Tree Automata and Tree Logics

Mikołaj Bojańczyk
(Warsaw)

1 Tree automata

What is a Tree Automaton?

Decision Problems

1 Tree automata
What is a Tree Automaton?
Decision Problems

2 Logic
Logic for Words
Logic for Trees
Transitive Closure Logic

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3 Temporal Logics
Temporal Logic for Words
Temporal Logic for Trees
XPath

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4 Tree-Walking Automata, 1

- Tree-Walking Automata
- Expressive Power
- Pebble Automata

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5 Tree-Walking Automata, 2

Tree-Walking Automata Cannot Be Determinized

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What is a Tree Automaton?

–definition and examples

–determinism, bottom-up vs top-down

–minimization

–closure properties

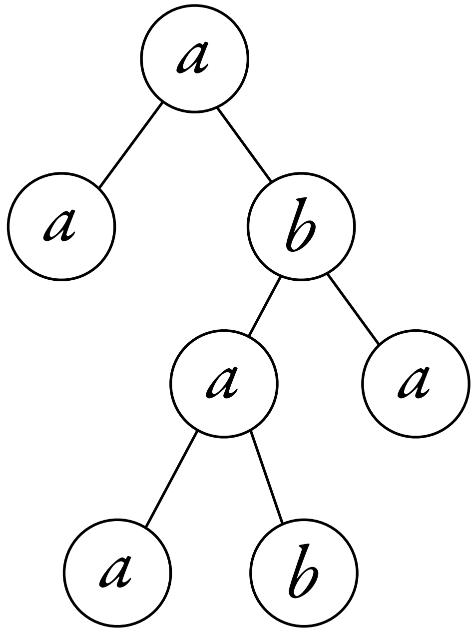
Decision Problems

–emptiness

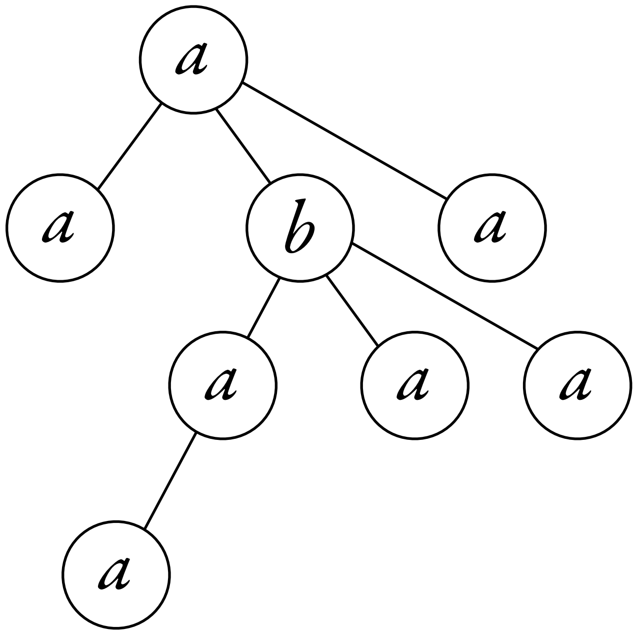
–membership

–universality

Trees are finite and labeled.



binary tree:
each node has 0 or 2 children



unranked tree

We will use automata to express properties such as...

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“there is at least one a in the tree”

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“there is at least one *a* in the tree”

“every node with label *a* has at most one *b* child”

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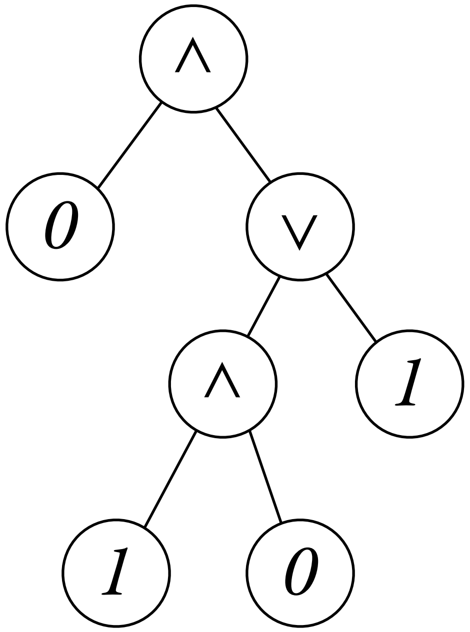

“boolean expressions with value *1*”

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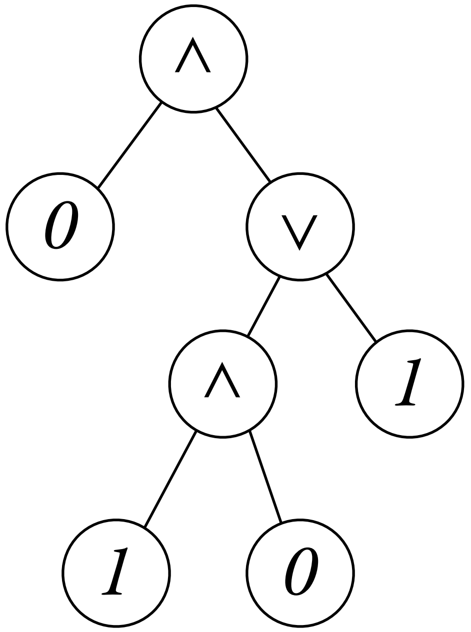
"there is at least one *a* in the tree"

"every node with label *a* has at most one *b* child"

"even number of nodes"



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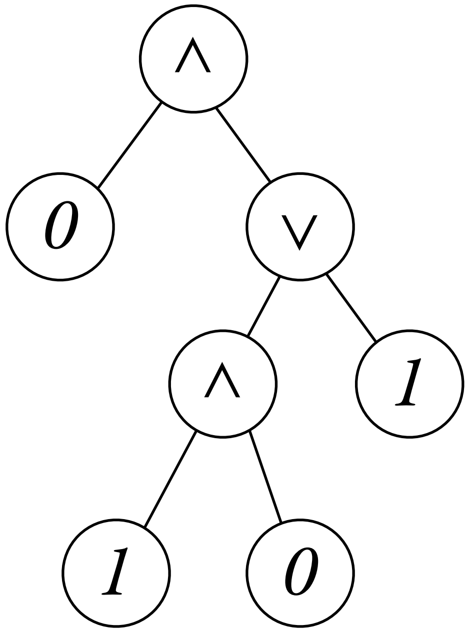

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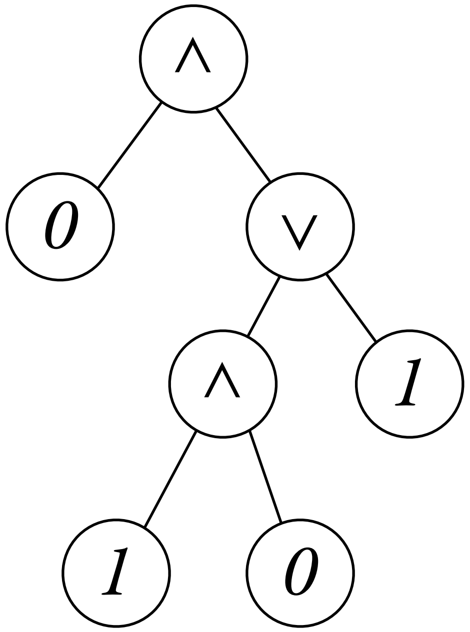

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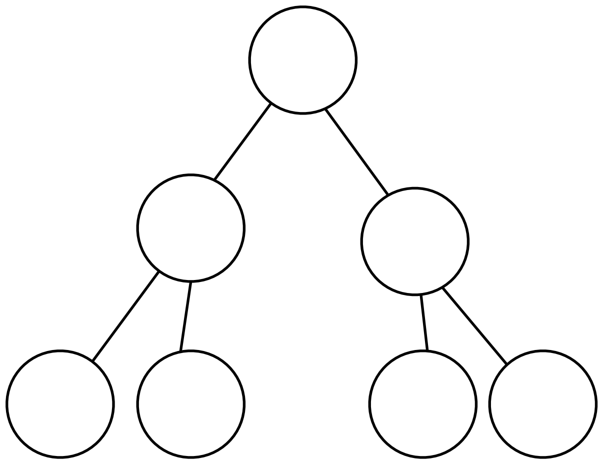


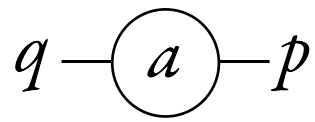
"balanced trees"

A word automaton

This definition best for nondeterministic automata.
No difference between bottom-up and top-down.

A word automaton

automaton transitions



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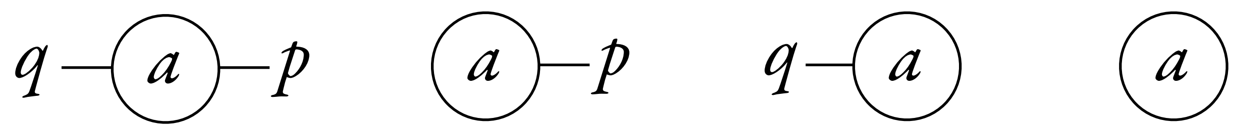

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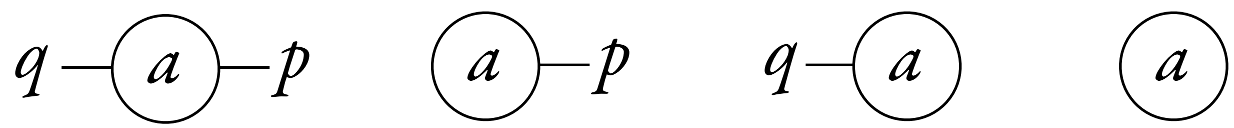

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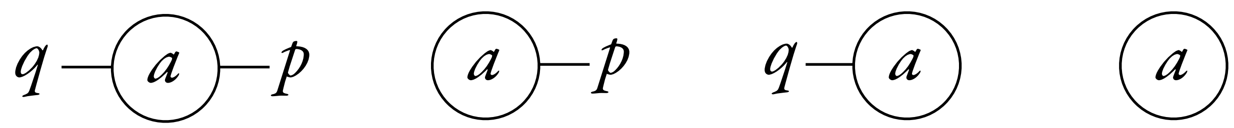

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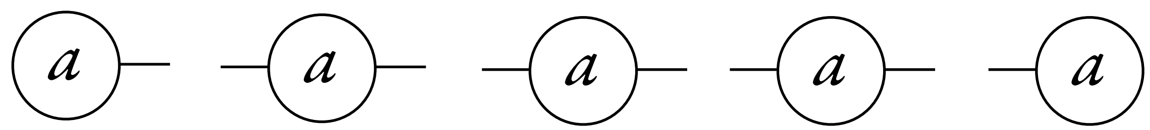

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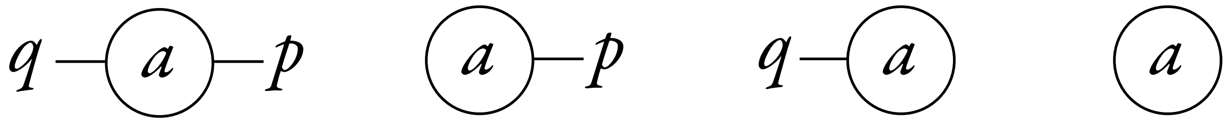

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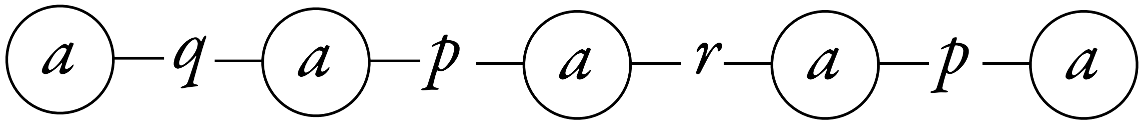

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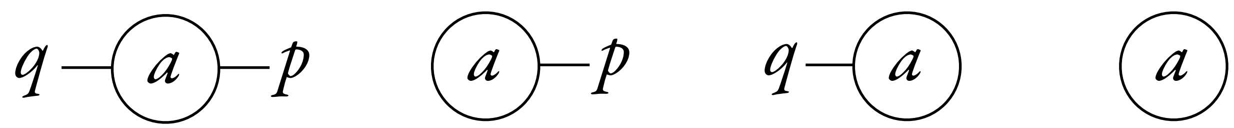

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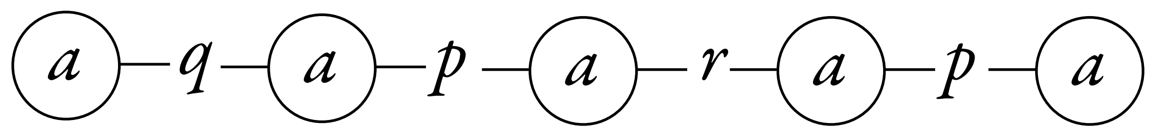

