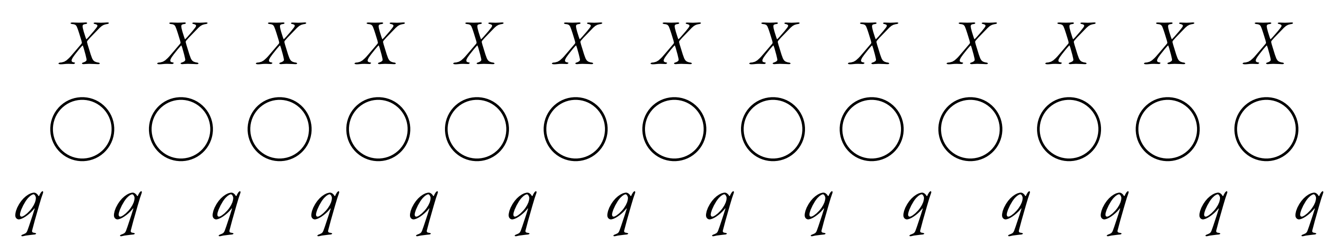

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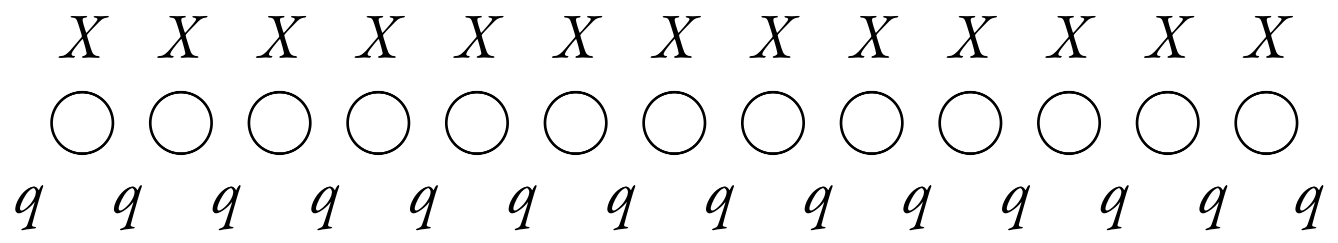

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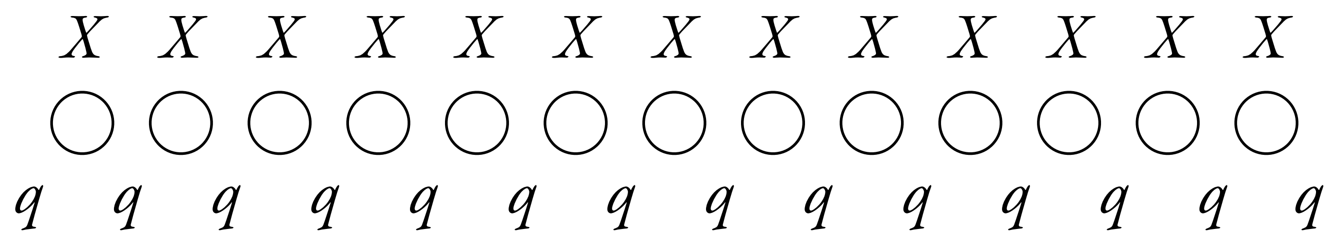

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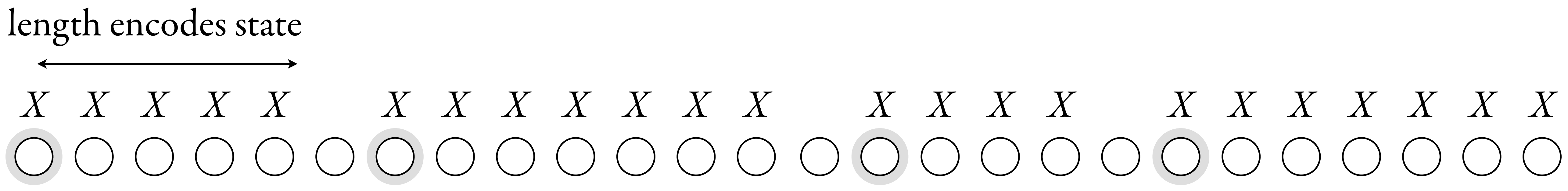

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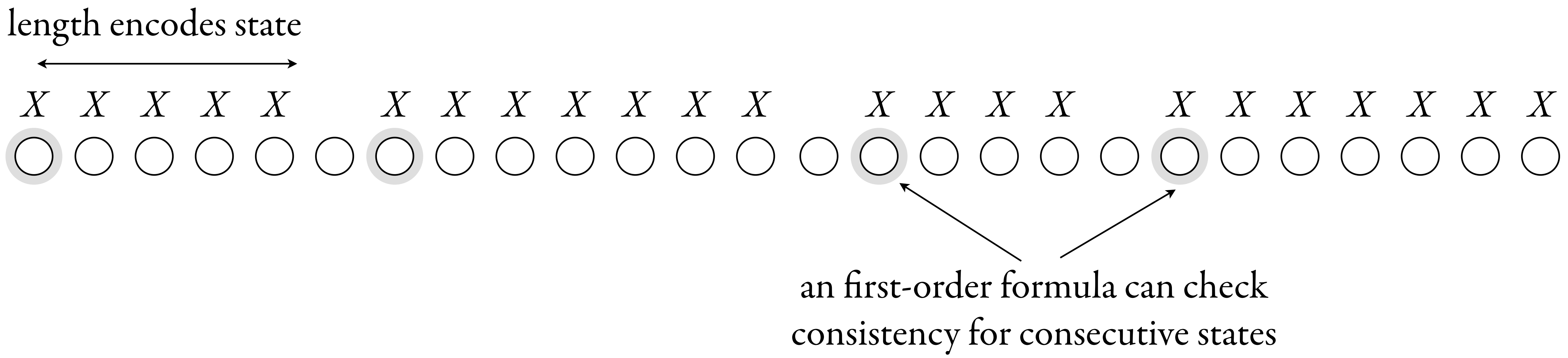
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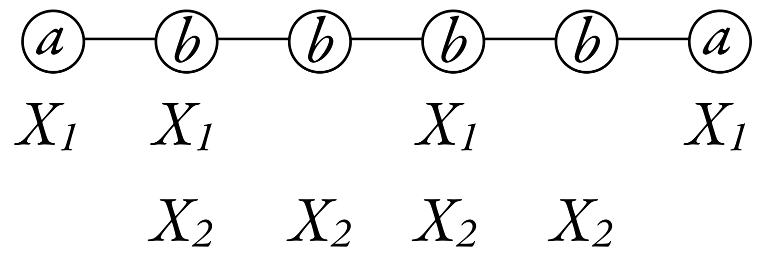

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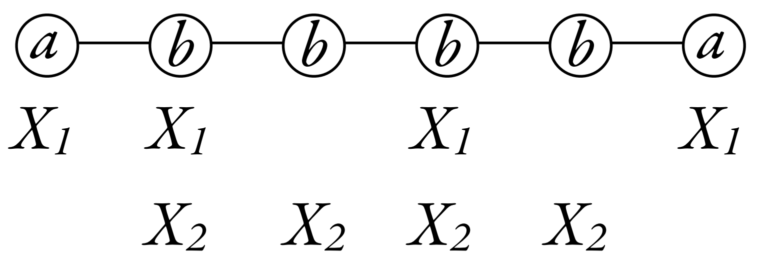

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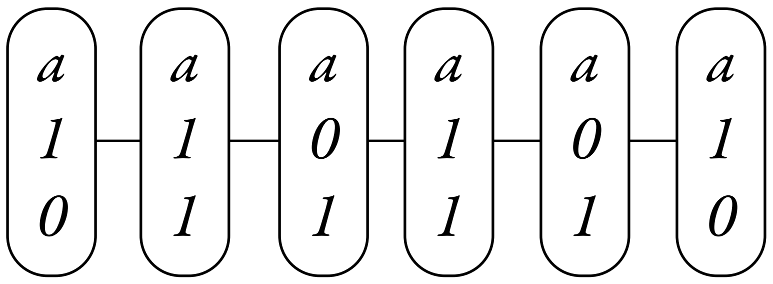

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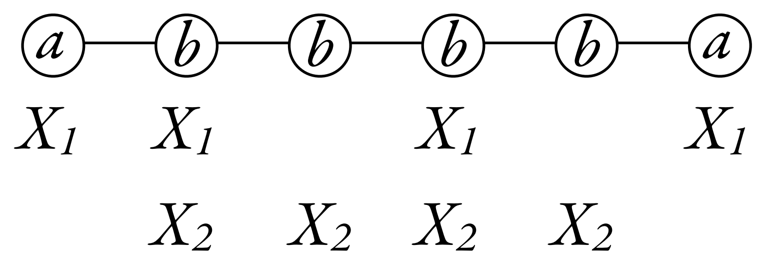

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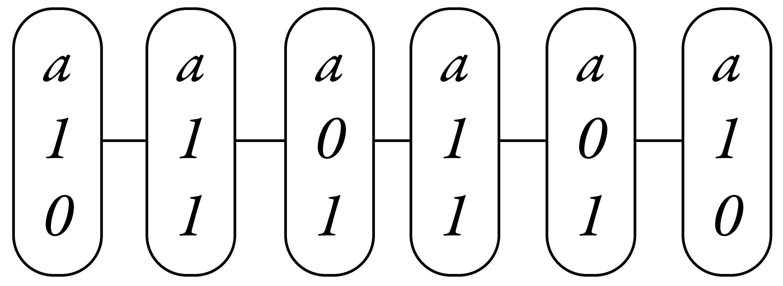

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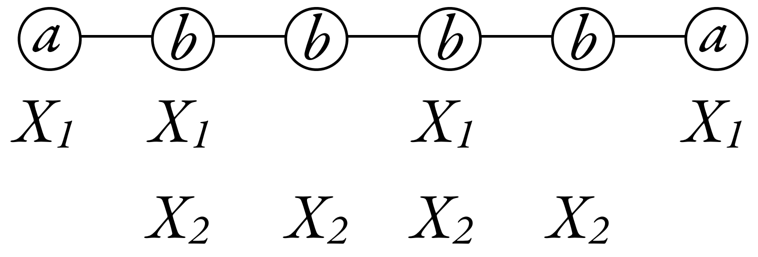

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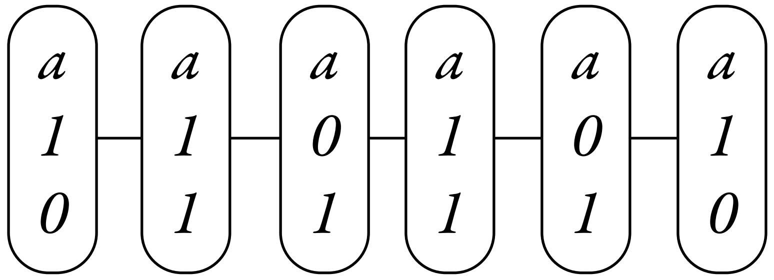

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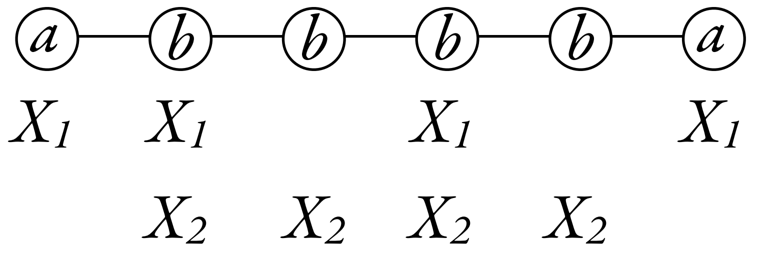

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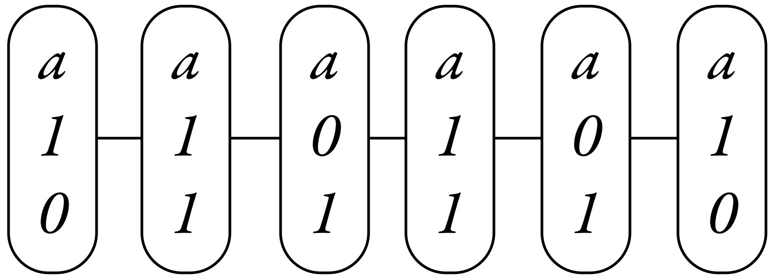

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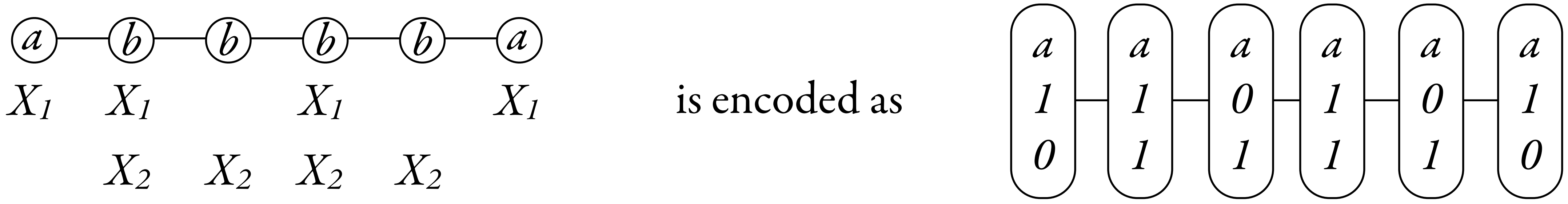

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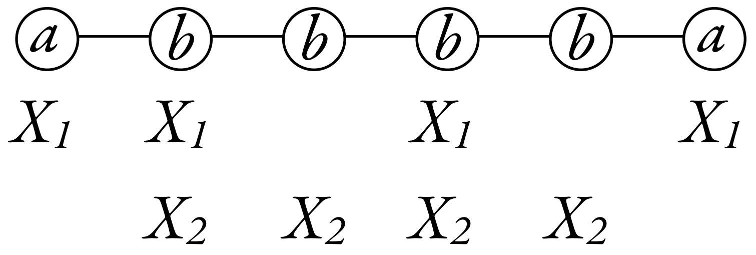

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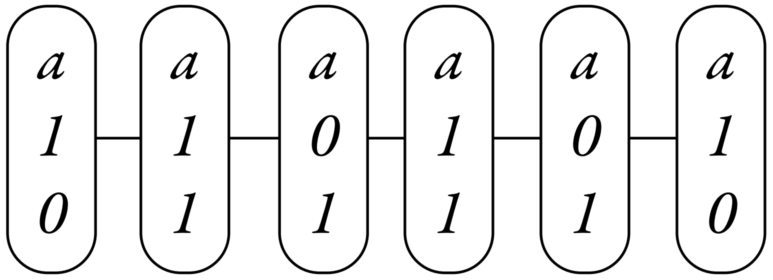

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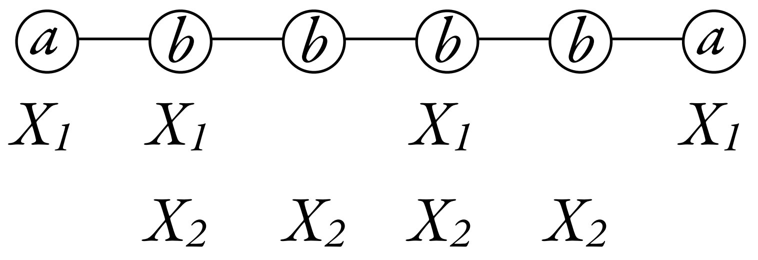