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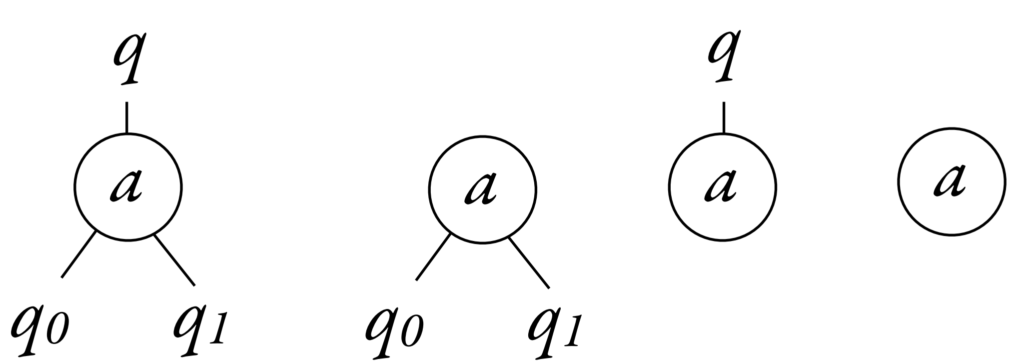

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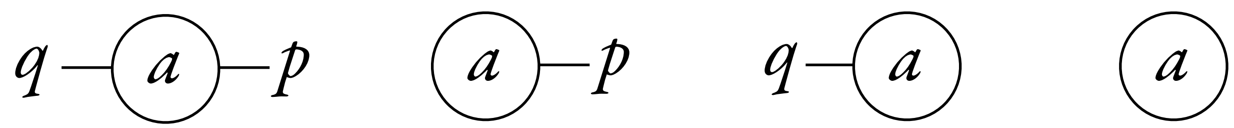

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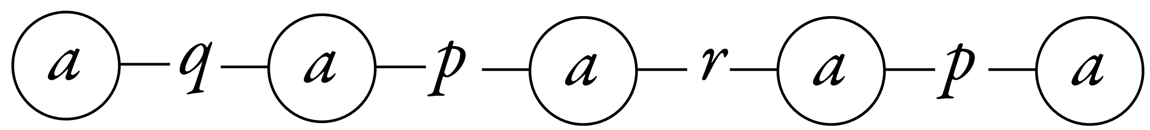

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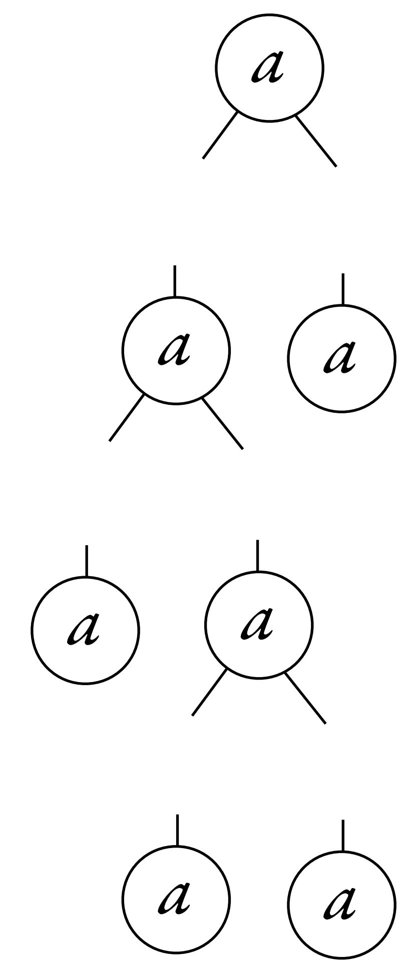

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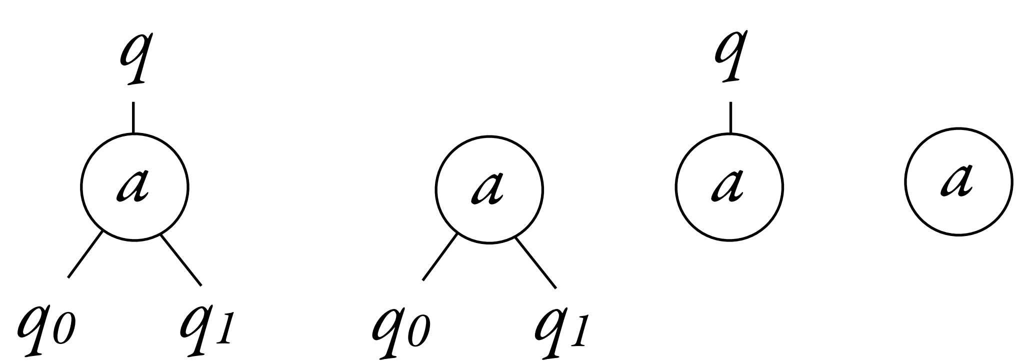

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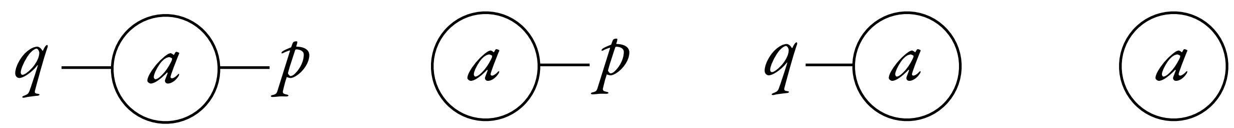

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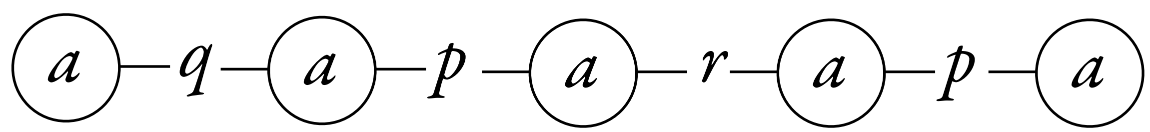

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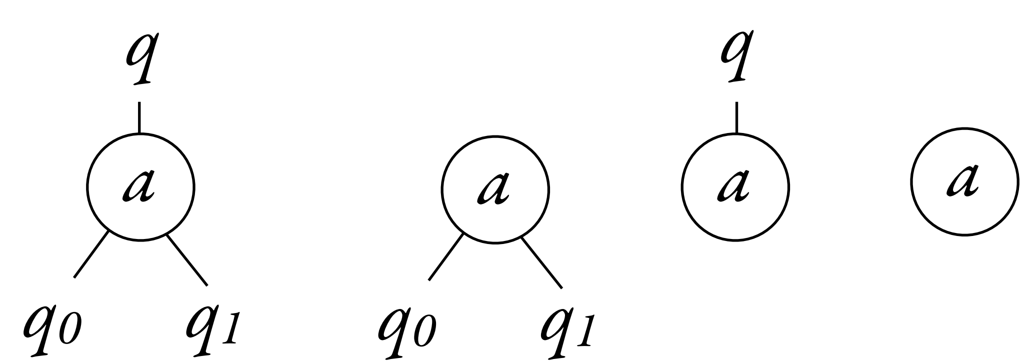

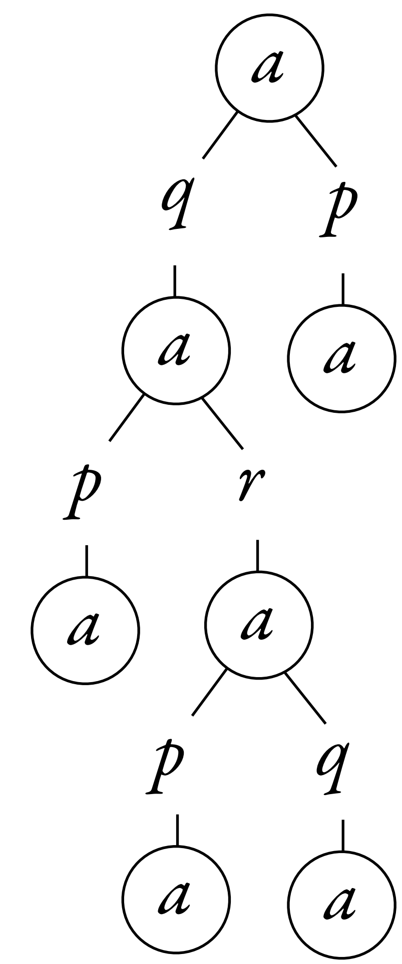

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Examples

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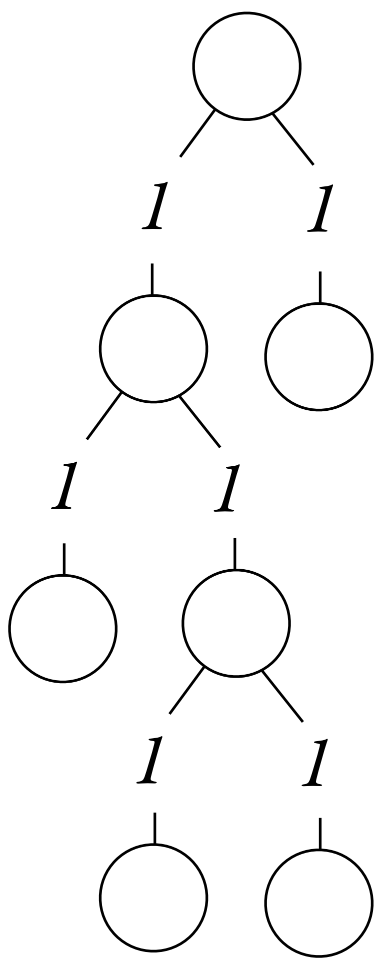
"even number of nodes"

states: 0,1



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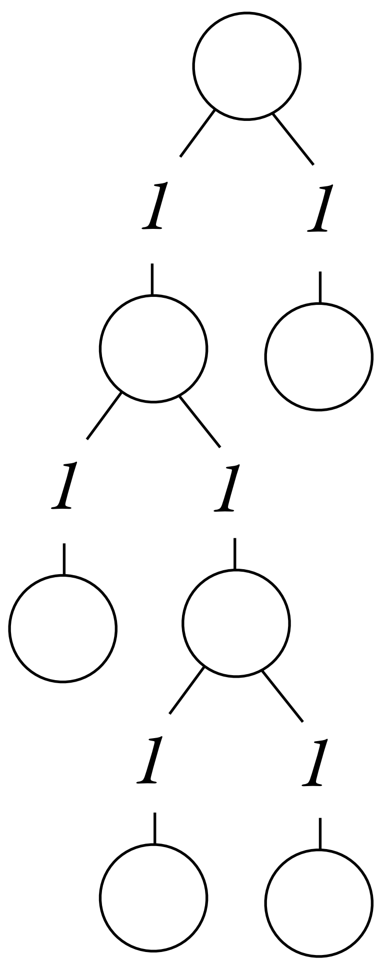

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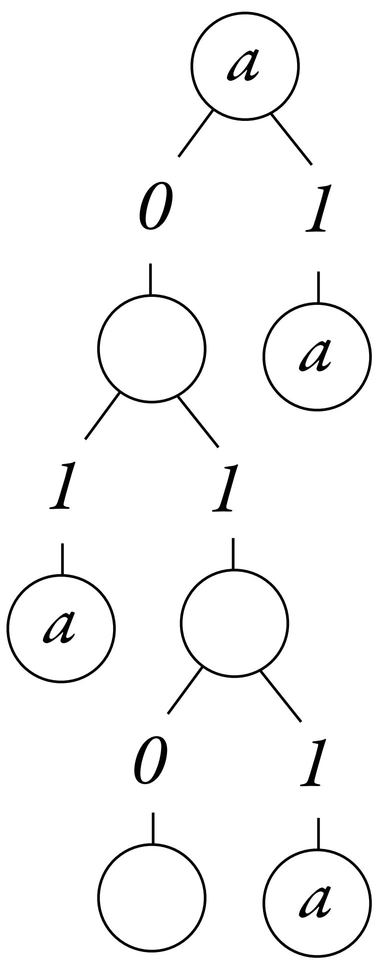

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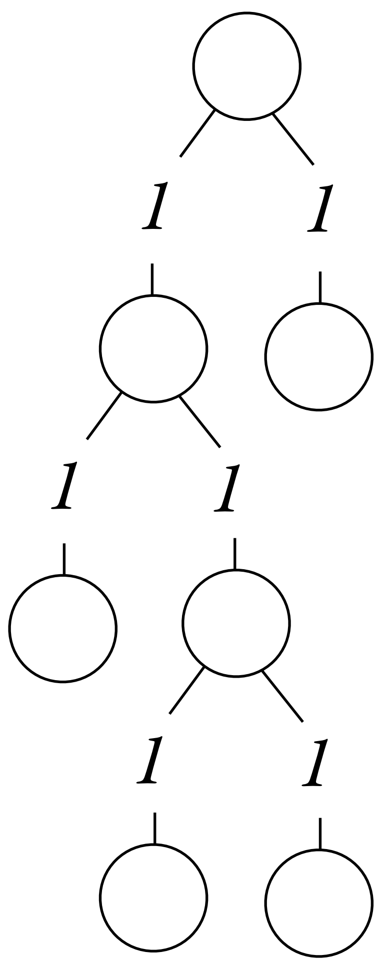

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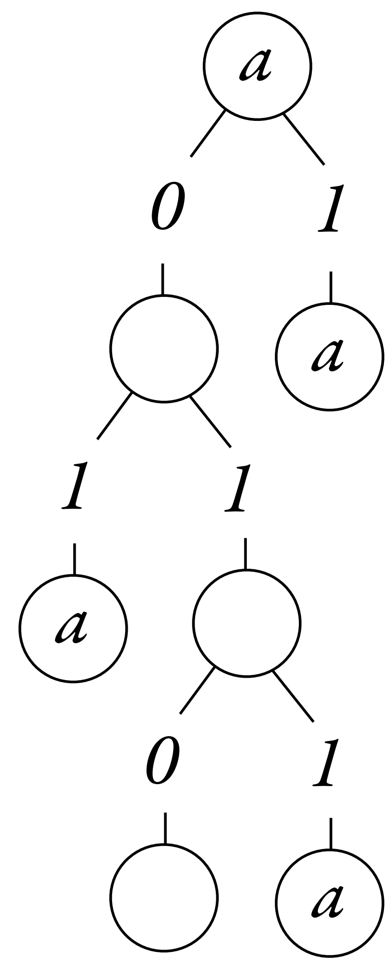

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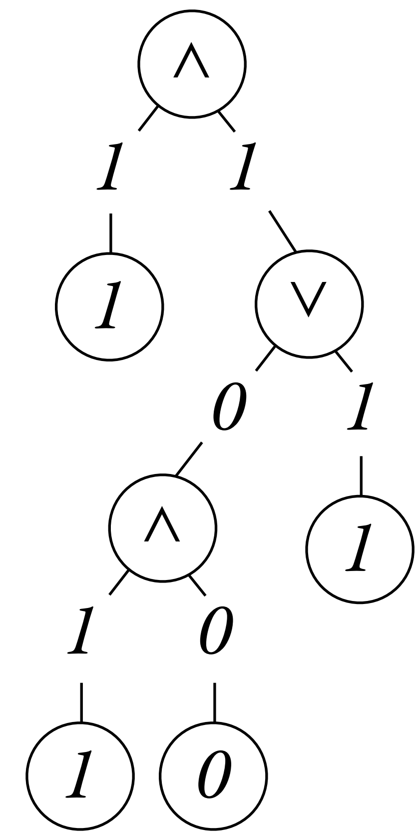
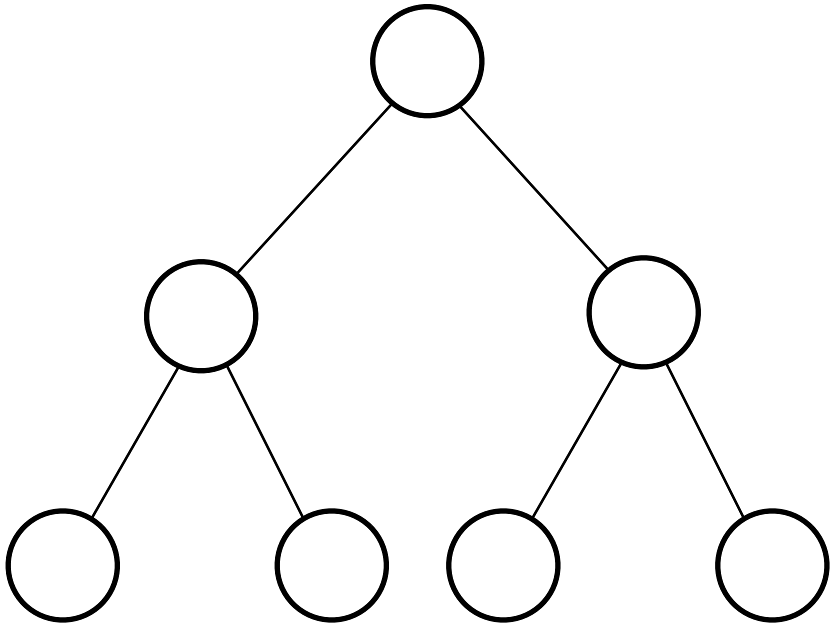


Deterministic automata

Deterministic automata

Top-down



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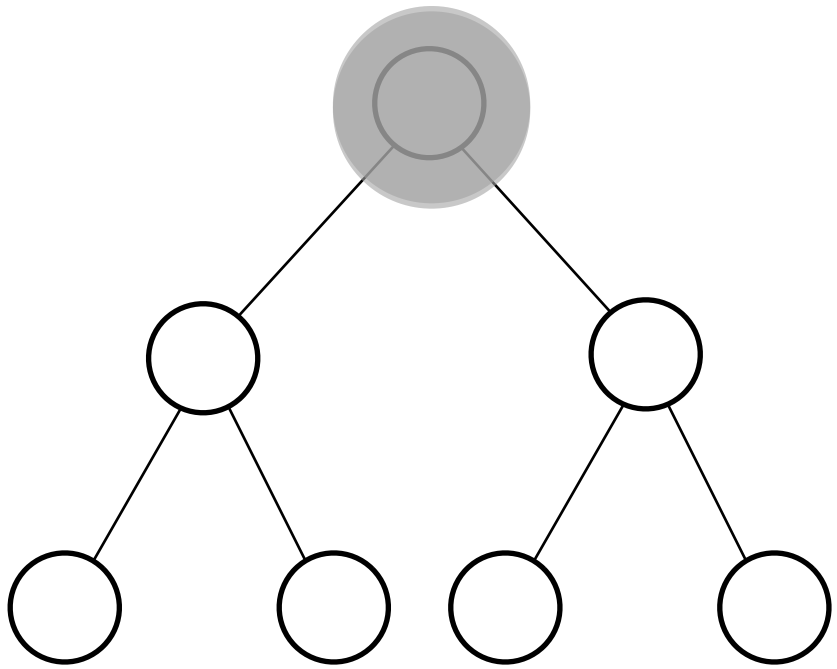

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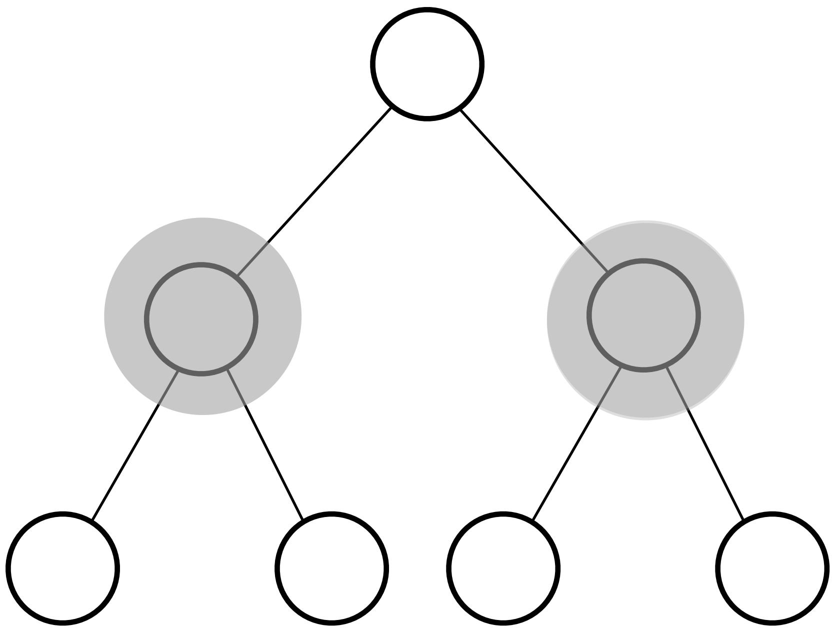

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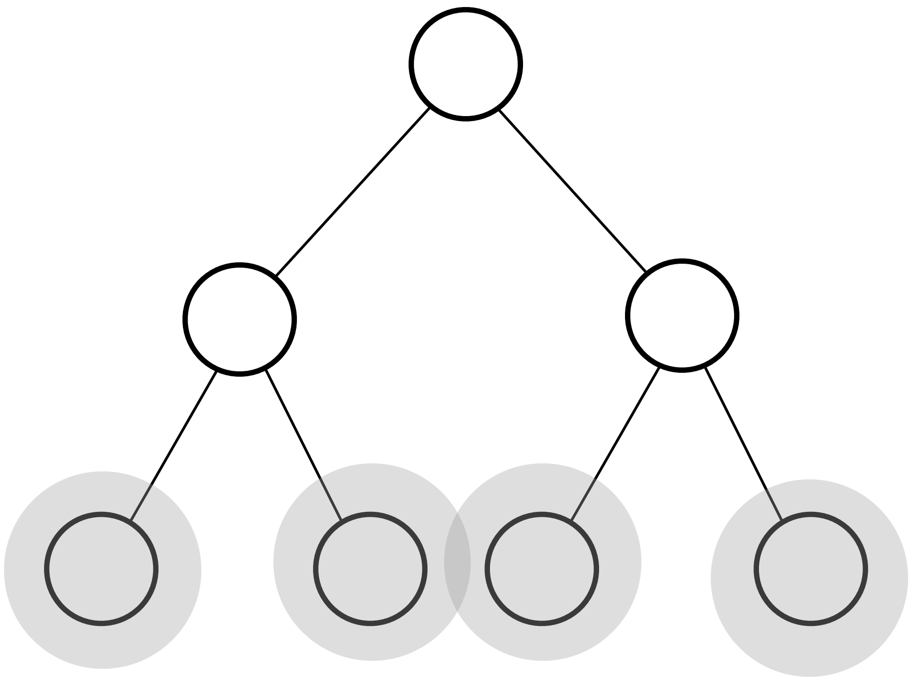

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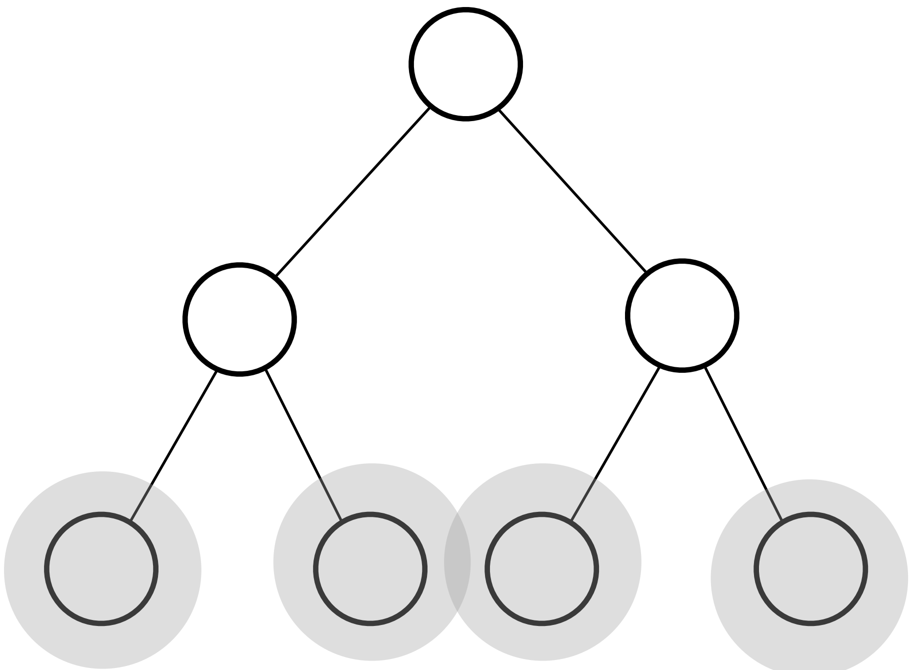