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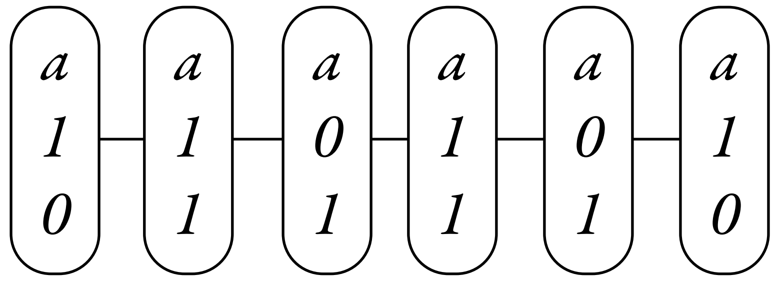

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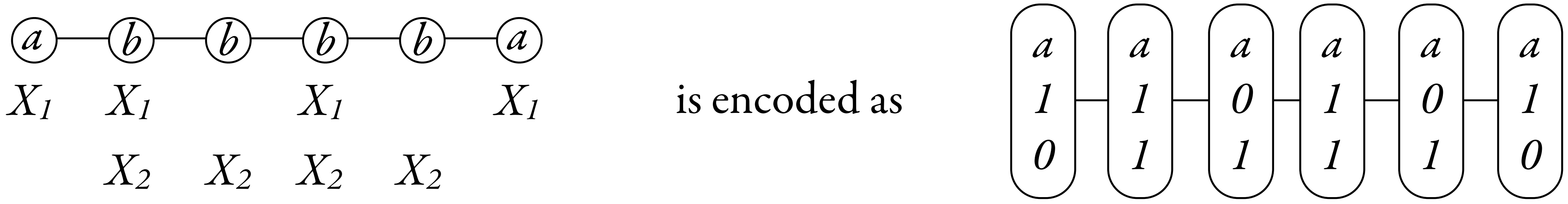

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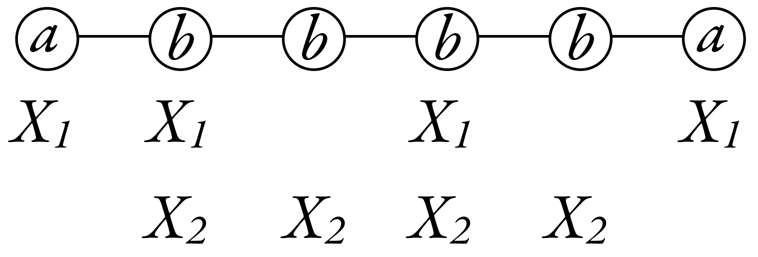
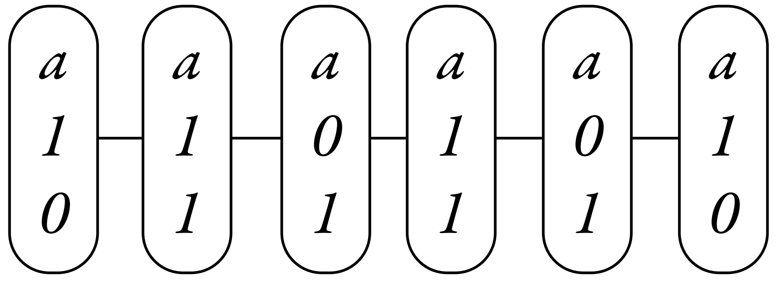
Language of $\exists X_m. \Psi(X_1,X_2,\ldots, X_m)$ = Projection under π of language of $\Psi(X_1,X_2,\ldots, X_m)$

Claim. For every formula $\Psi(X_1,X_2,\ldots,X_m)$ of MSO, the set L_Ψ is regular.

How do we define L_Ψ for formulas with free set variables $X_1,\ldots,X_n$?
A word $w\in A^*$ together with valuations for sets $X_1,\ldots,X_n$ is represented as a word over $A\times\{0,1\}^n$.



is encoded as



Under this encoding, L_Ψ is a language over $A\times\{0,1\}^n$.

Induction proof of claim.

induction base.

simple. Eg. $X_i\subseteq X_j$ is the regular language “if true on bit i then true on bit j”

boolean operations

standard constructions for automata.

existential quantification $\exists X_m.\,\Psi(X_1,X_2,\ldots,X_m)$

run the automaton for $\Psi(X_1,X_2,\ldots,X_m)$, nondeterministically guessing values for X_m.

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$$\pi : (A\times\{0,1\}^n)^* \longrightarrow (A\times\{0,1\}^{n-1})^*$$

Monadic- and First-Order Logic for Words

definition

weakness of first-order logic

MSO=regular

Monadic- and First-Order Logic for Trees

Transitive Closure Logic and Regular Expressions

Monadic- and First-Order Logic for Words

- definition
- weakness of first-order logic
- MSO=regular

Monadic- and First-Order Logic for Trees

- definition
- problems with parity
- problems with aperiodicity

Transitive Closure Logic and Regular Expressions

MSO for Trees

A binary tree has an even number of nodes

iff

there is a set of positions

$\exists X$

$\left\{ \right.$

that contains no leaf

$\forall x \exists y \;\; y \geq x \;\wedge\; y \notin X$

but contains the root

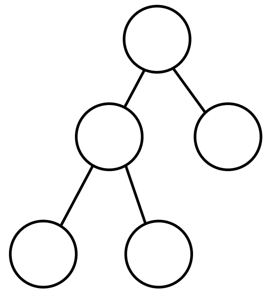
$\forall x \exists y \;\; y \leq x \;\wedge\; y \in X$

and contains a node iff exactly on of its children is in X

$\forall x \forall y_0 \forall y_1 \; \big(suc_0(x, y_0) \wedge suc_1(x, y_1)\big) \;\Rightarrow\; \big(x \notin X$

iff

false



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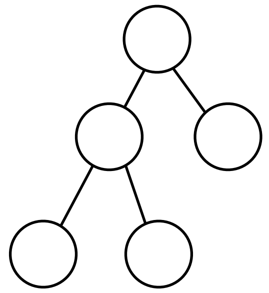

Thm. (Thatcher, Wright `68)

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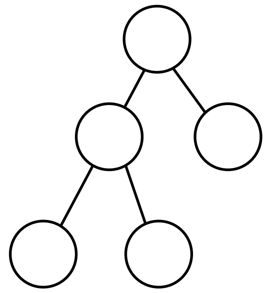
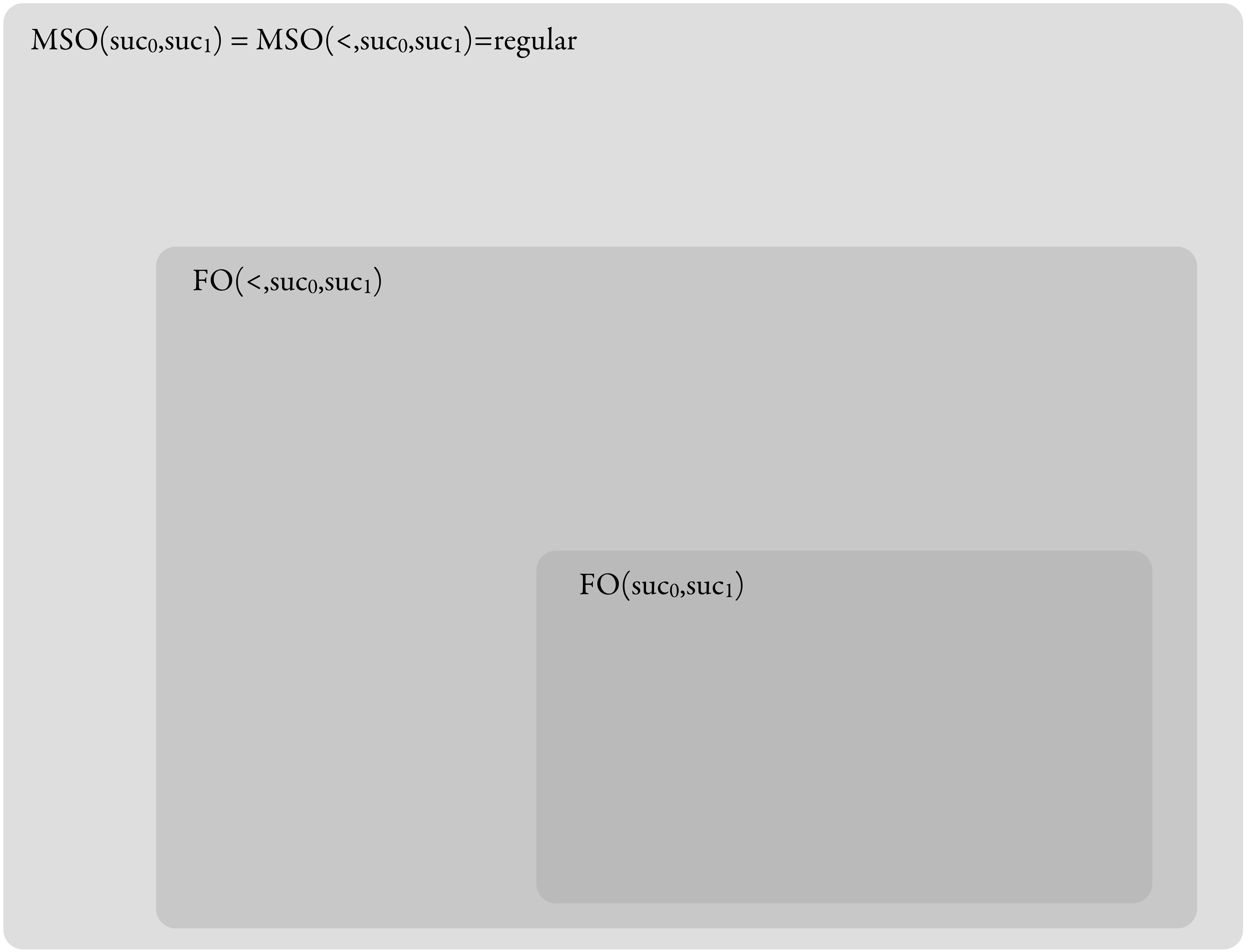
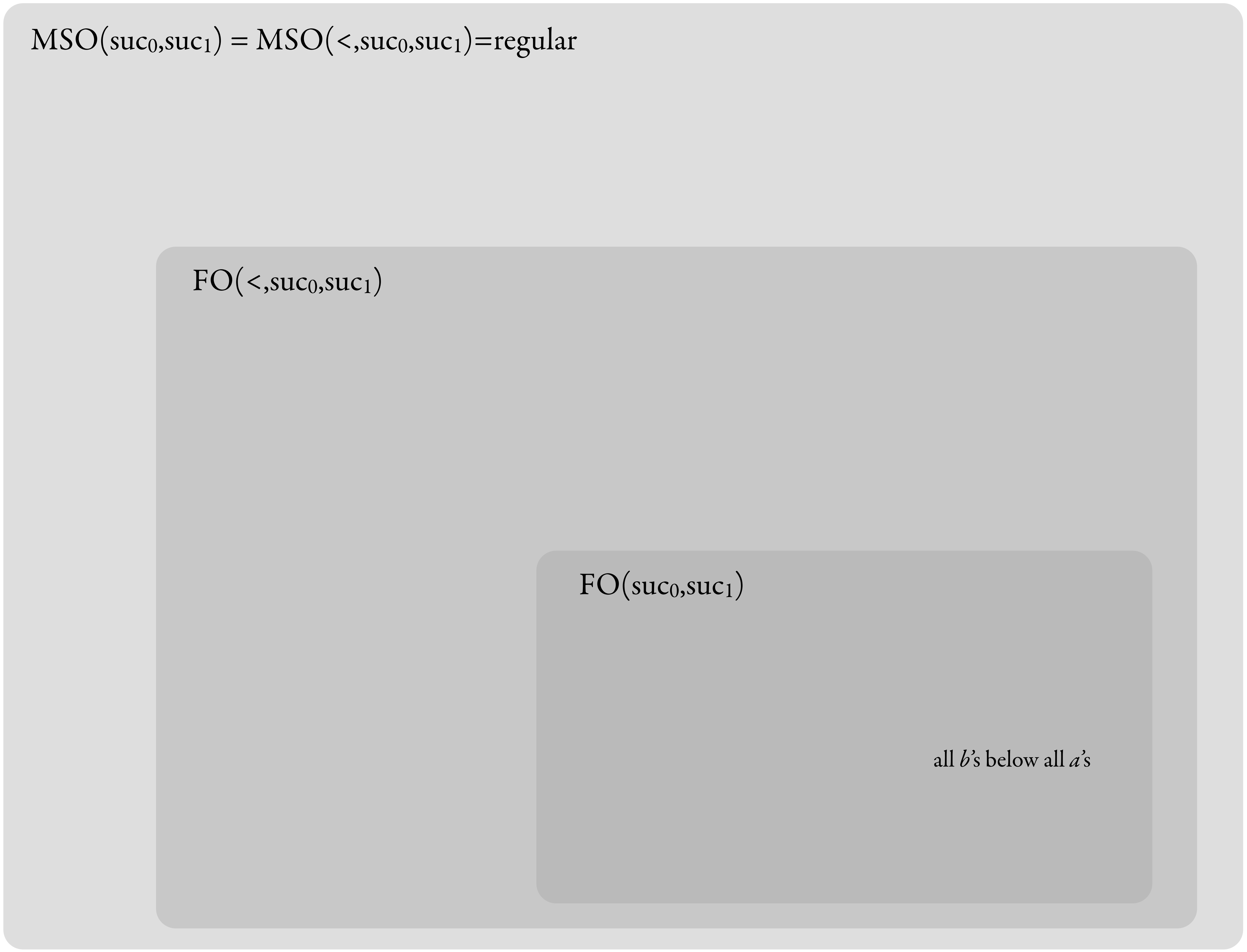


Thm. (Thatcher, Wright `68)
MSO = regular languages for finite trees.