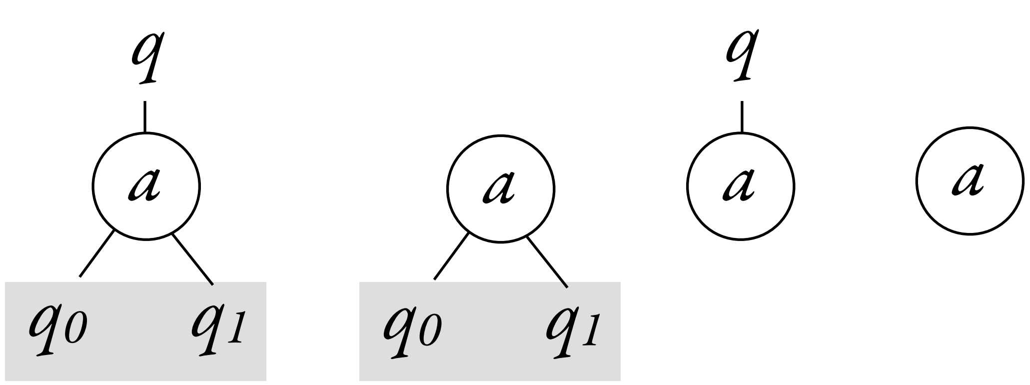

determined by other components

Deterministic automata

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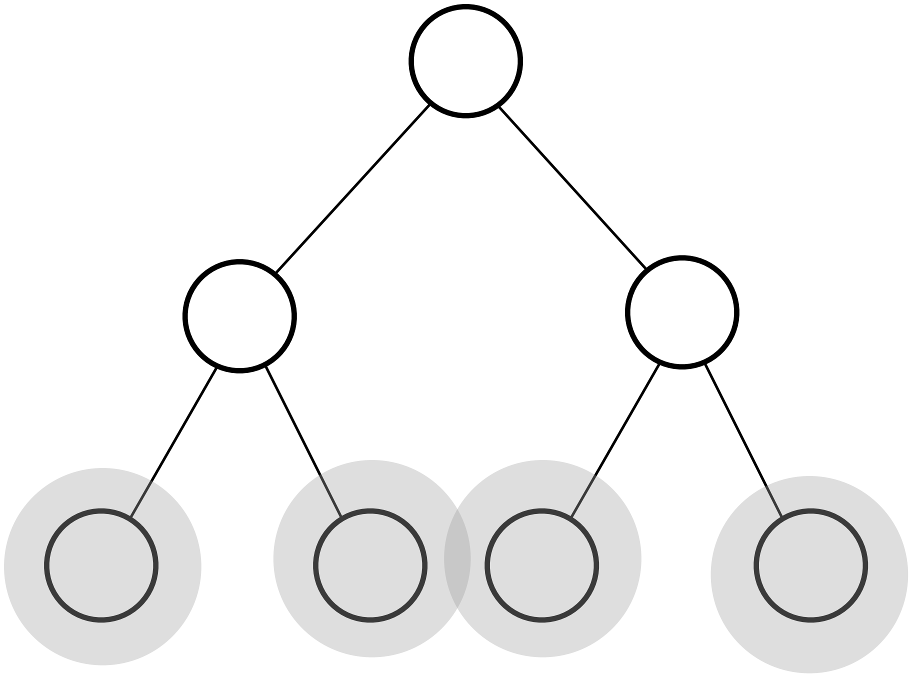
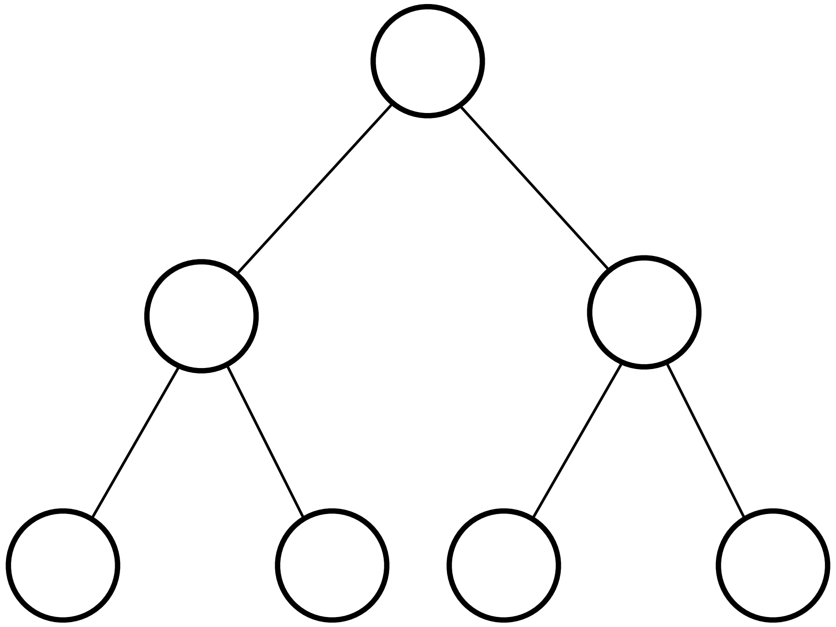
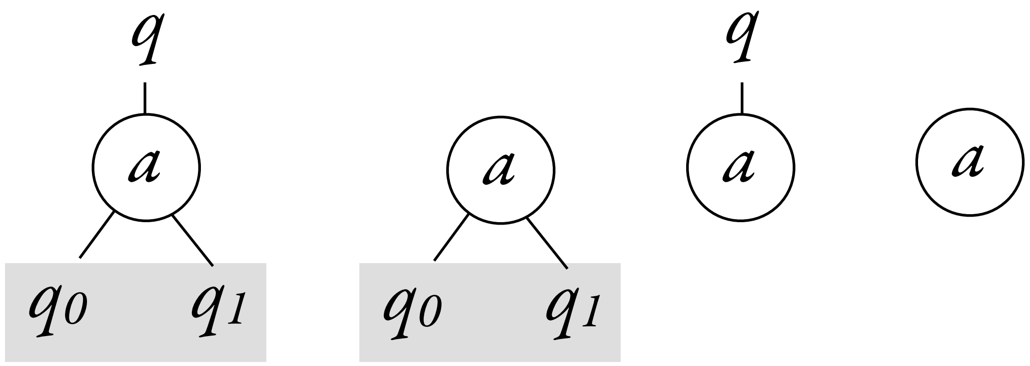


Bottom-up





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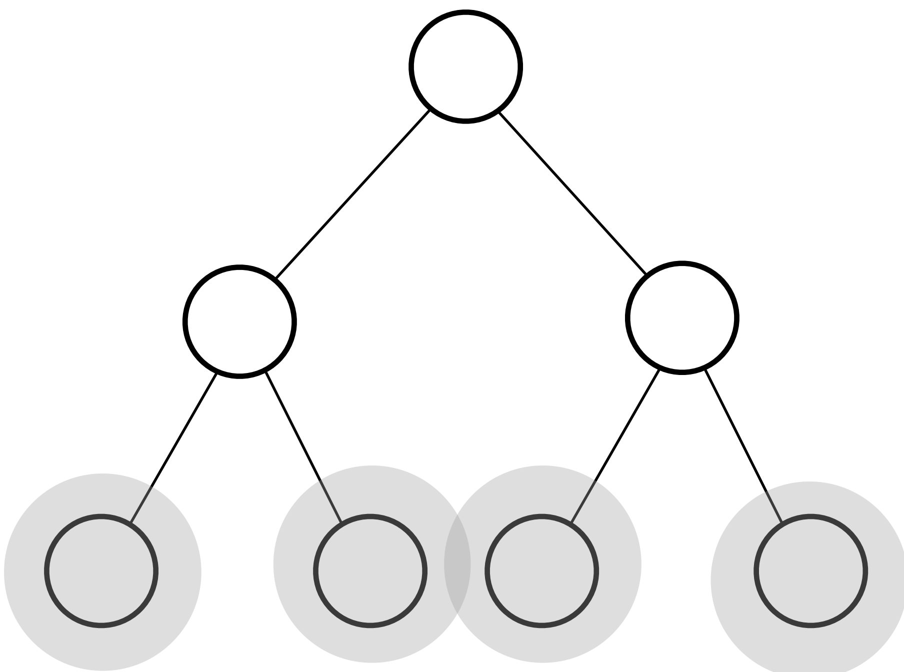

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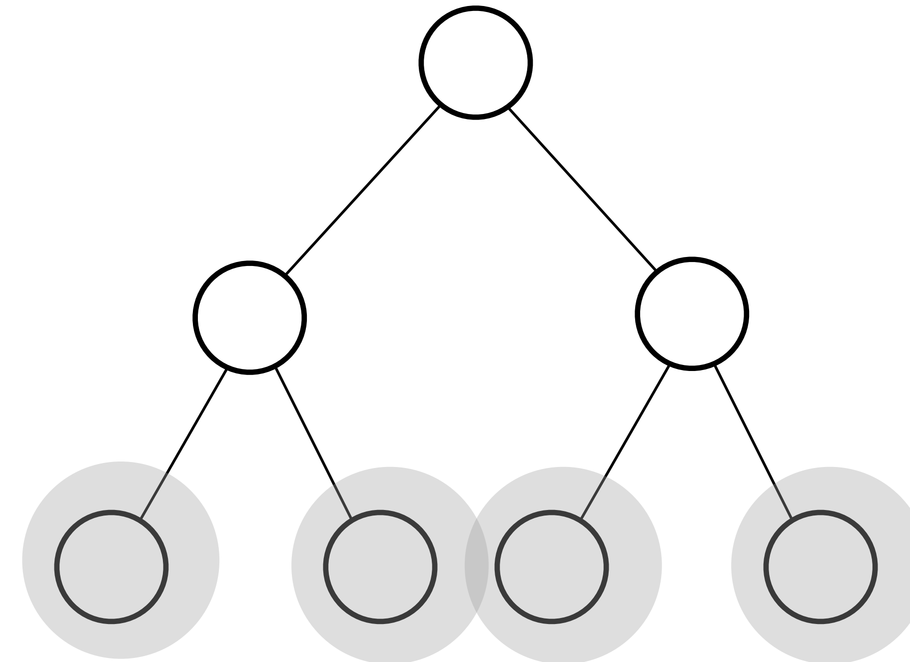

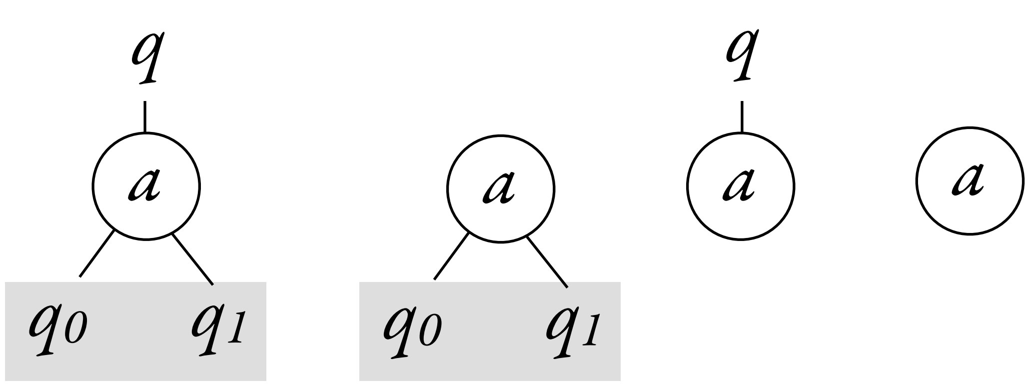

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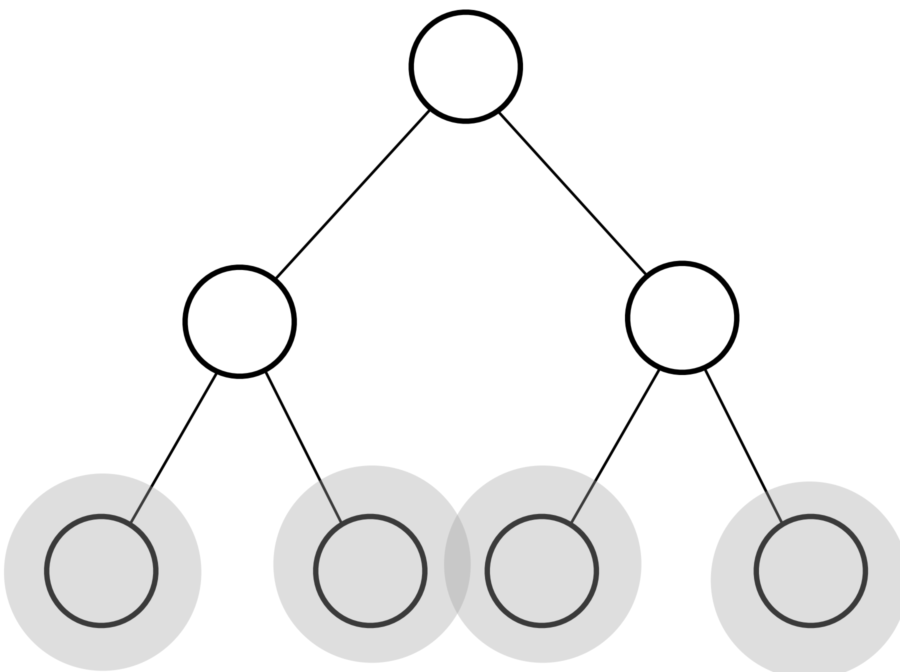

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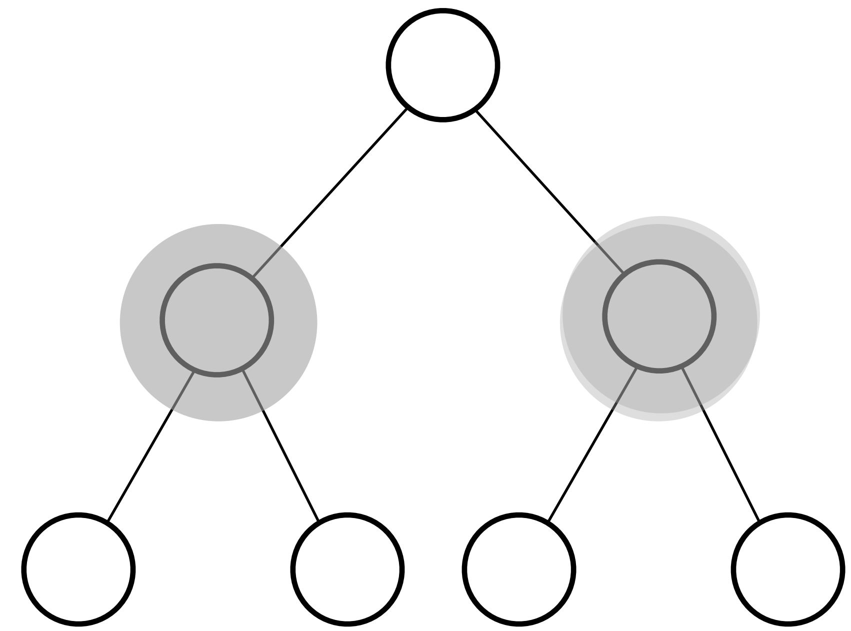

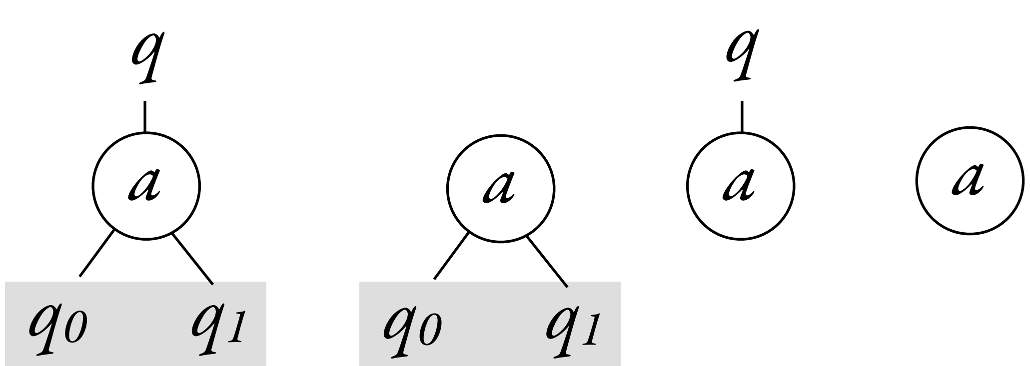

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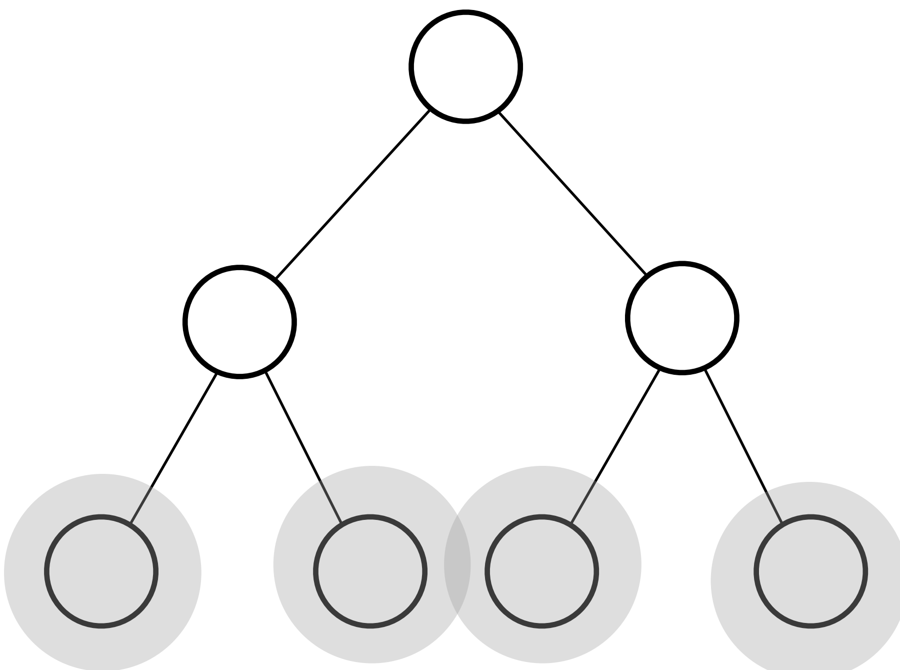

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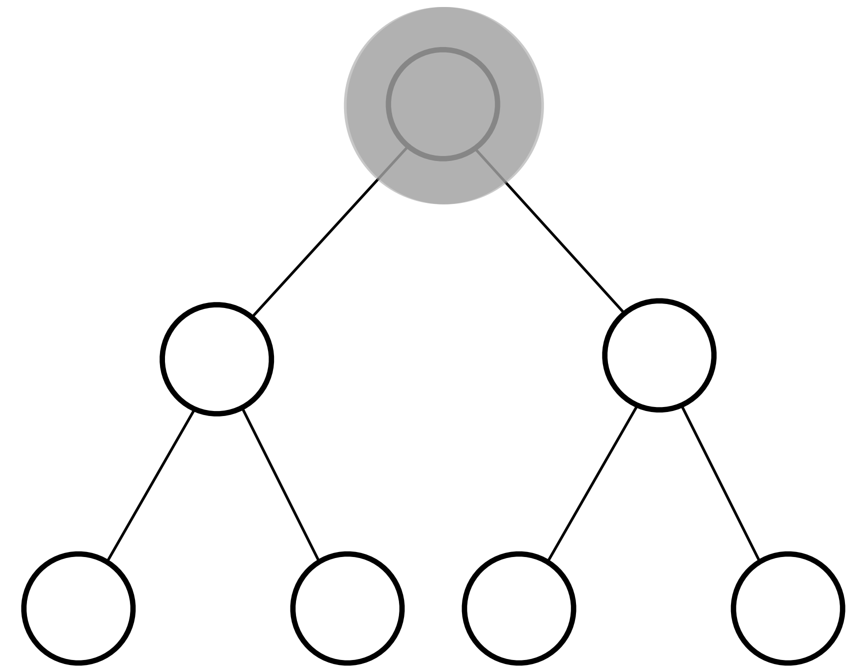

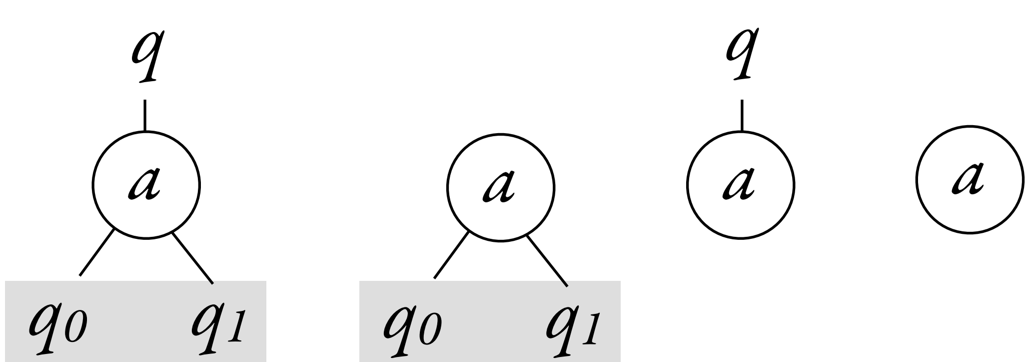

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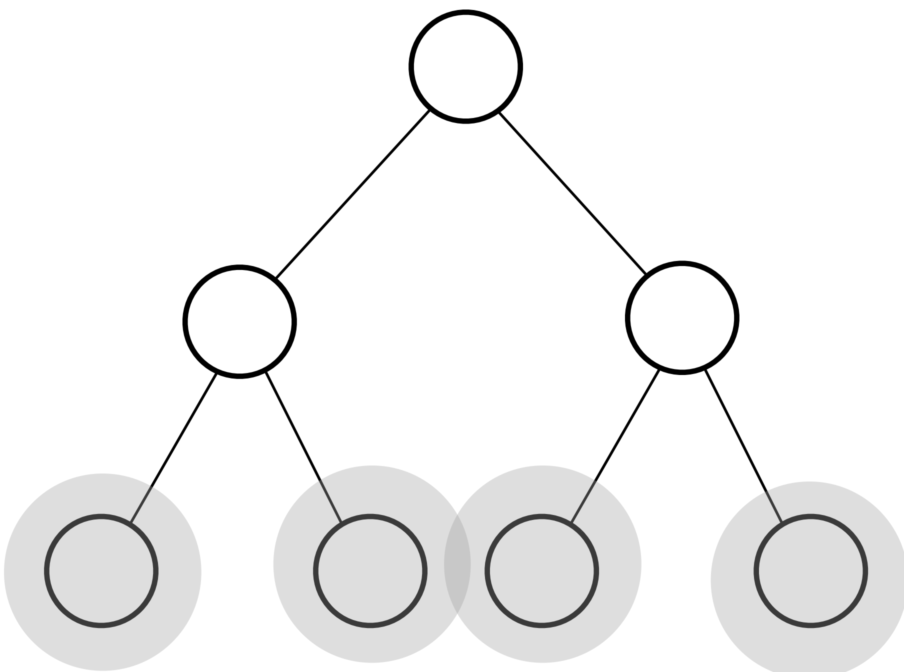

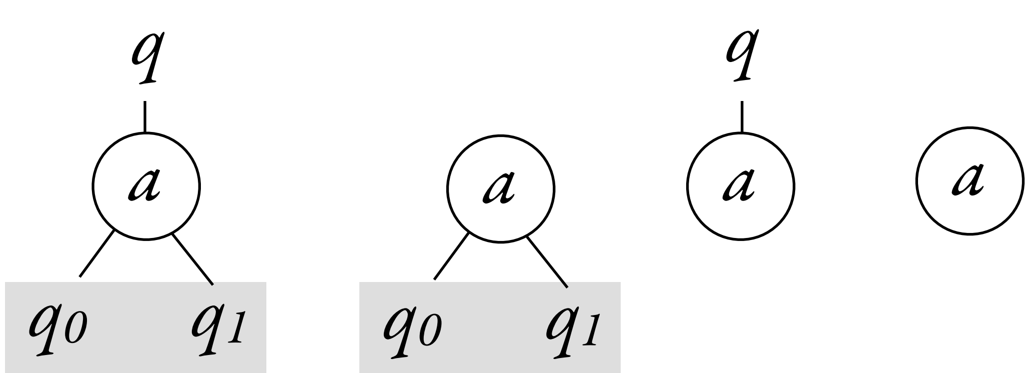

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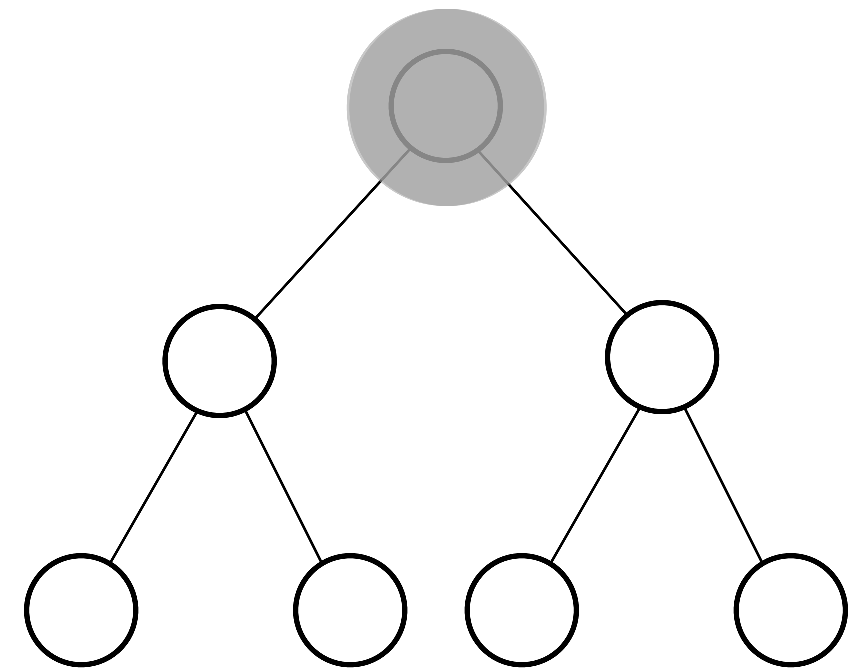