$\mathrm{MSO}(\mathrm{suc}_0,\mathrm{suc}_1) = \mathrm{MSO}(<,\mathrm{suc}_0,\mathrm{suc}_1)=\mathrm{regular}$
$\mathrm{FO}(<,\mathrm{suc}_0,\mathrm{suc}_1)$
$\mathrm{FO}(\mathrm{suc}_0,\mathrm{suc}_1)$


MSO(suc_0,suc_1) = MSO(<,suc_0,suc_1)=regular
FO(<,suc_0,suc_1)
FO(suc_0,suc_1)
all *b*’s below all *a*’s

MSO(suc_0,suc_1) = MSO(<,suc_0,suc_1)=regular

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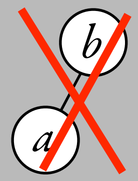

MSO(suc_0,suc_1) = MSO(<,suc_0,suc_1)=regular

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for alphabet *a,b,c*

FO(suc_0,suc_1)

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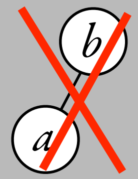

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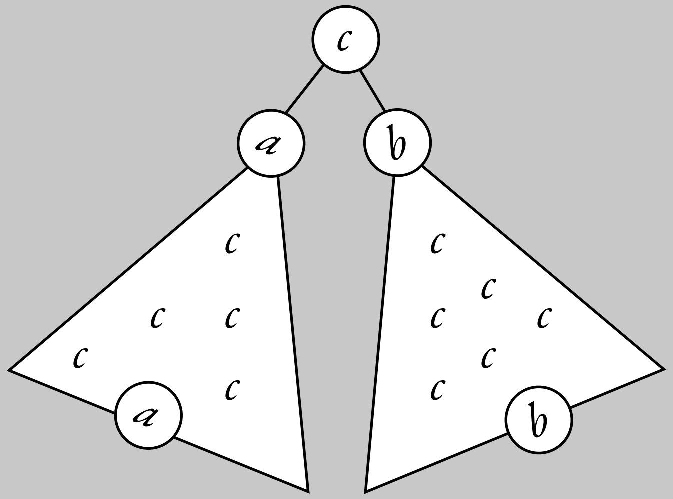

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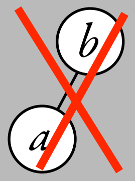

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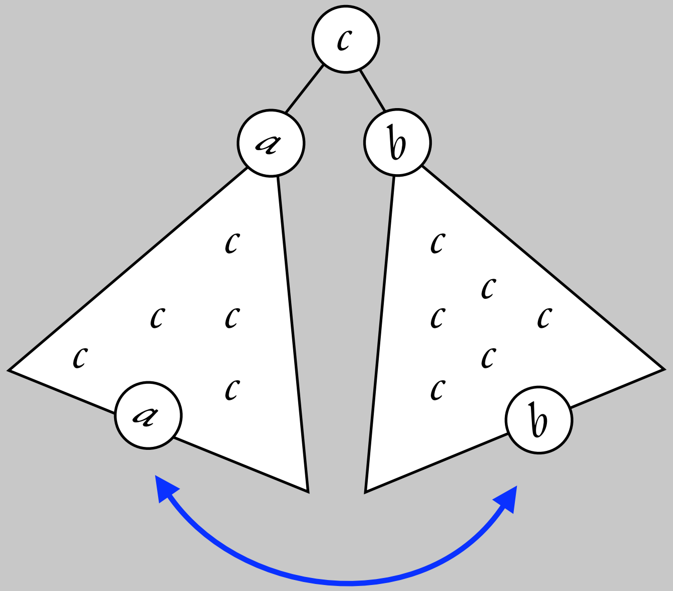
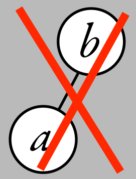


FO(suc_0,suc_1)

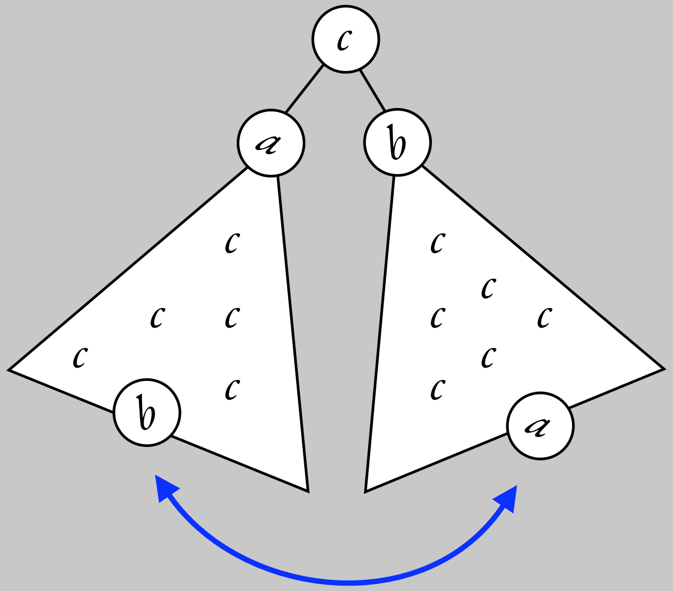
all b's below all a's



MSO(suc_0,suc_1) = MSO($<$,suc_0,suc_1)=regular

FO($<$,suc_0,suc_1)

all *b*'s below all *a*'s
for alphabet *a*,*b*,*c*



FO(suc_0,suc_1)

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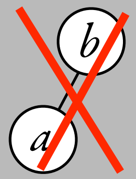
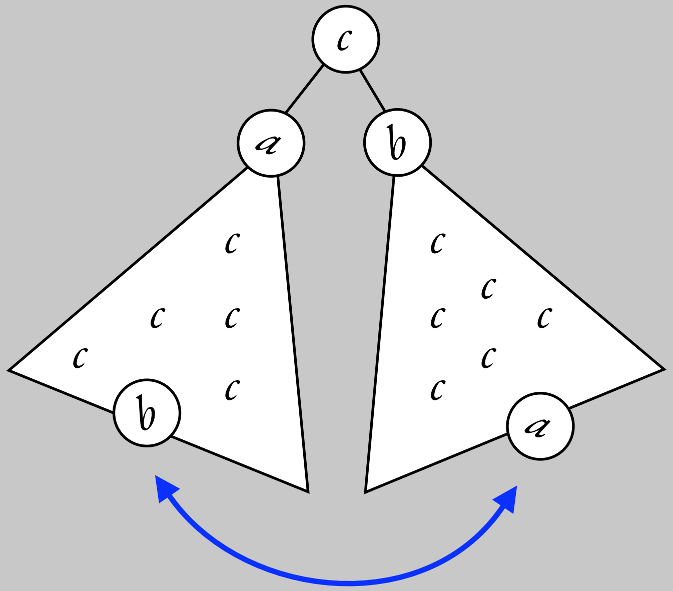


MSO(suc_0,suc_1) = MSO(<,suc_0,suc_1)=regular

parity

FO(<,suc_0,suc_1)

all *b*'s below all *a*'s
for alphabet *a*,*b*,*c*



FO(suc_0,suc_1)

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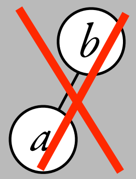
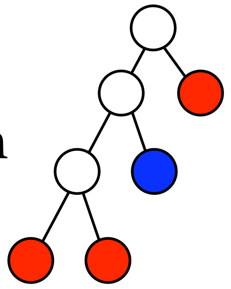


Parity

Parity

L=Exists a leaf at even depth



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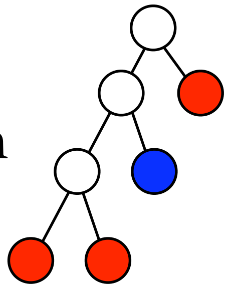

Surprise (Potthof)

This language is definable in FO($<$,suc_0,suc_1)

Parity

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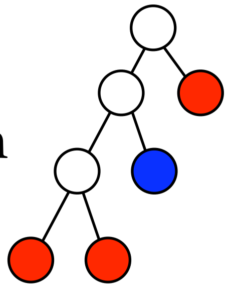

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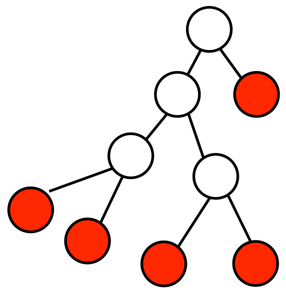
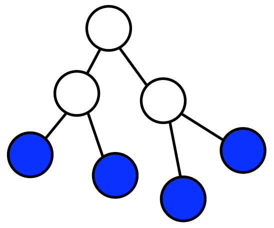
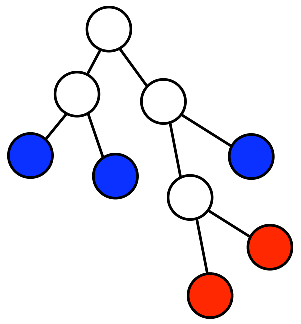


all leaves at odd depth



both parities



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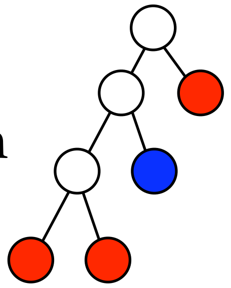

Surprise (Potthof)
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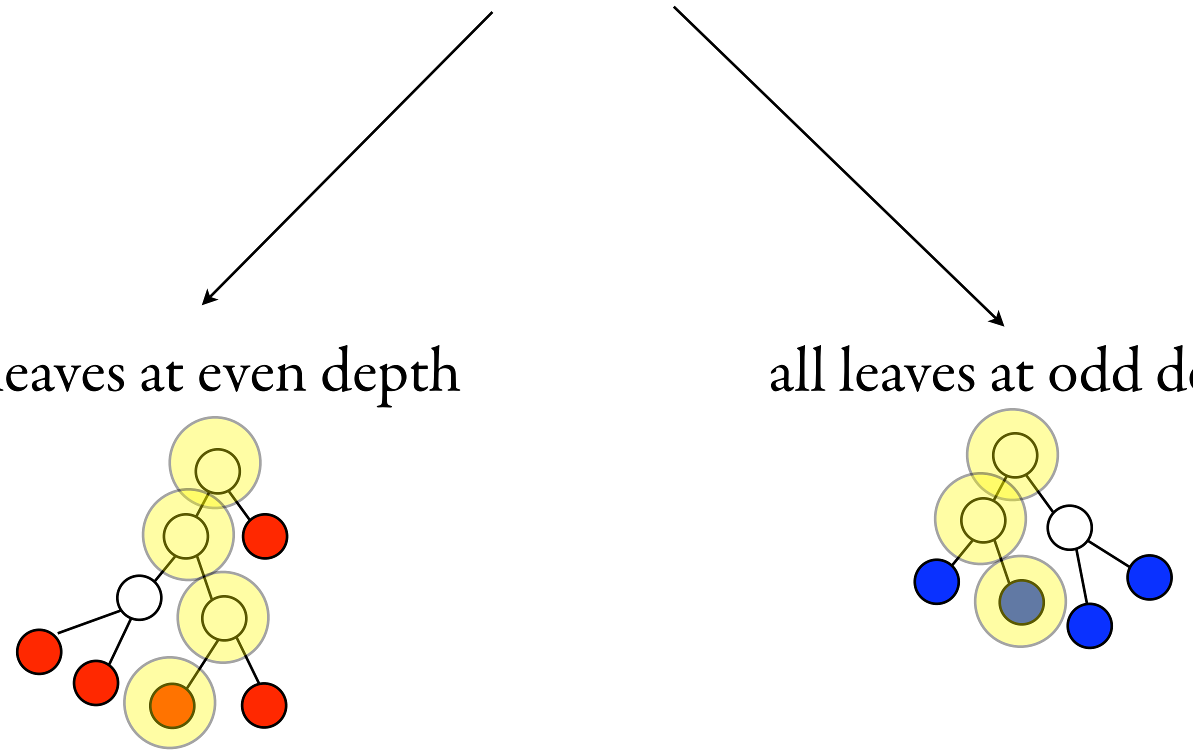
to disinguish between these two,
follow the left zigzag



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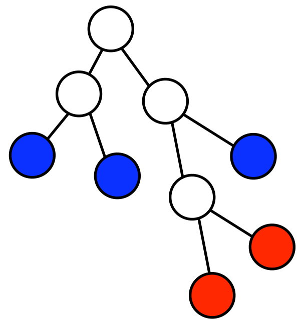

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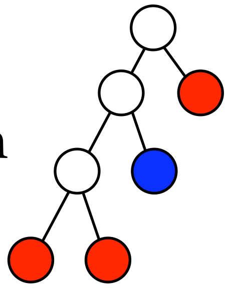

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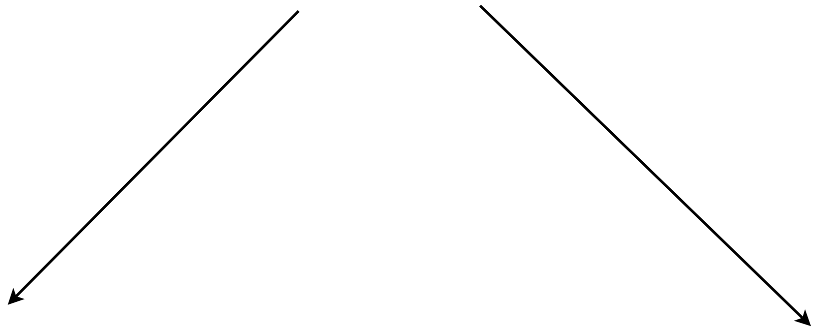

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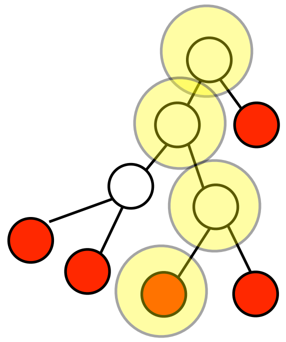
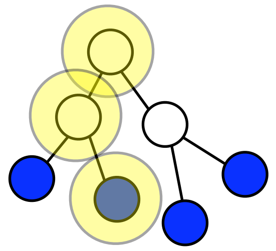


all leaves at odd depth



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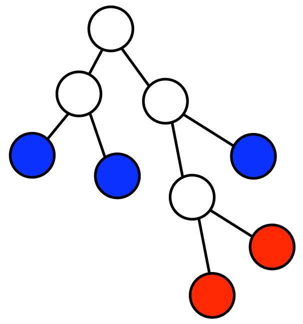

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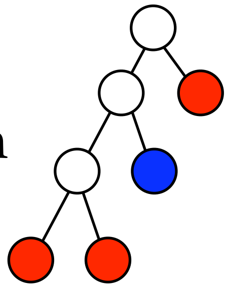

Surprise (Potthof)
This language is definable in FO($<$,suc$_0$,suc$_1$)

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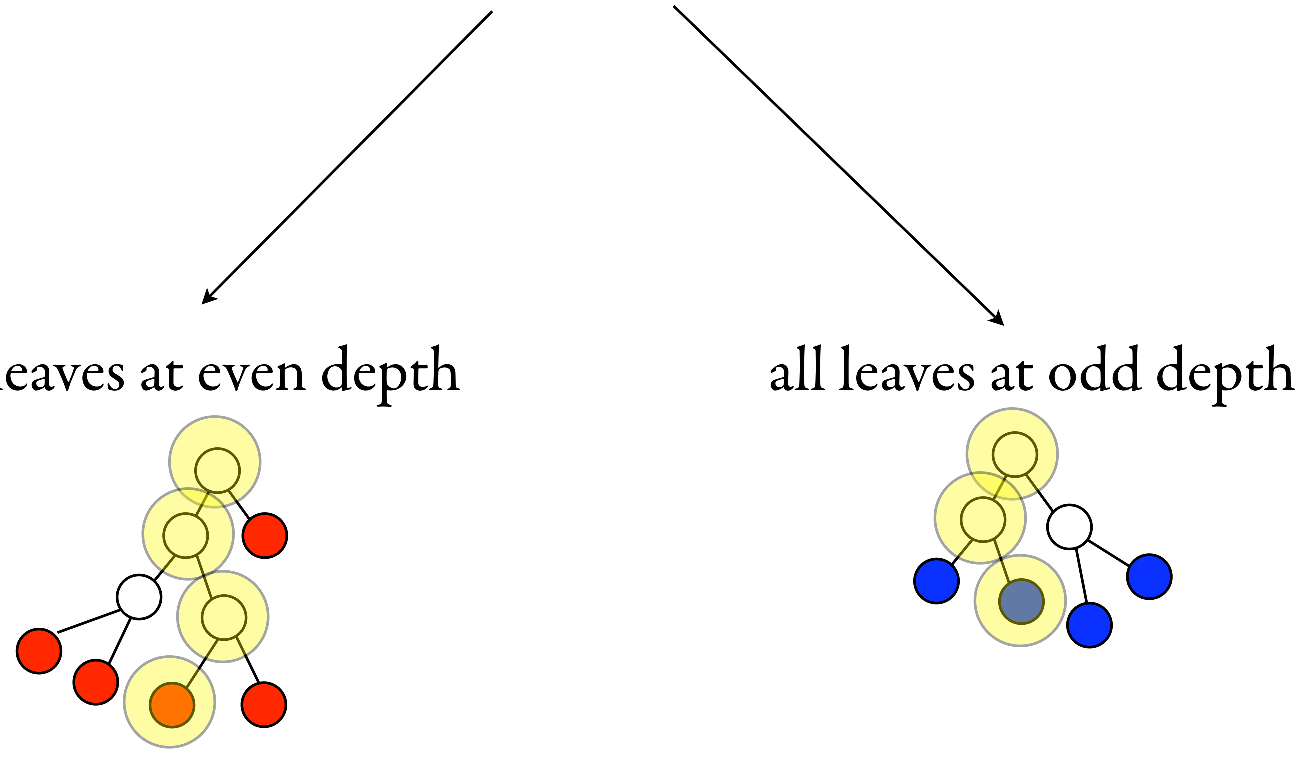
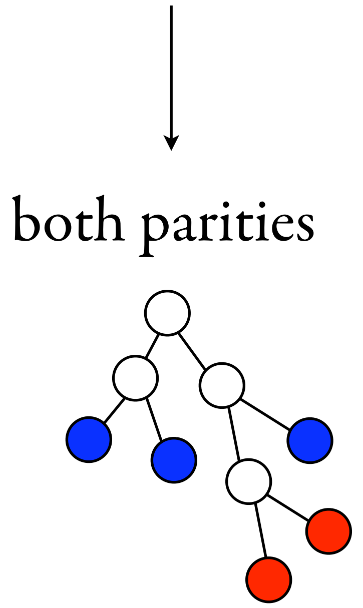
to disinguish between these two, follow the left zigzag

to detect this one, search for conflicting zigzags





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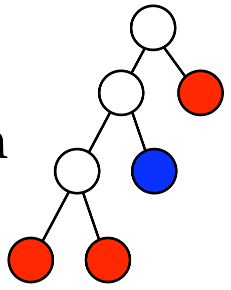