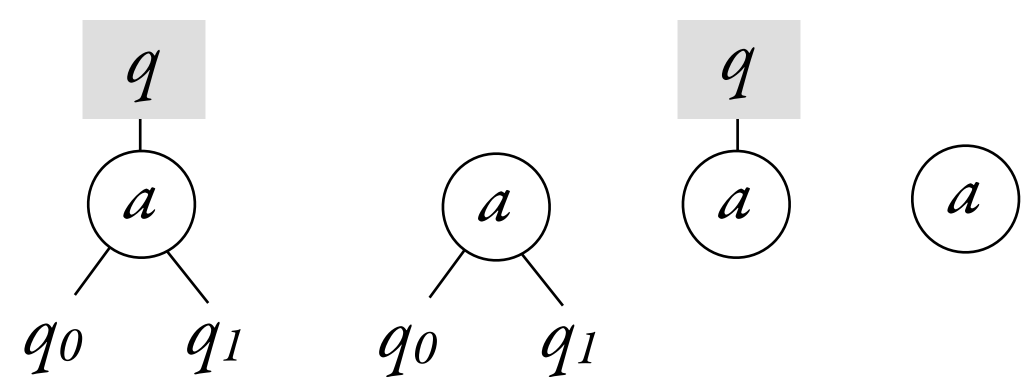

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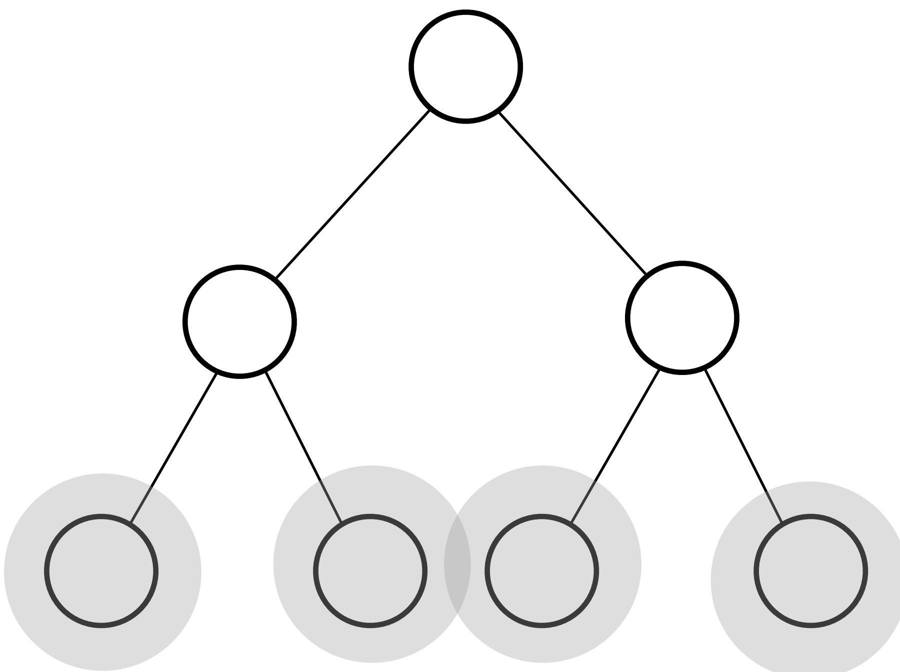
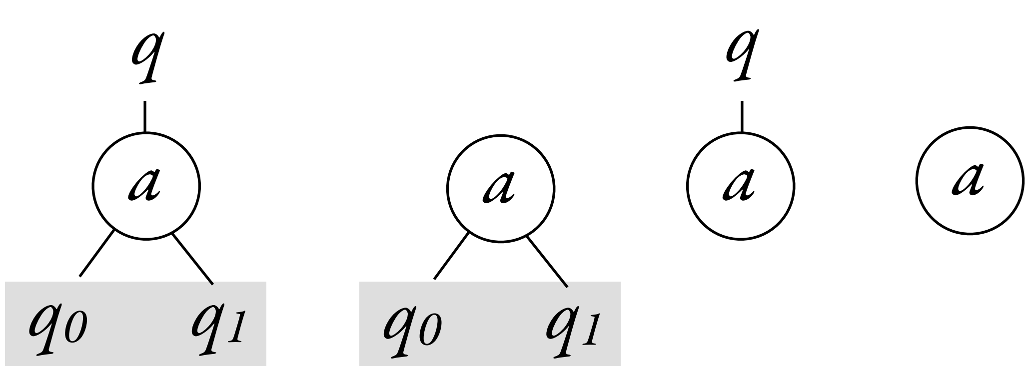
Deterministic automata

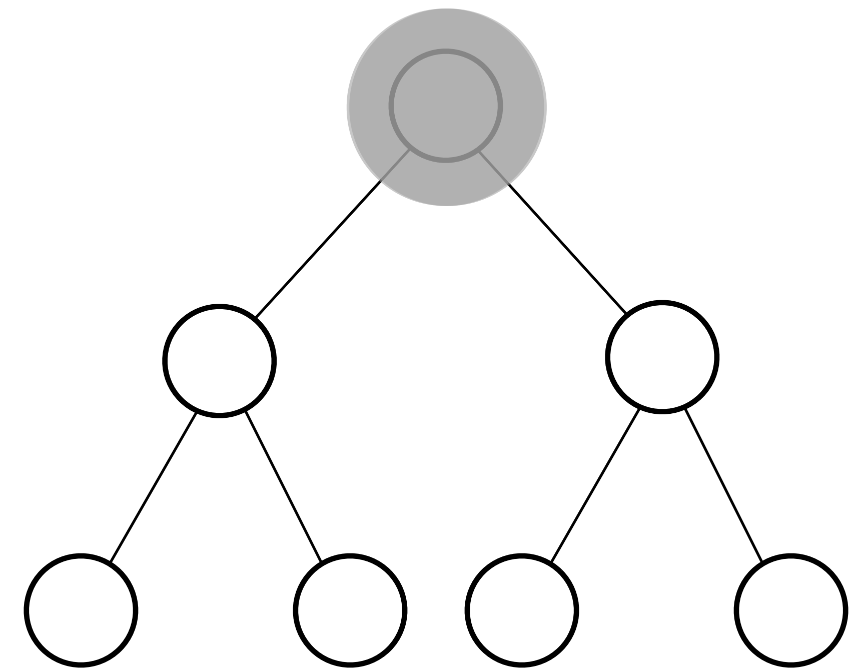
Top-down **bad**

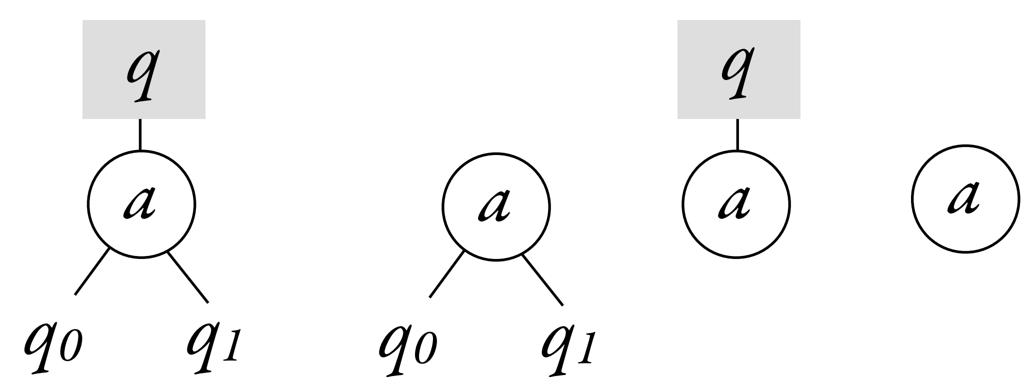




determined by other components

good Bottom-up





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Bottom-up determinism

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for all possible runs

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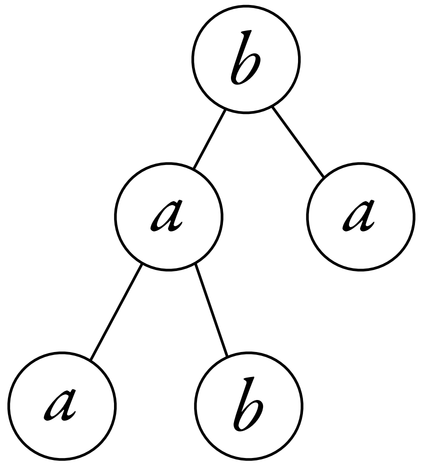

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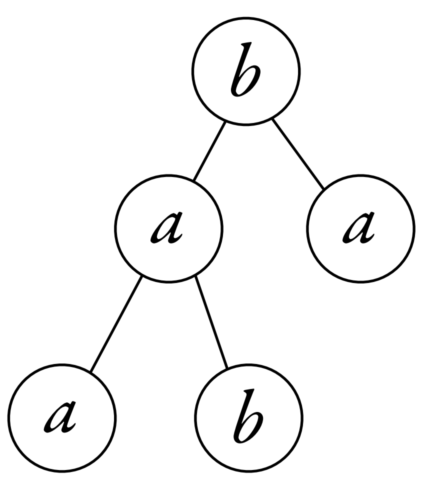
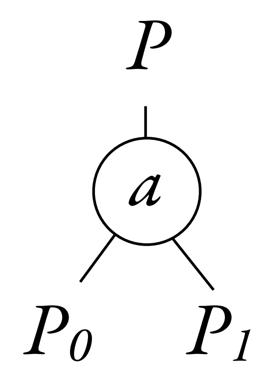
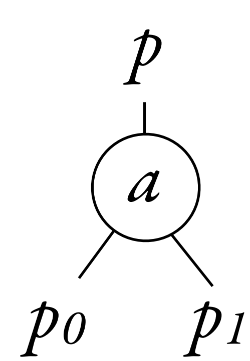




$P = \{\, p :$  holds for some $p_0 \in P_0,\, p_1 \in P_1 \,\}$

Def. A *regular* tree language is one recognized by a nondeterministic tree automaton or, equivalently, by a deterministic bottom up tree automaton.

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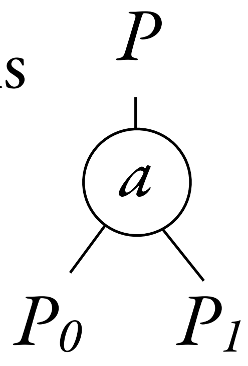
subset construction fails
for top-down
determinism



Top-down determinism

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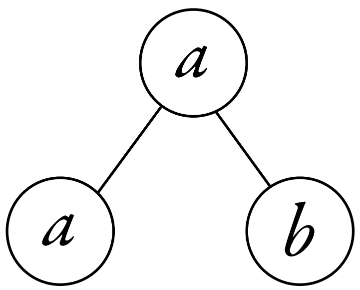
$L = \{$  $\}$

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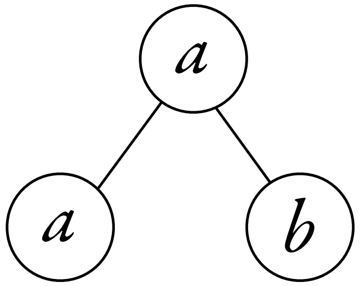
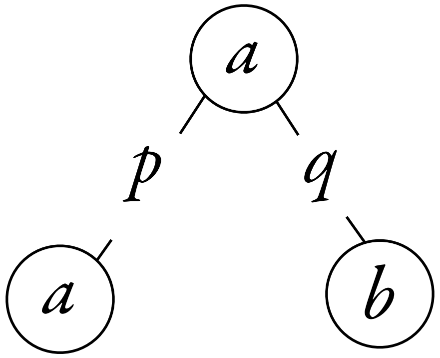
$L = \{$  $\}$



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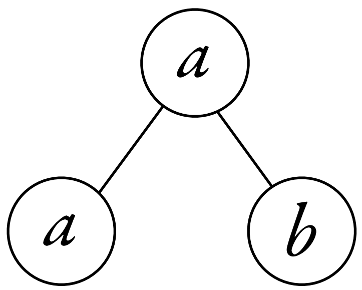
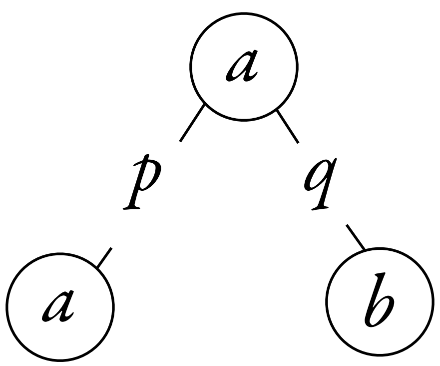
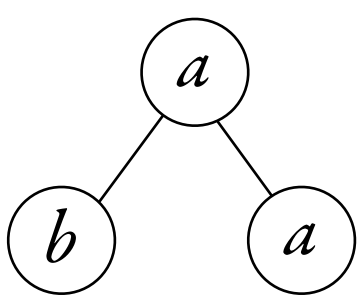
$L = \{$  $\}$