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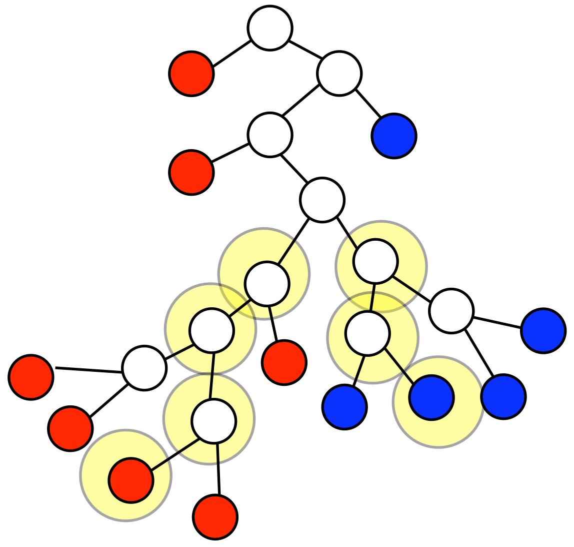

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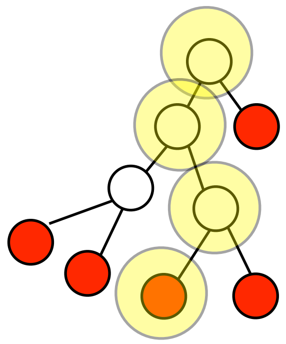
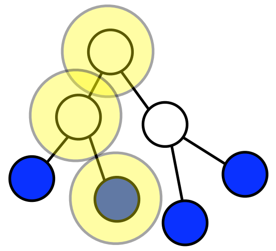
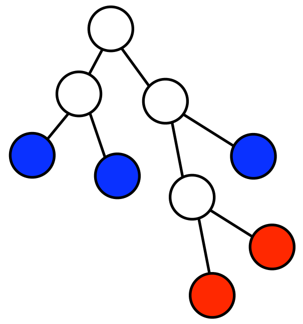


all leaves at odd depth



both parities



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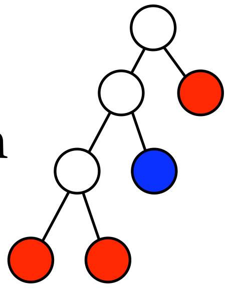

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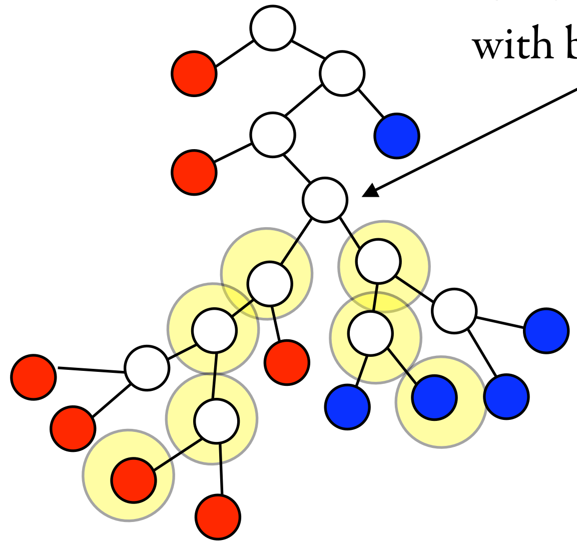

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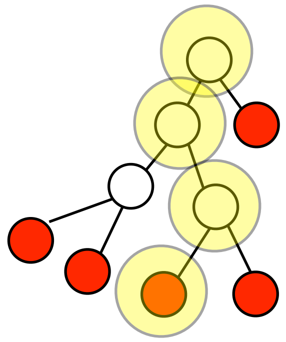

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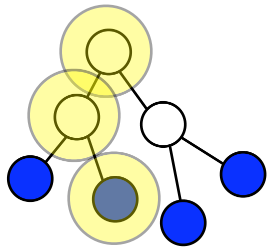

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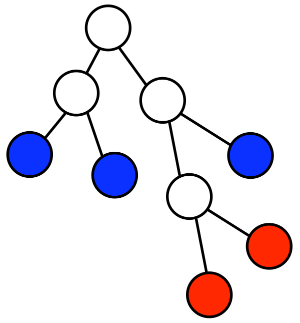

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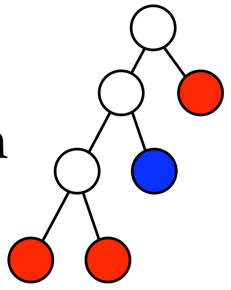

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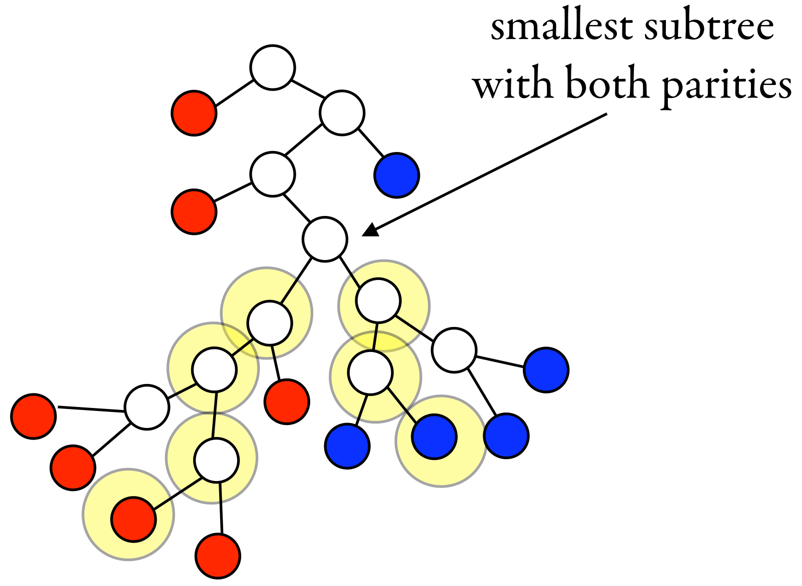

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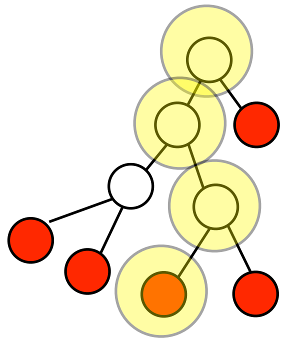

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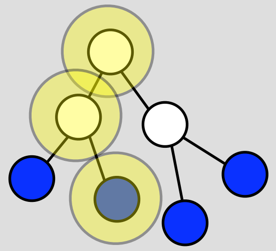

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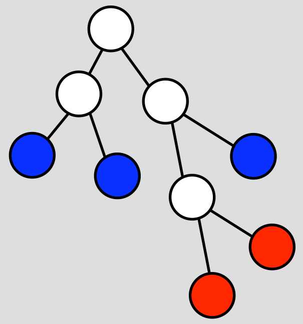
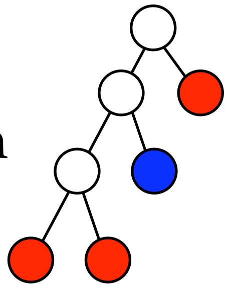


FO(<) $\not\subseteq$ FO(<,suc_0,suc_1)

+

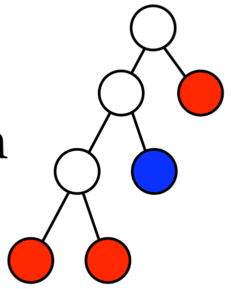
commutative children

L=Exists a leaf at even depth



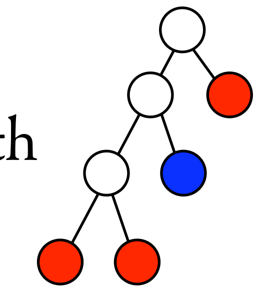
$$\mathrm{FO}(<) \not\subseteq \begin{array}{c}\mathrm{FO}(<,\mathrm{suc}_0,\mathrm{suc}_1)\\ + \\ \text{commutative children}\end{array}$$

L=Exists a leaf at even depth



This language is definable in FO(<,suc$_0$,suc$_1$)...

FO(<) $\not\subset$ FO(<,suc$_0$,suc$_1$) + commutative children

L=Exists a leaf at even depth 

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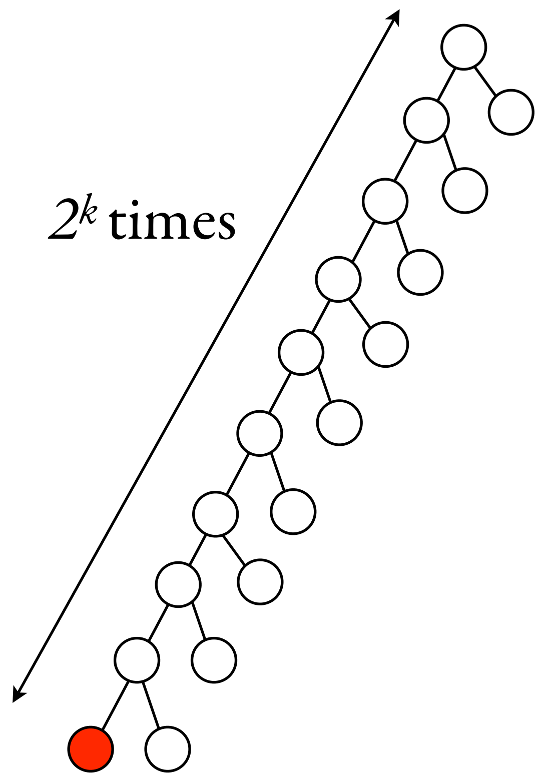

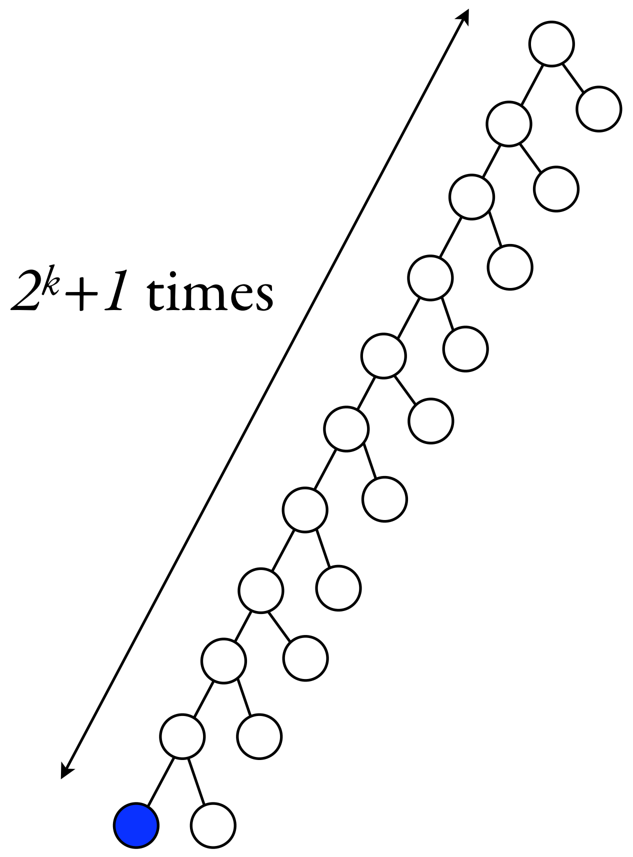

Duplicator survives the k round game on trees

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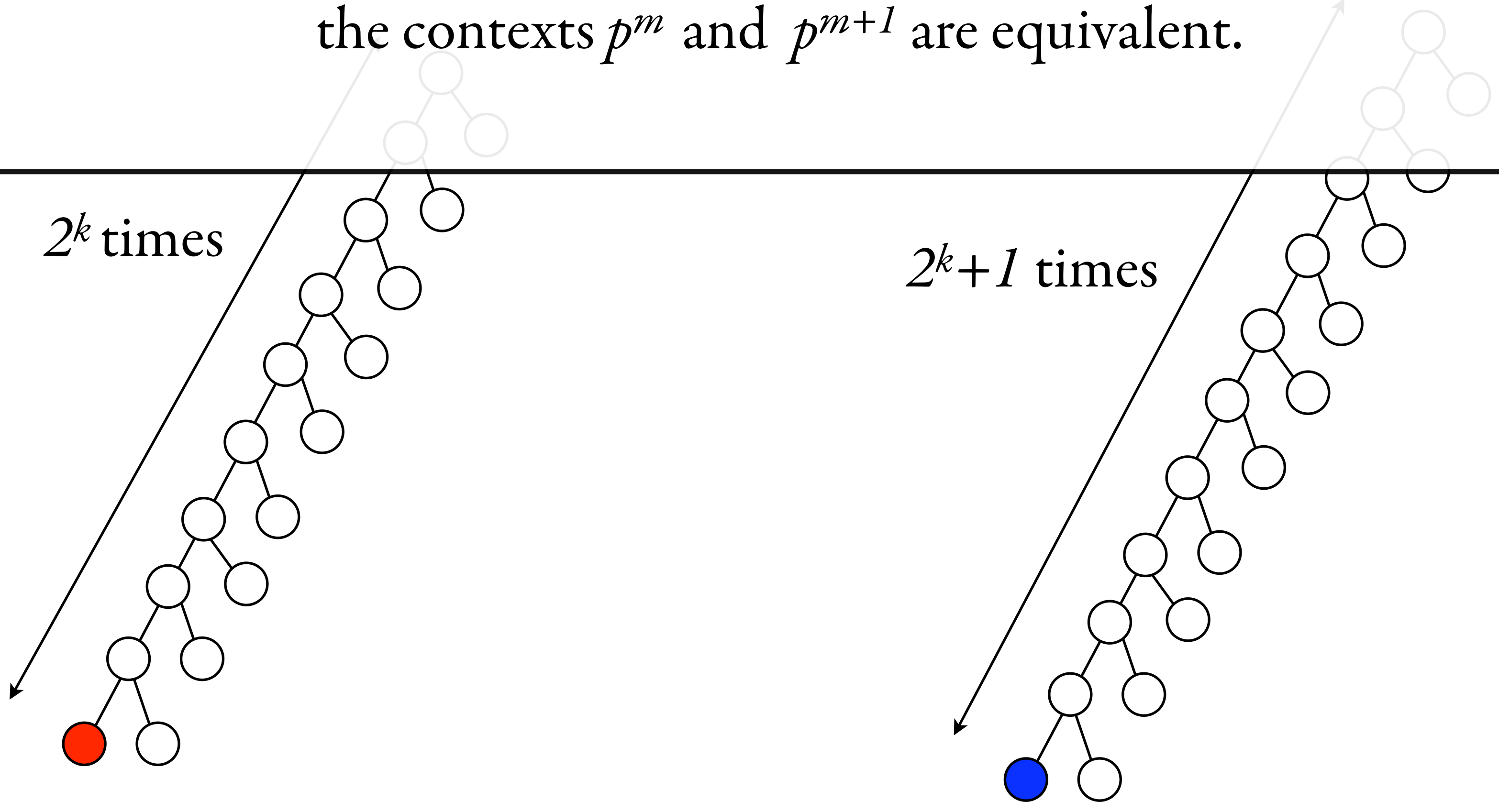

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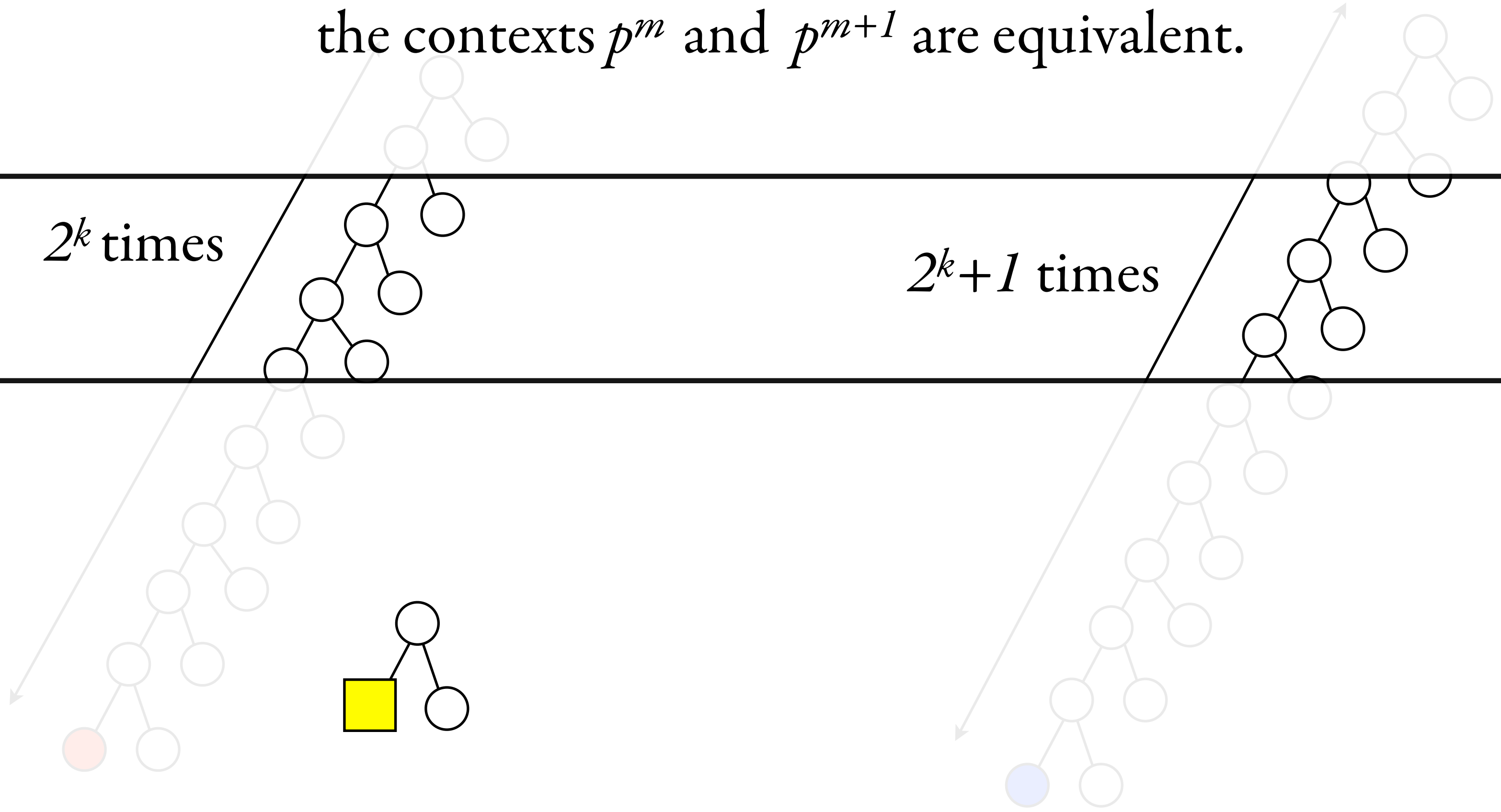

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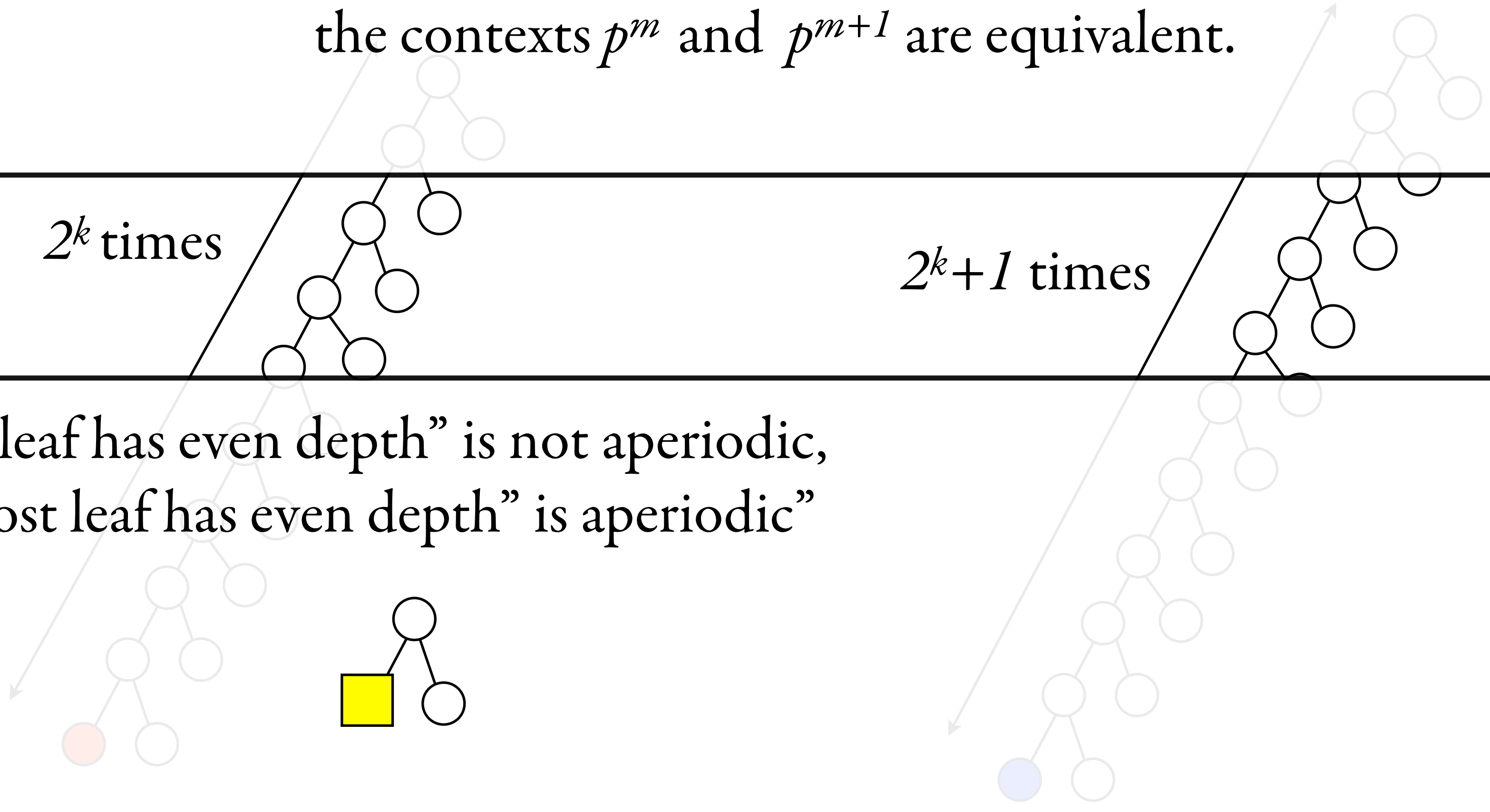

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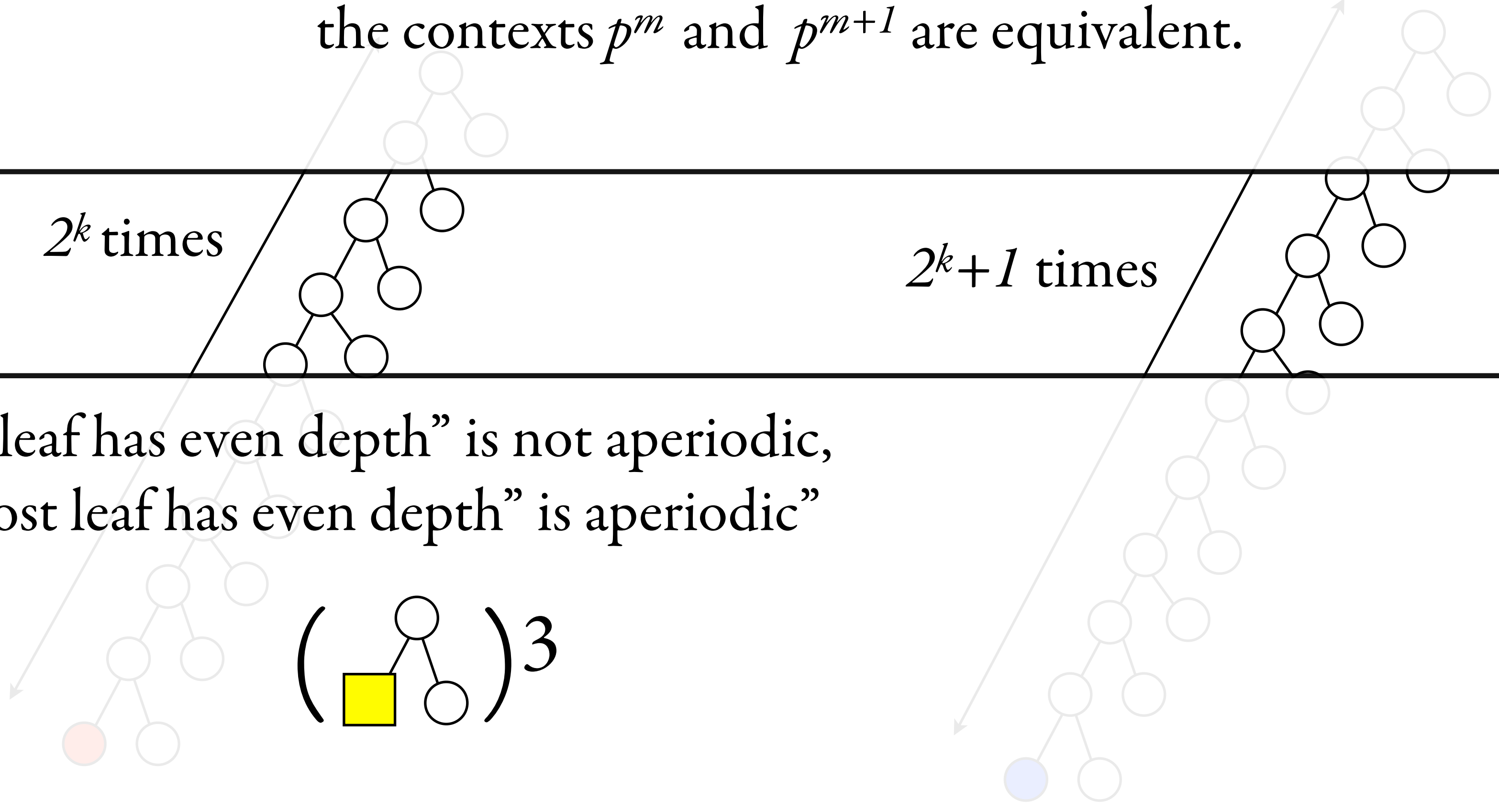

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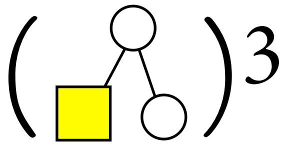

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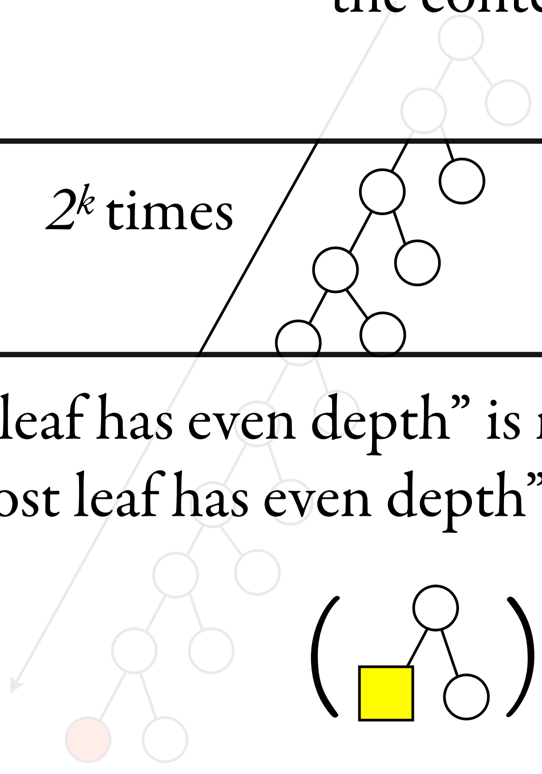

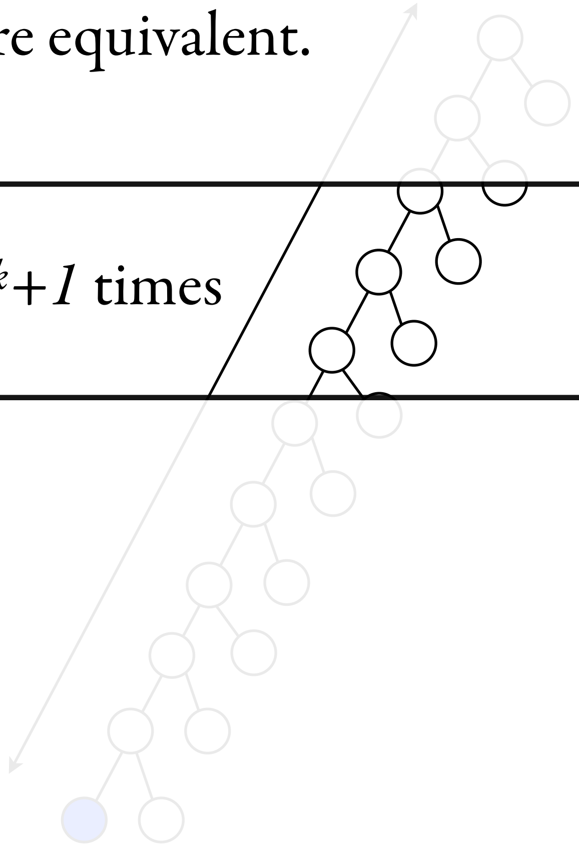

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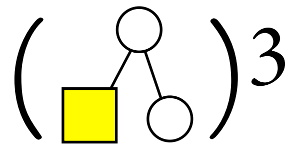