$K = \{$  $\}$

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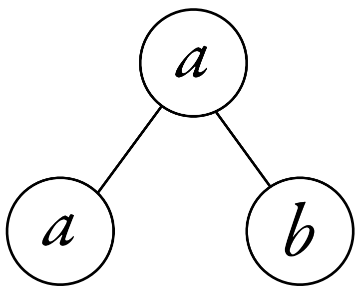
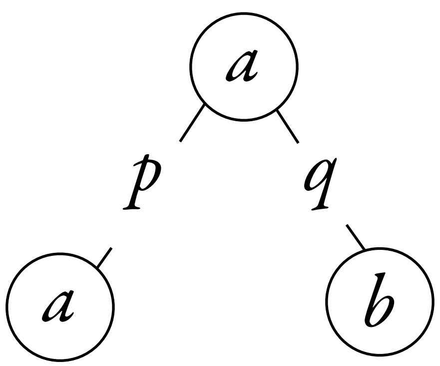
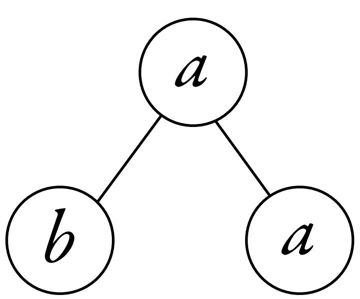
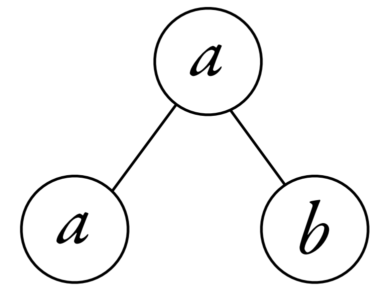
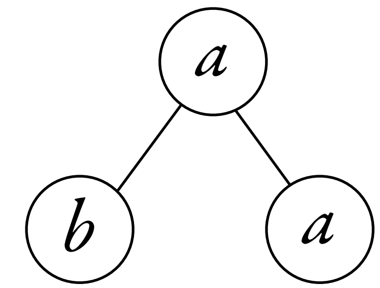
$L = \{$  $\}$



$K = \{$  $\}$

$L \cup K = \{$  ,  $\}$

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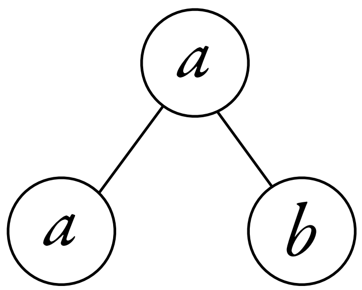
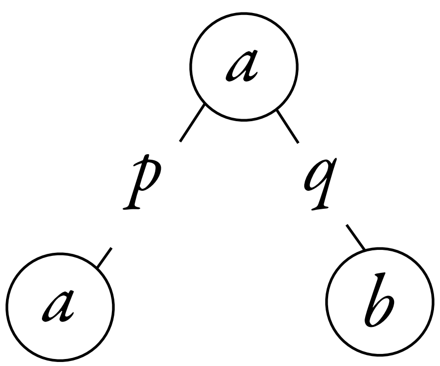
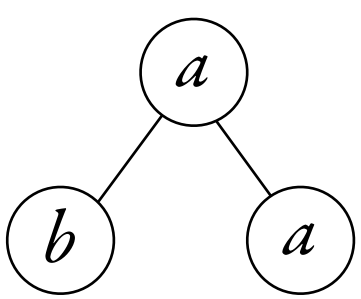
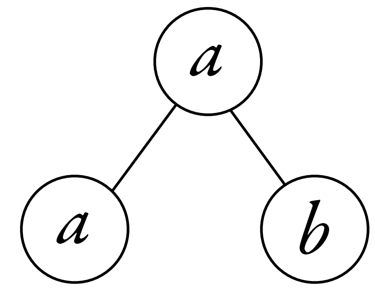
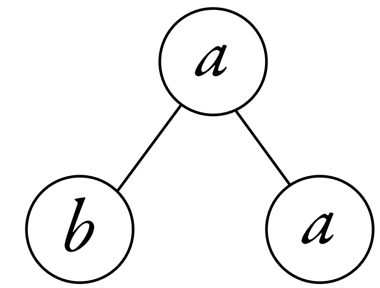
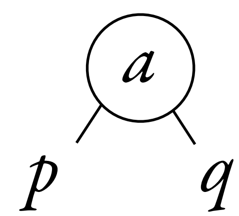
$L = \{$  $\}$



$K = \{$  $\}$

$L \cup K = \{$  ,  $\}$

There must be a transition,



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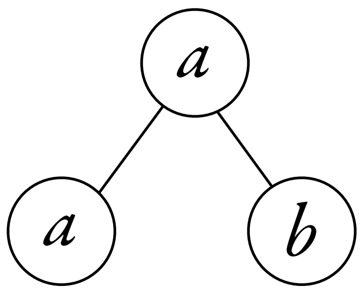
$L = \{$ 

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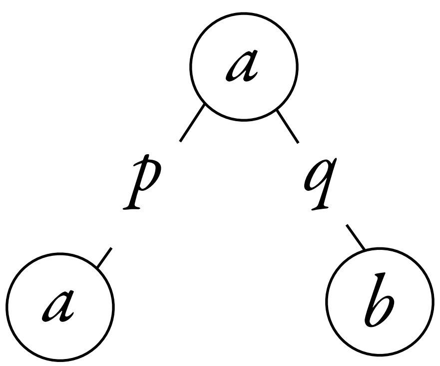
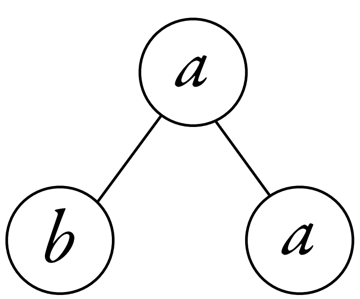


$K = \{$ 

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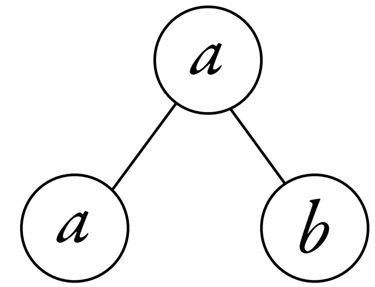
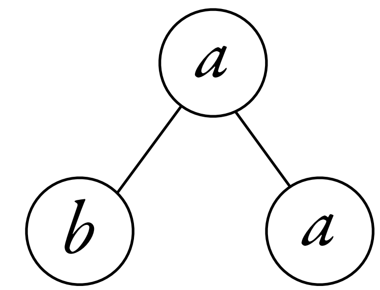
$L \cup K = \{$ 

, 

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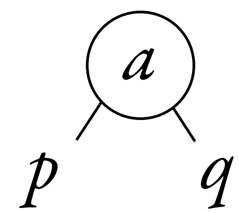
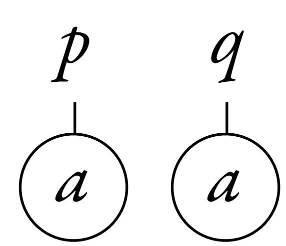


and hence also



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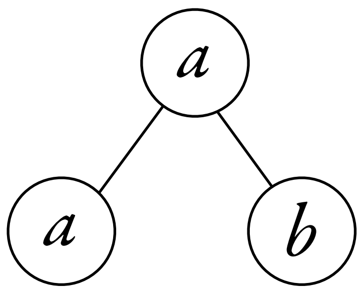
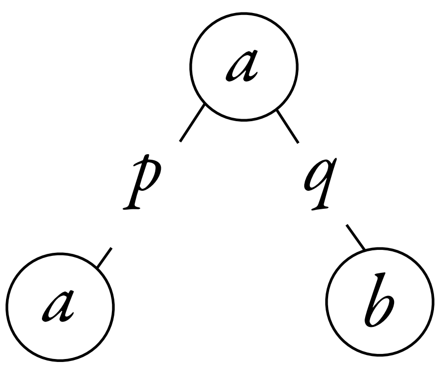
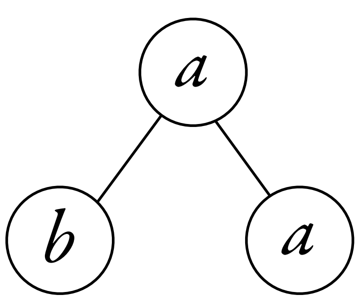
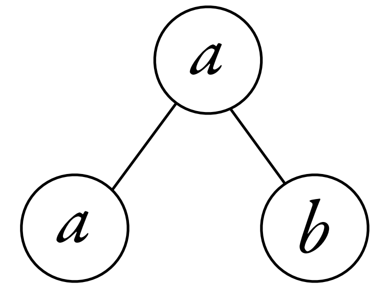
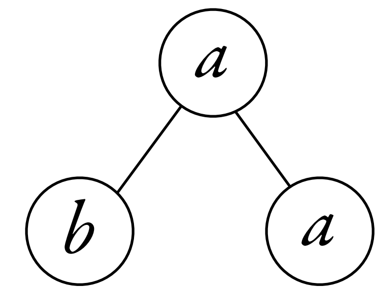
$L = \{$  $\}$



$K = \{$  $\}$

$L \cup K = \{$  ,  $\}$

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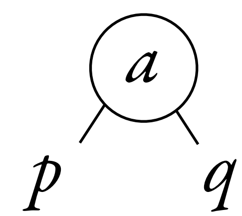

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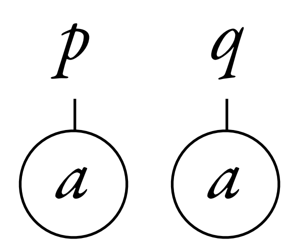
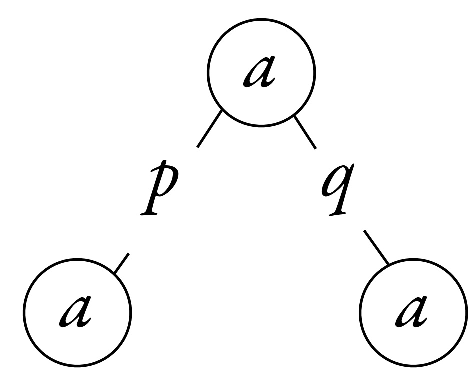
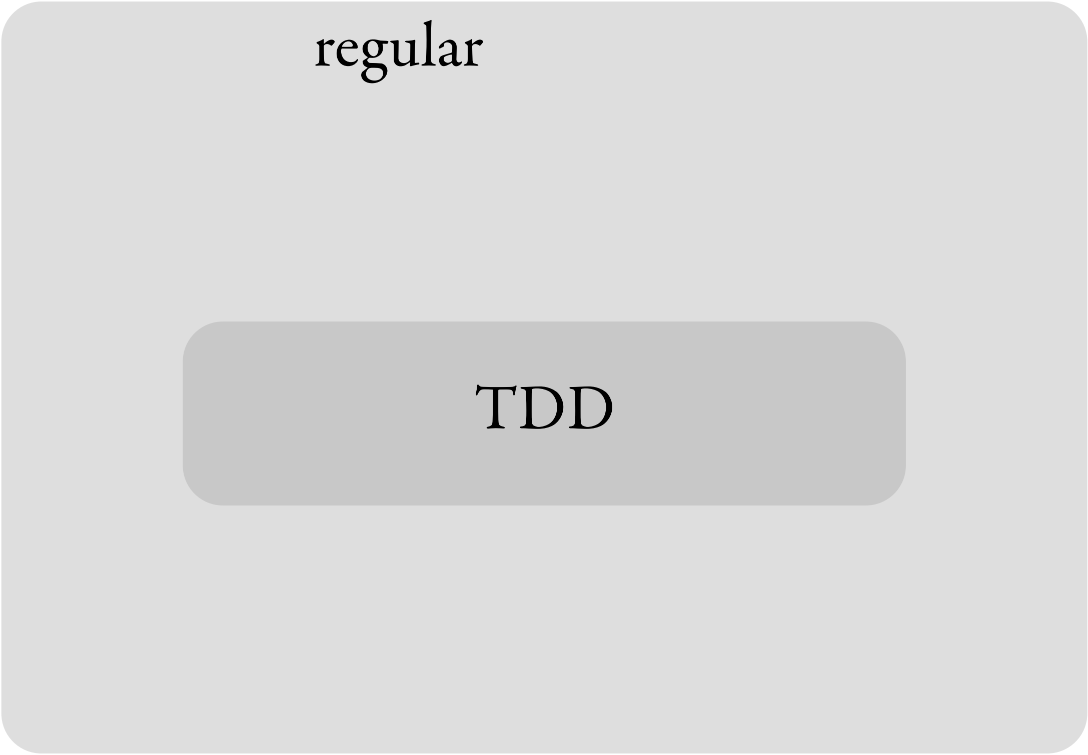


...but then: 

Top-down determinism, continued.