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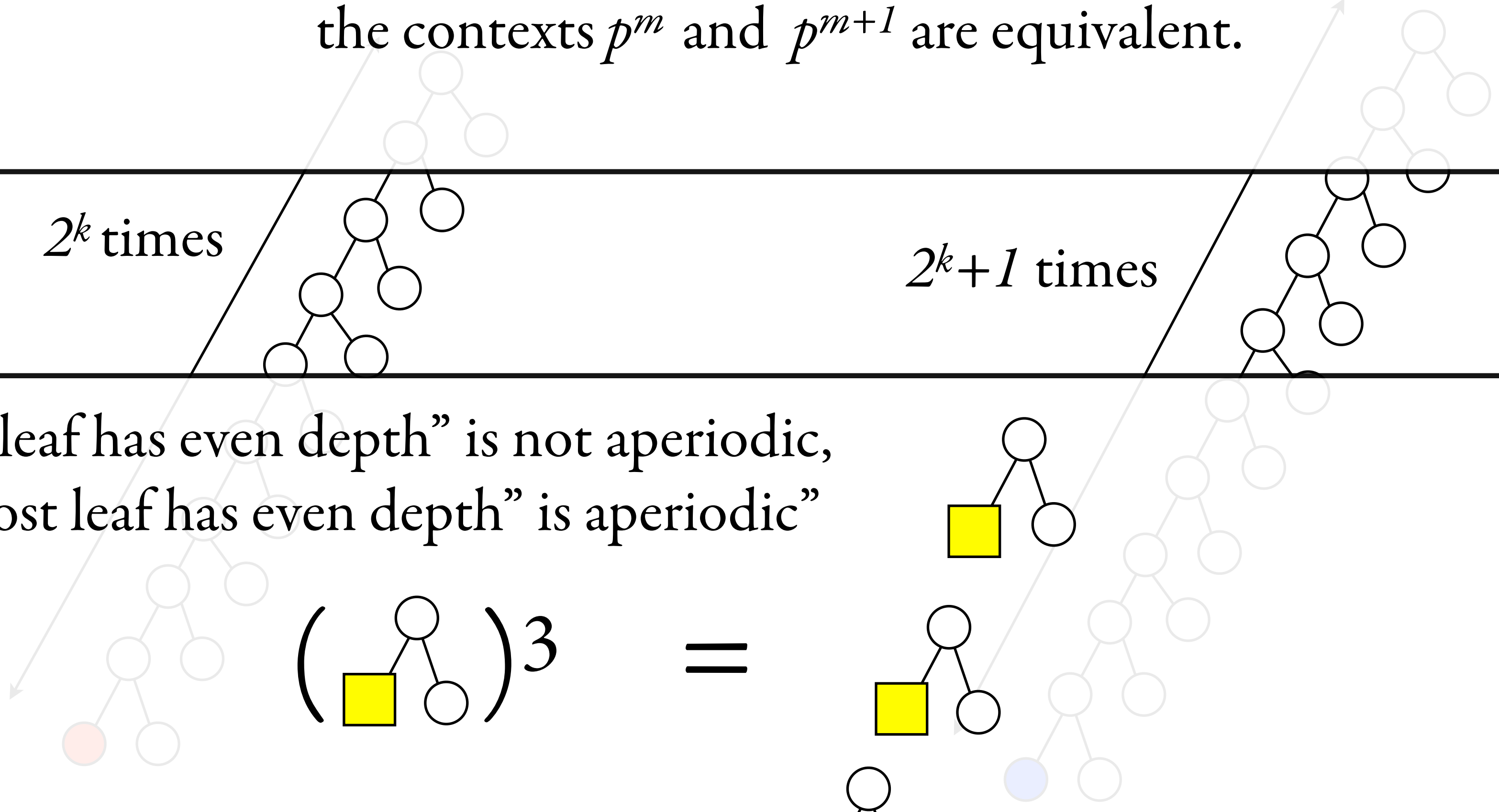

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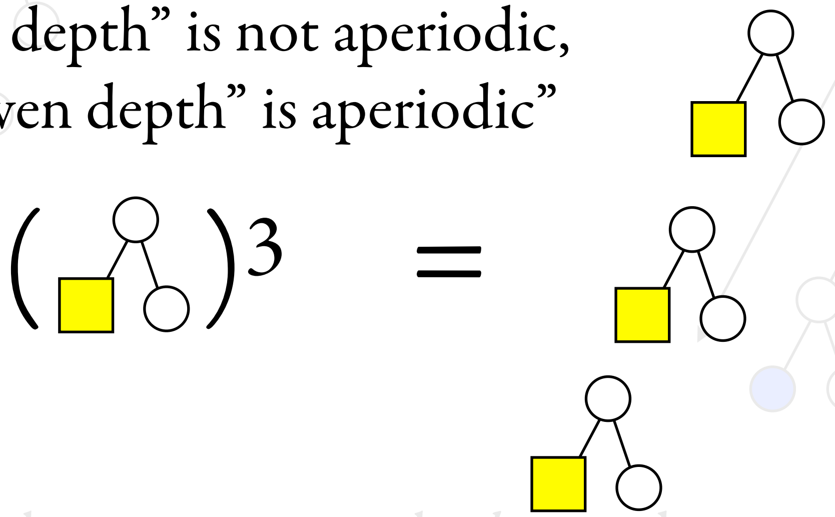

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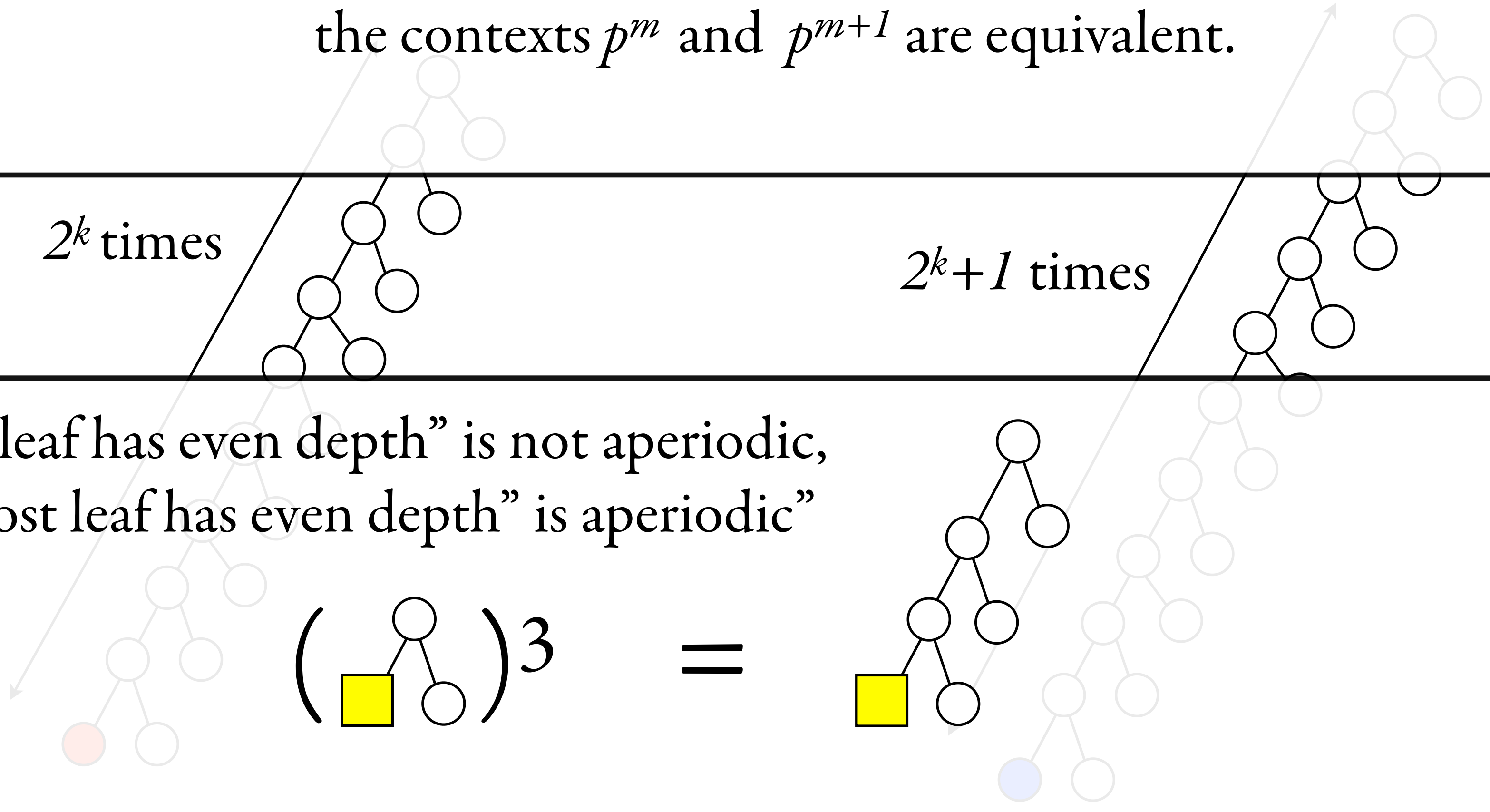

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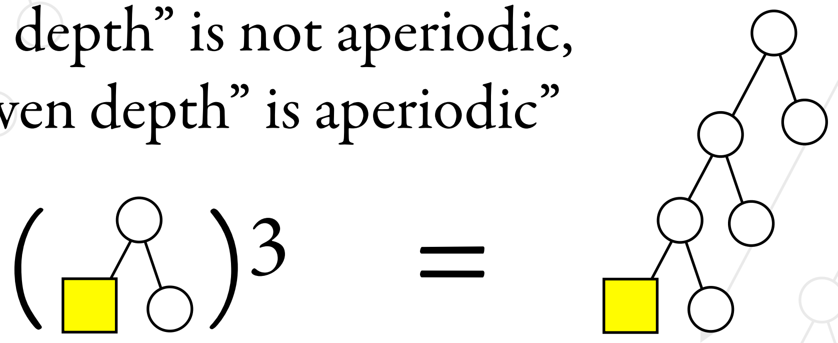

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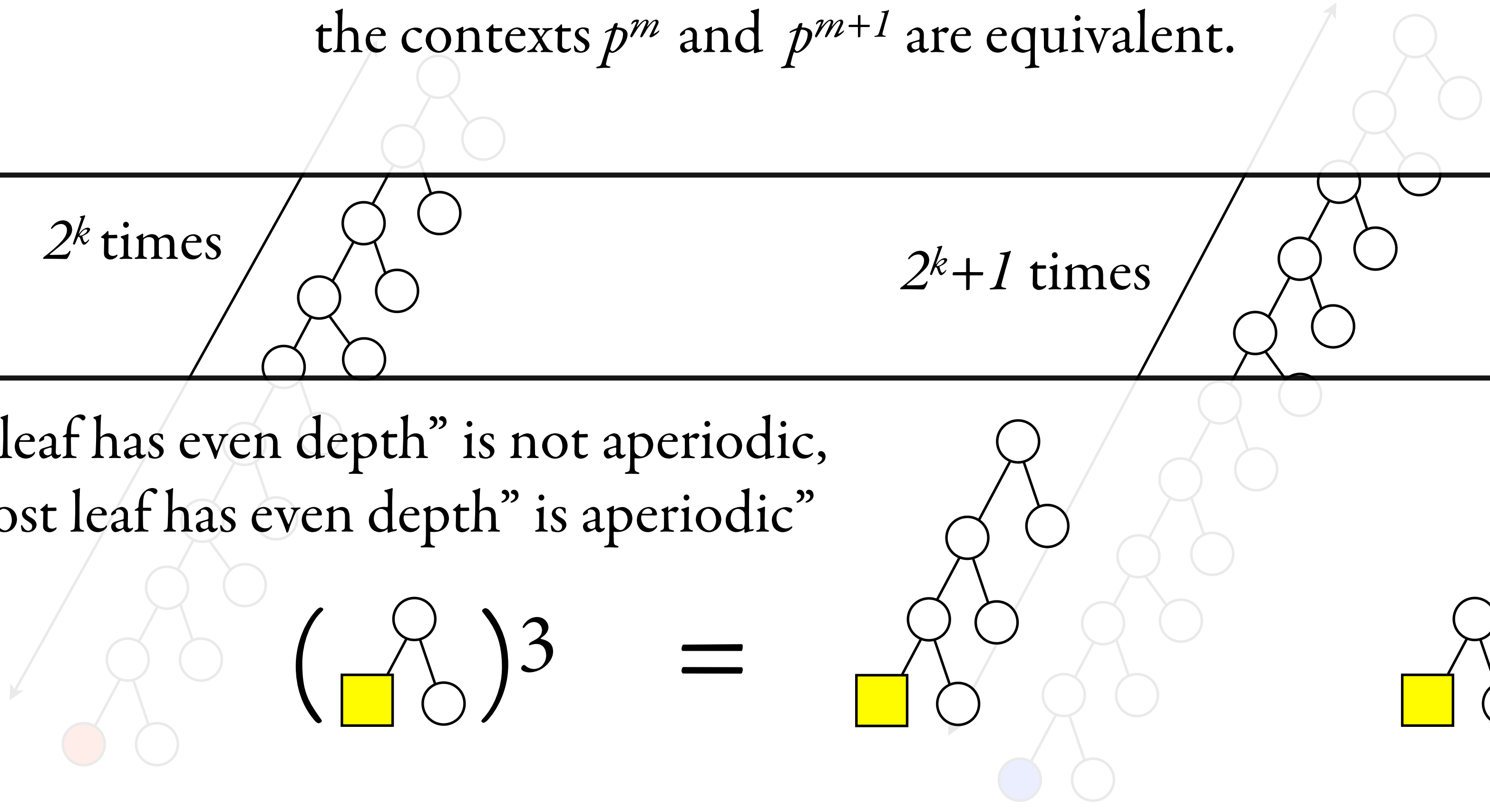

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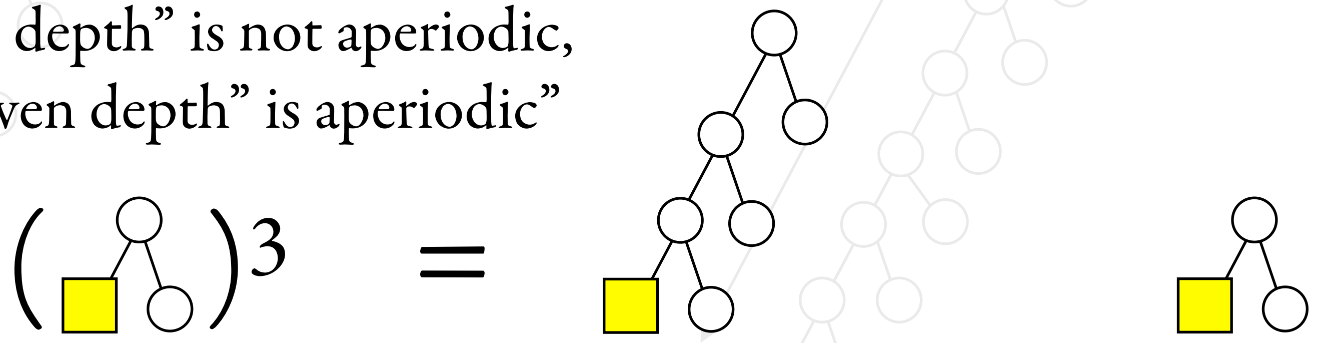
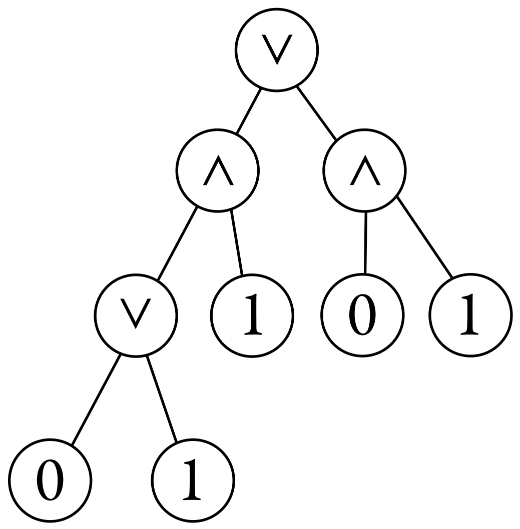


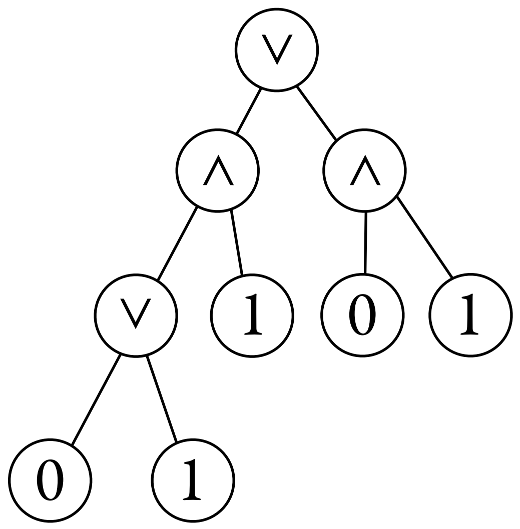
Duplicator survives the k round game on trees

Boolean Expressions



L = “Boolean expressions with value 1”

Boolean Expressions



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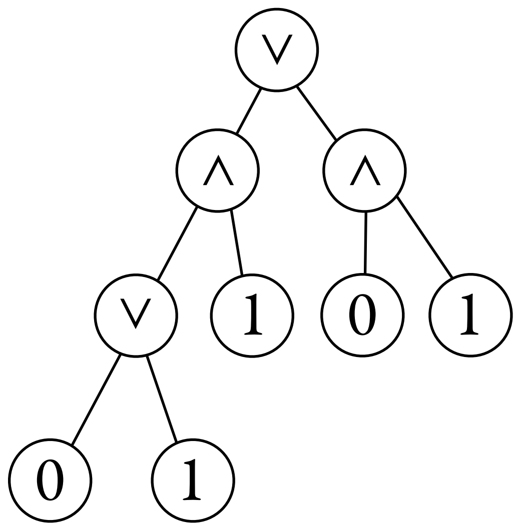

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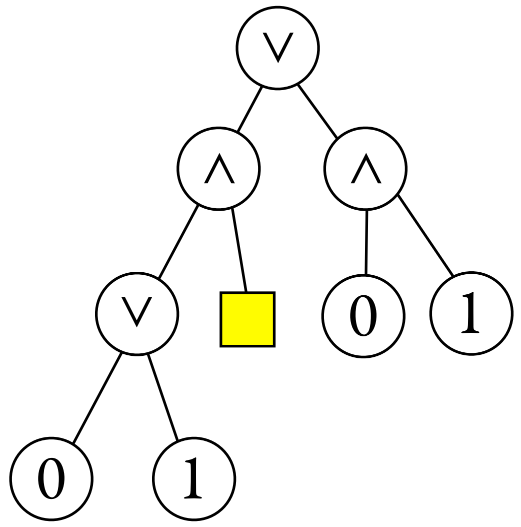

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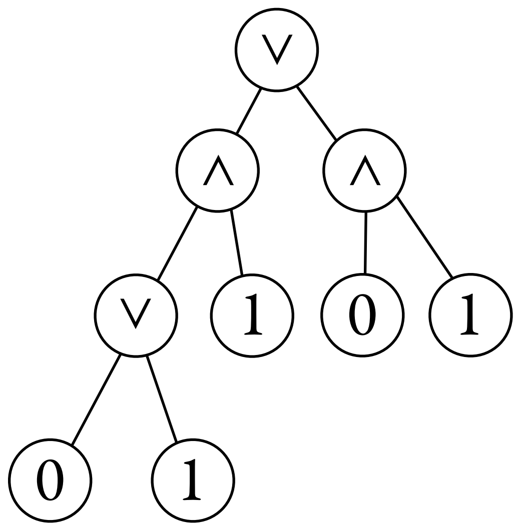

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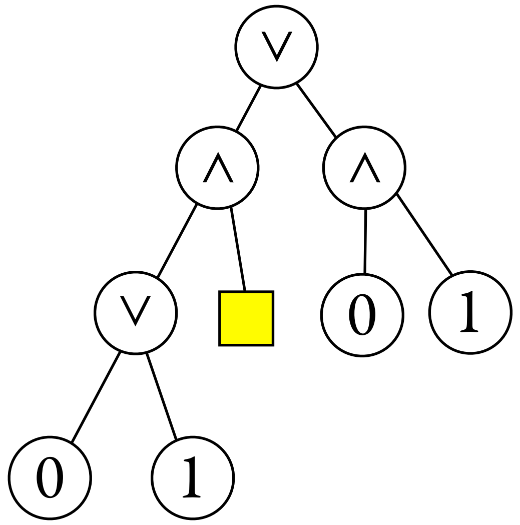
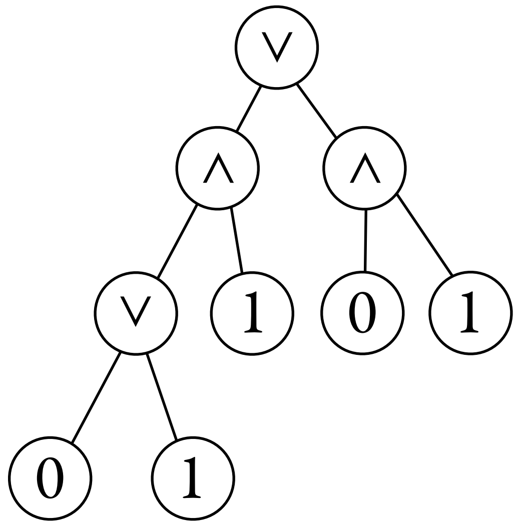


identity

Boolean Expressions



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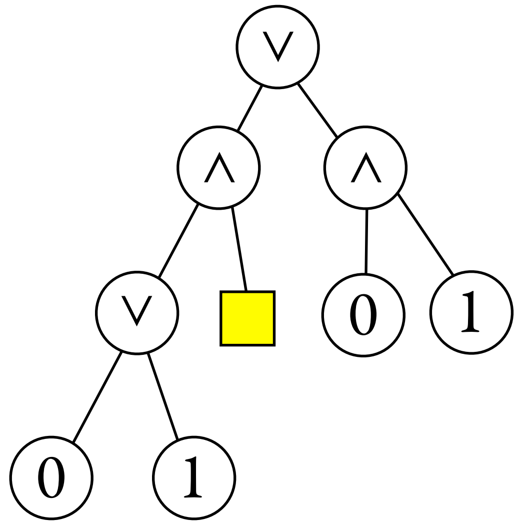

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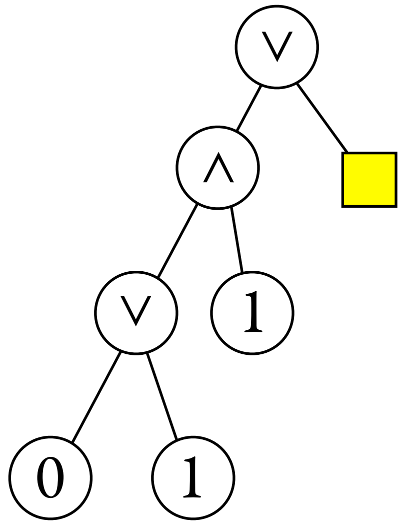

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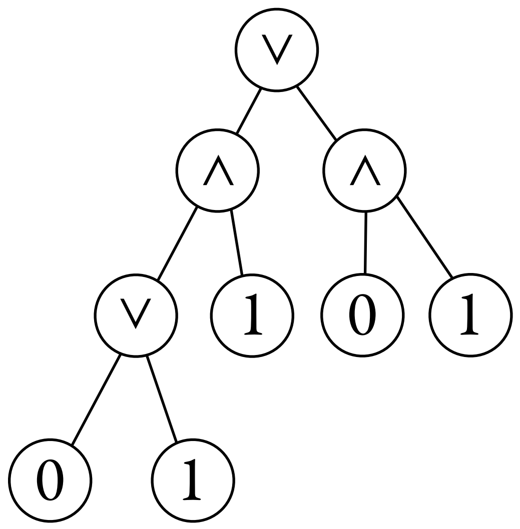

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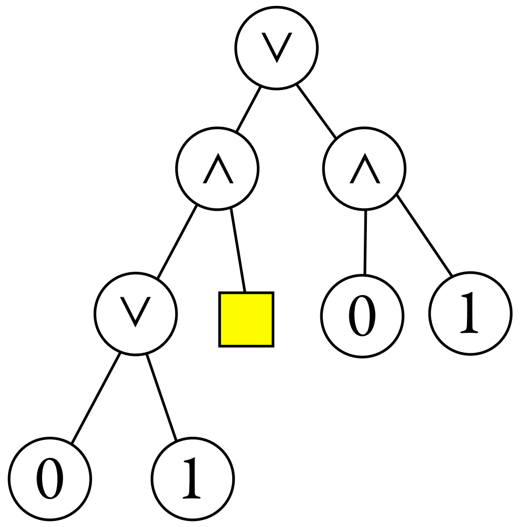

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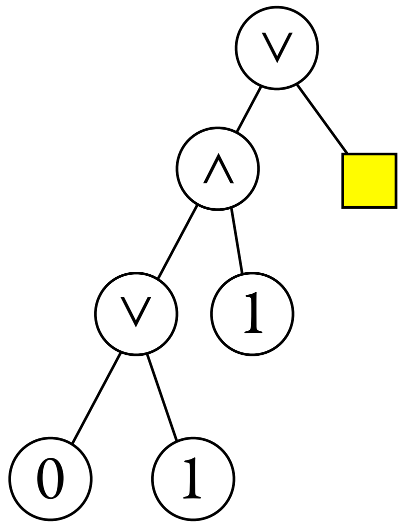
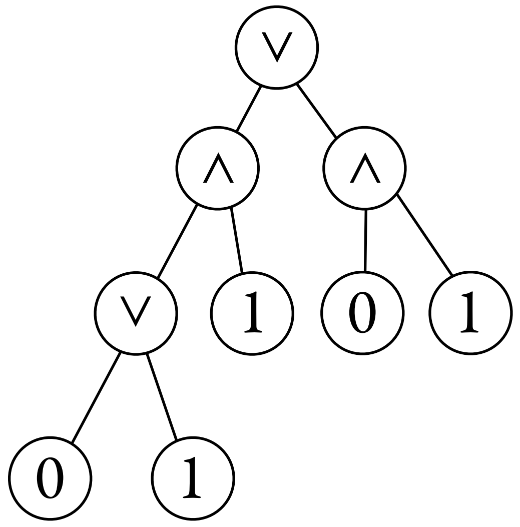


constant 1

Boolean Expressions



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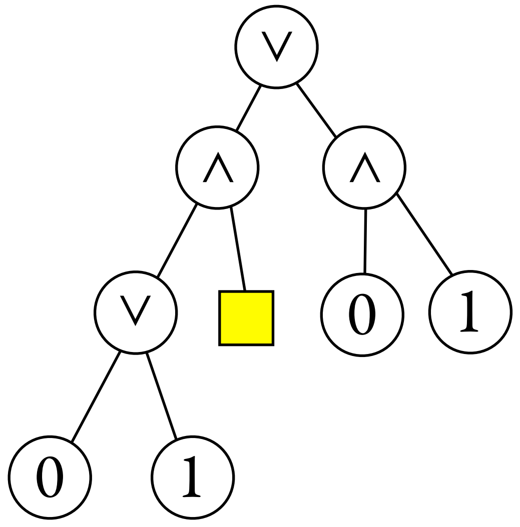

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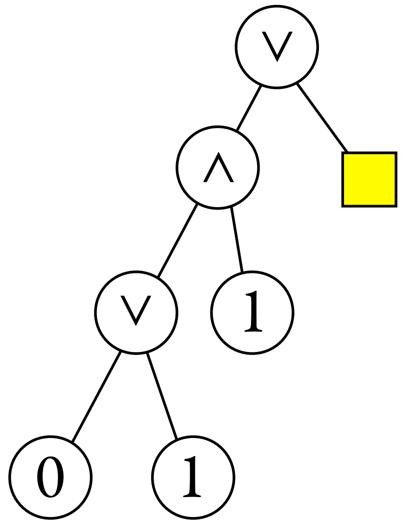

constant 1

generally, monotone functions, which are an aperiodic set.

Monadic- and First-Order Logic for Words

- definition
- weakness of first-order logic
- MSO=regular

Monadic- and First-Order Logic for Trees

- definition
- problems with parity
- problems with aperiodicity

Transitive Closure Logic and Regular Expressions

Monadic- and First-Order Logic for Words

definition
weakness of first-order logic
MSO=regular

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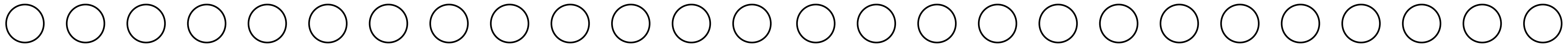
Transitive Closure Logic and Regular Expressions

transitive closure logic for words...
...and for trees
regular expressions for trees

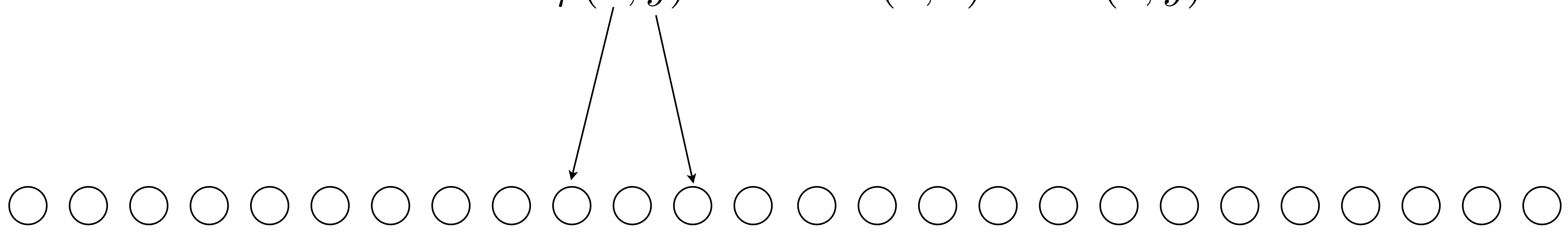
Transitive closure logic

Transitive closure logic



Transitive closure logic

$$\varphi(x, y) = \exists z\ suc(x, z) \wedge suc(z, y)$$



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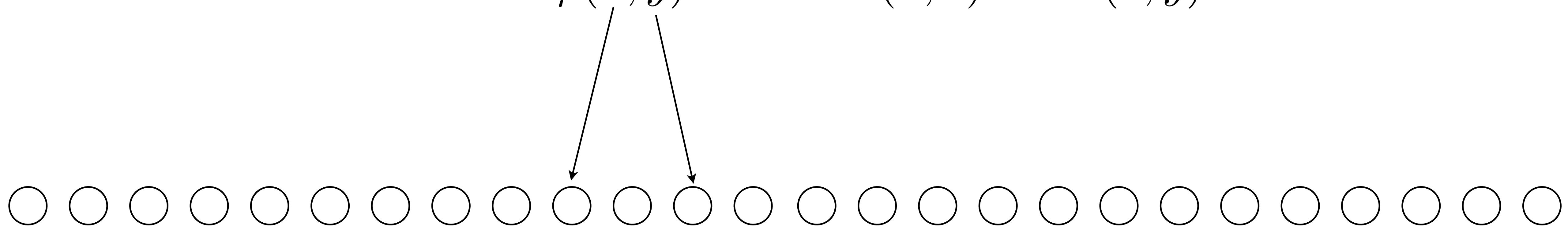

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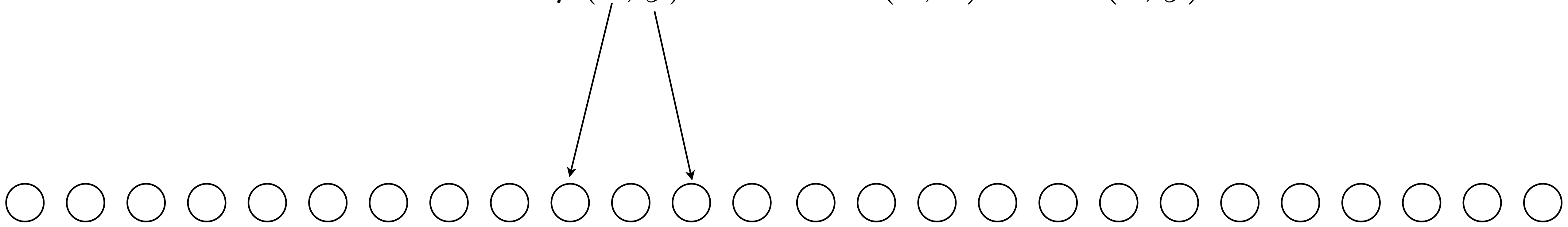

$$\varphi^*(x, y)$$
$$\big(TC\varphi(x, y)\big) \ (x, y)$$

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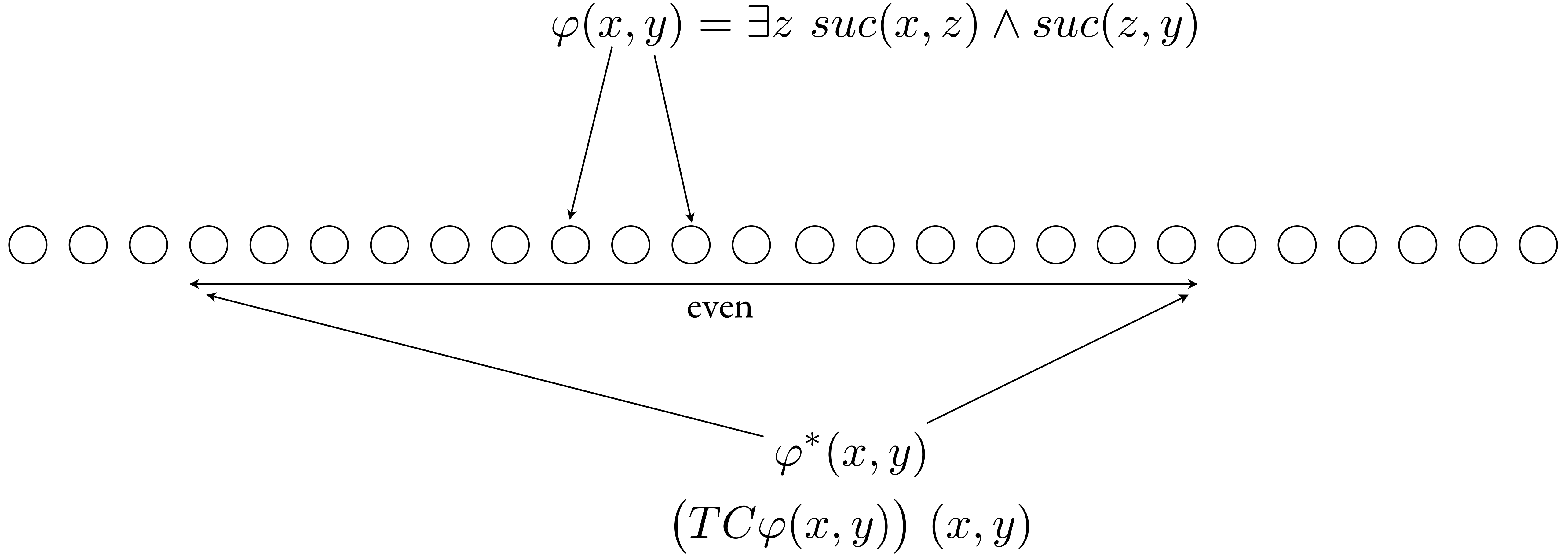

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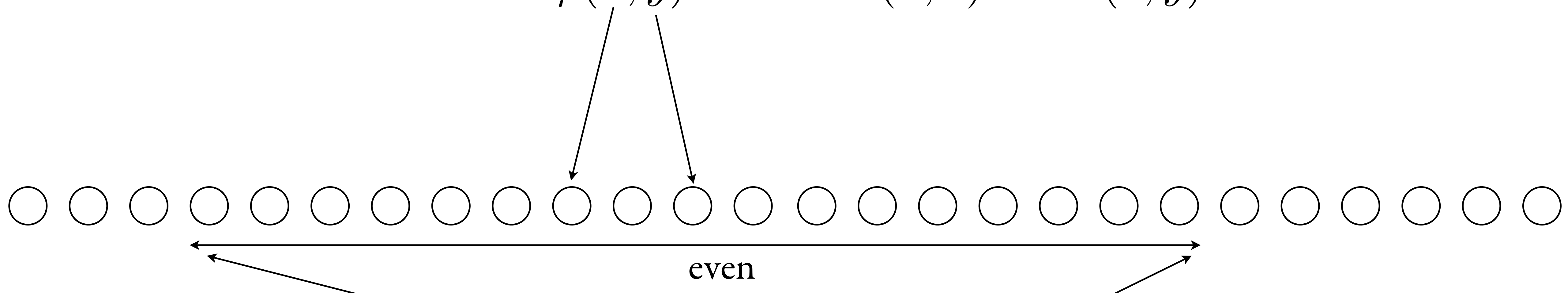

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Fact.
Transitive closure logic is a fragment of MSO.

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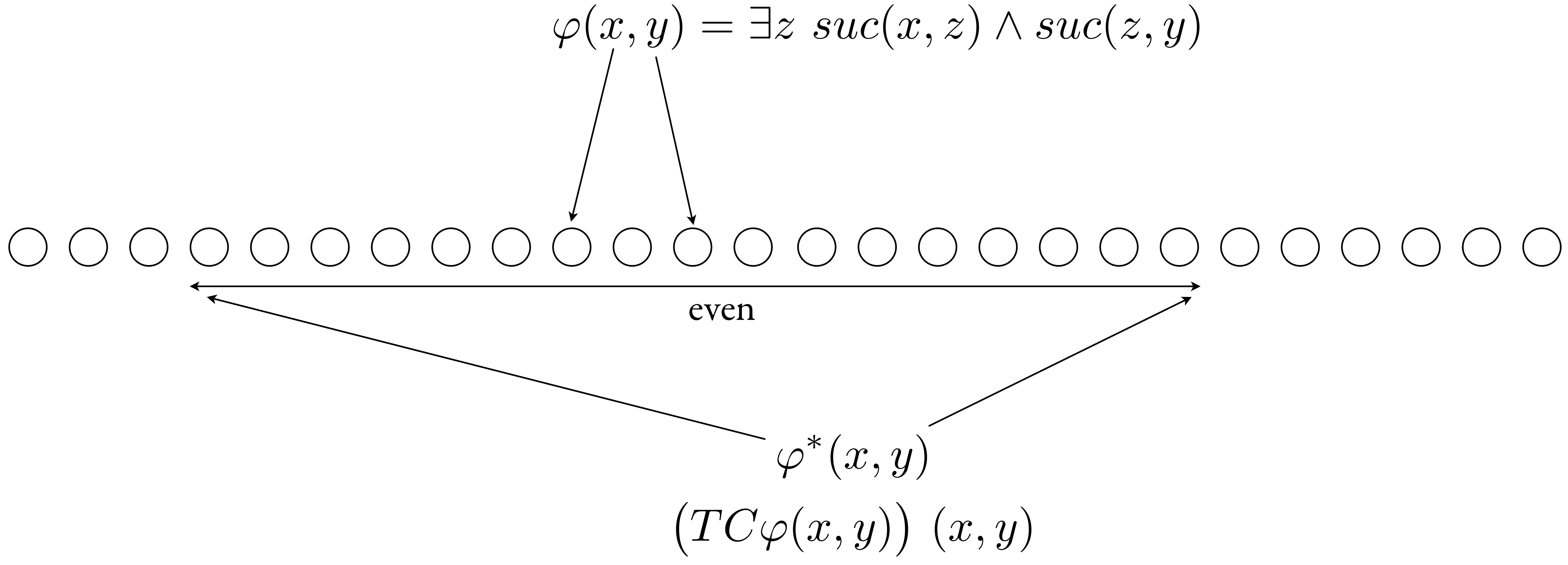

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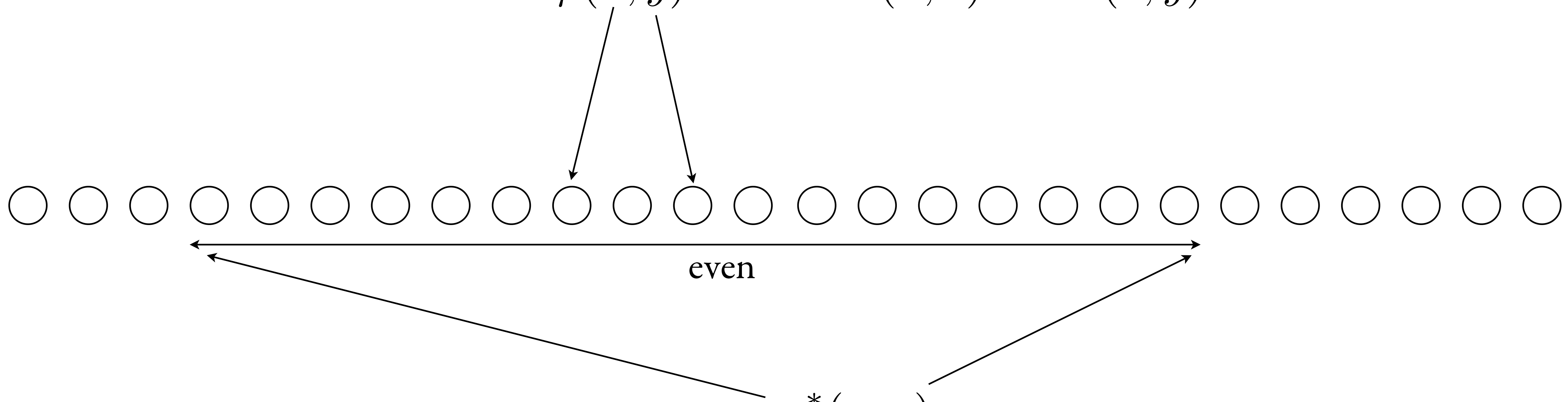
X is closed under φ

$$\varphi^*(x,y) \quad \text{iff} \quad \forall X \left\{ \begin{array}{c} \forall z_1 \forall z_2\ \ \big(\varphi(z_1,z_2)\ \wedge\ z_1 \in X\big)\ \Rightarrow\ z_2 \in X \\ \Downarrow \\ x \in X\ \Rightarrow y\ \in X \end{array} \right.$$

Transitive closure logic

$$\varphi(x, y) = \exists z \; suc(x, z) \wedge suc(z, y)$$



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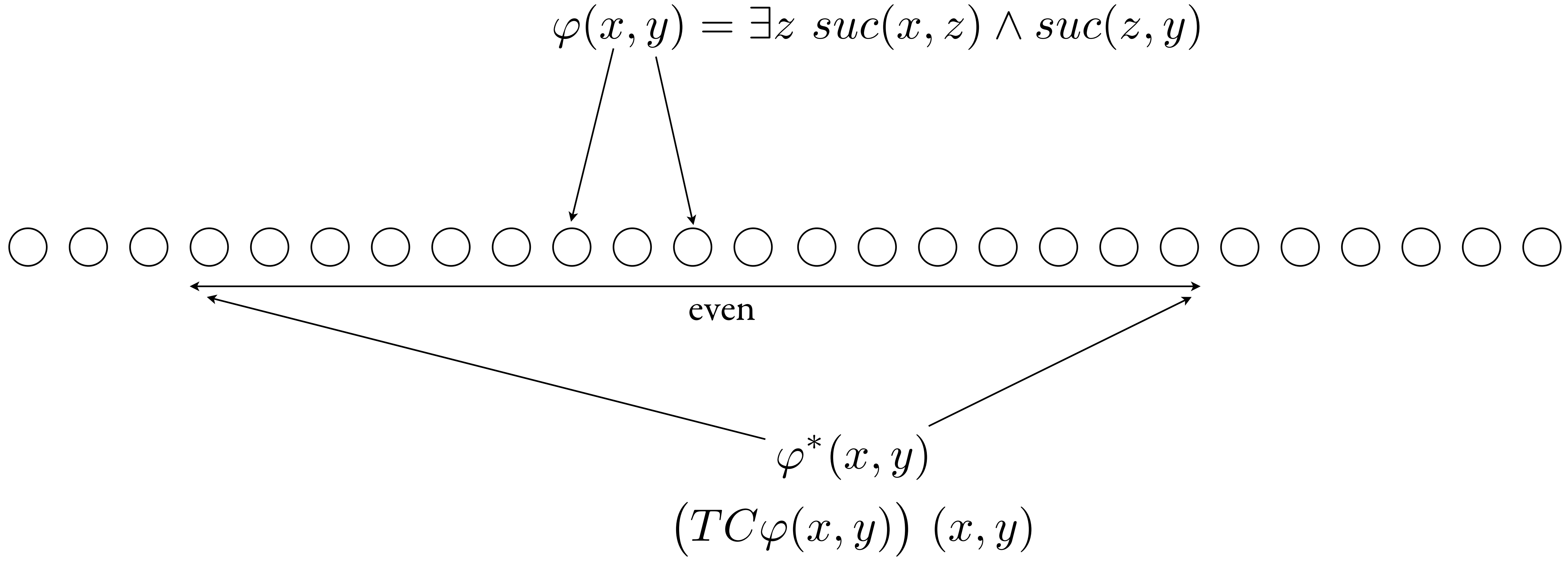
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Fact.
Transitive closure logic is a fragment of MSO.

For every regular expression (on words), there is an equivalent formula of transitive closure logic. Hence, transitive closure logic = MSO for words.

Transitive closure logic

For trees, transitive closure logic is closely related to tree-walking pebble automata, and shares their weaknesses.

Thm. ten Cate, Segoufin `08
For trees, transitive closure logic is less expressive than MSO.

Meta-Corollary.
There is no nice regular expression syntax for regular tree languages.