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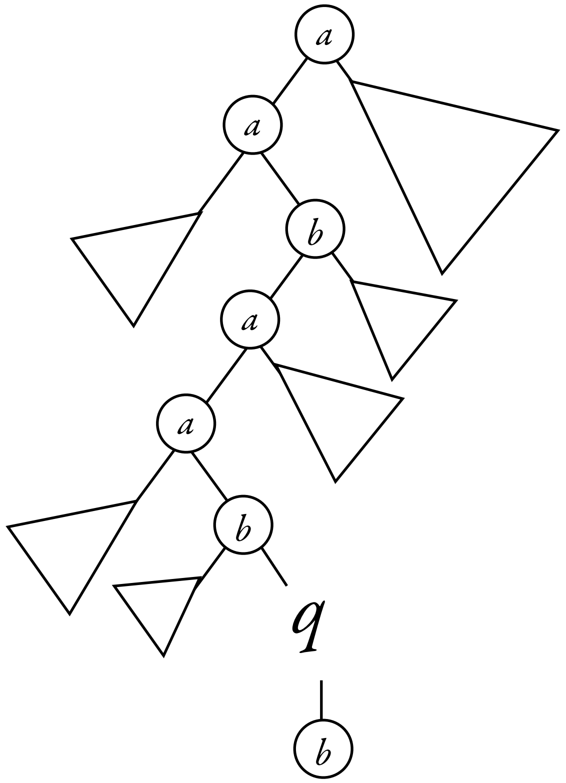

the run is accepting,
if every leaf state is
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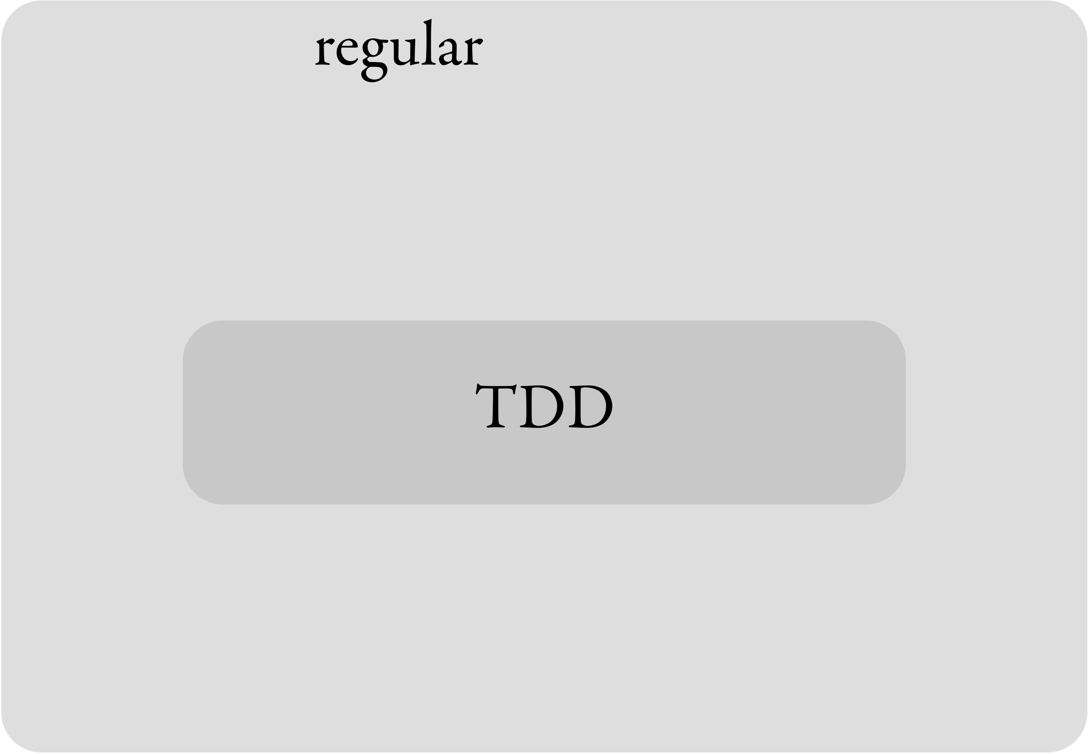

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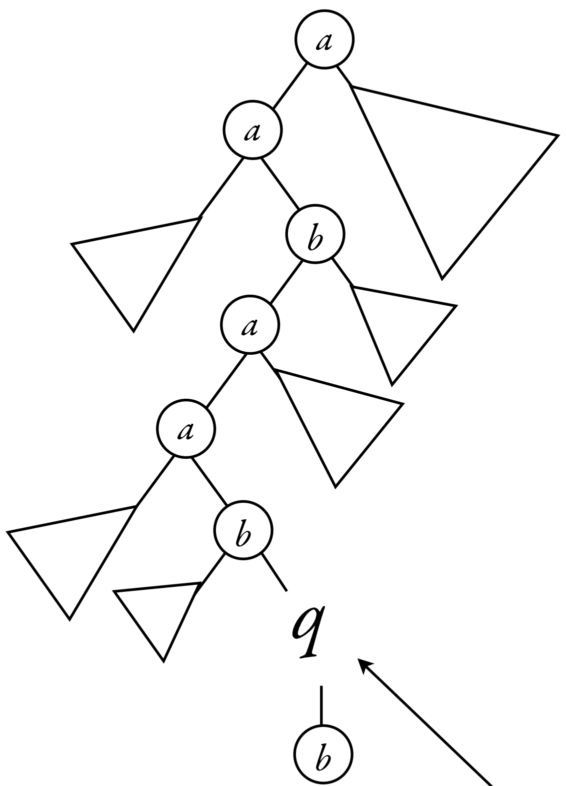

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what does the leaf
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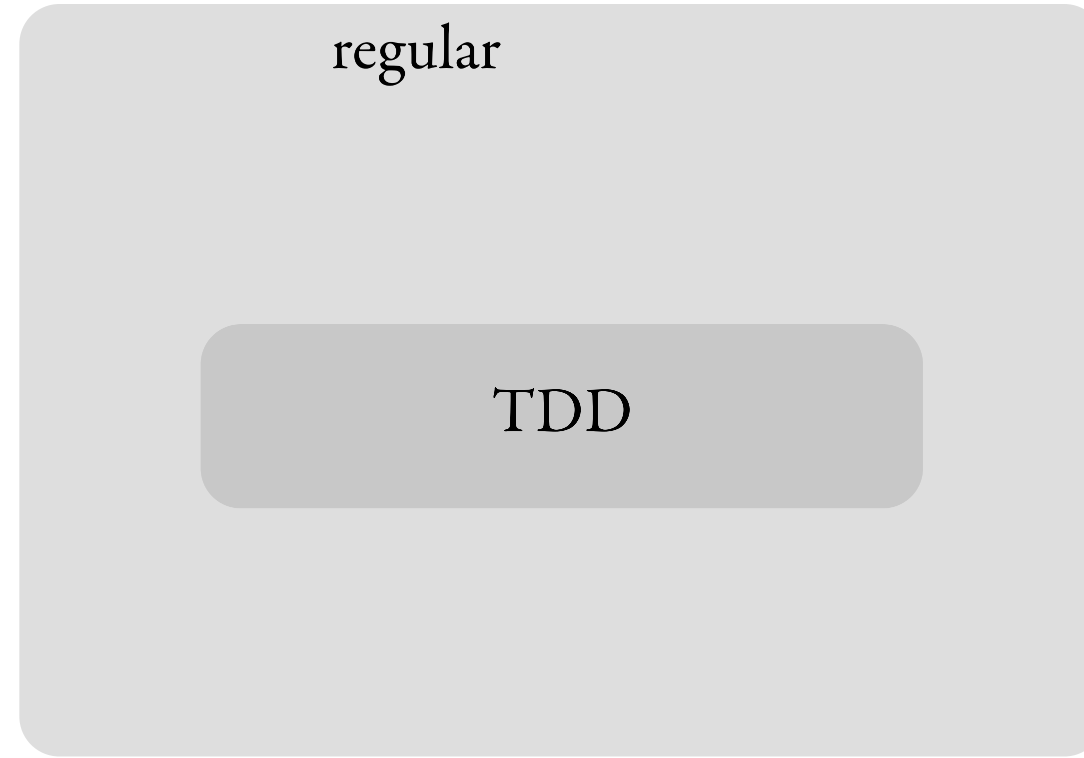

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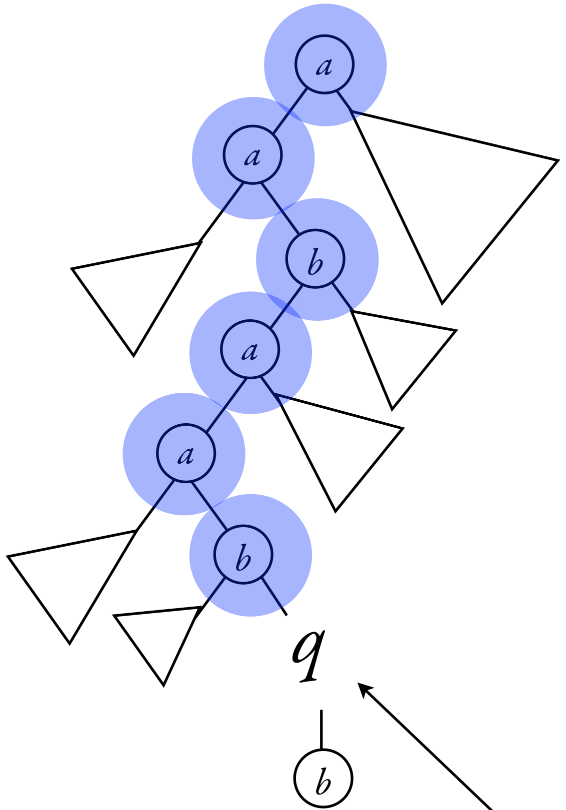

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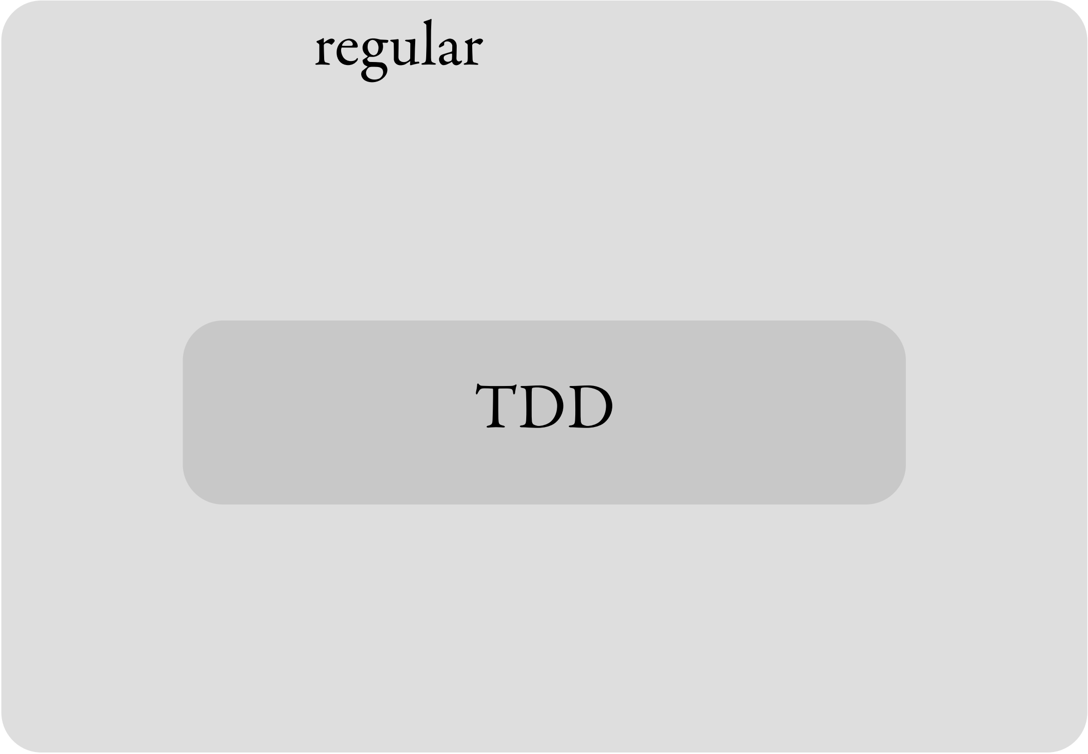

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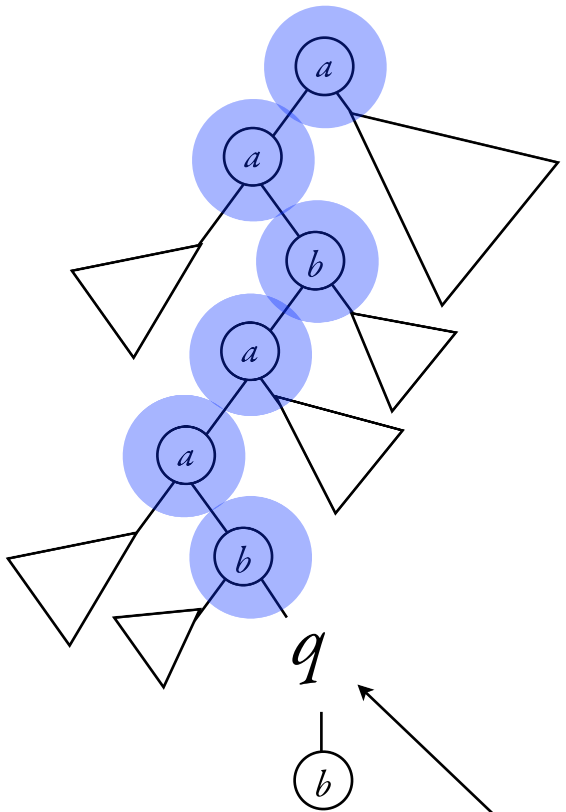

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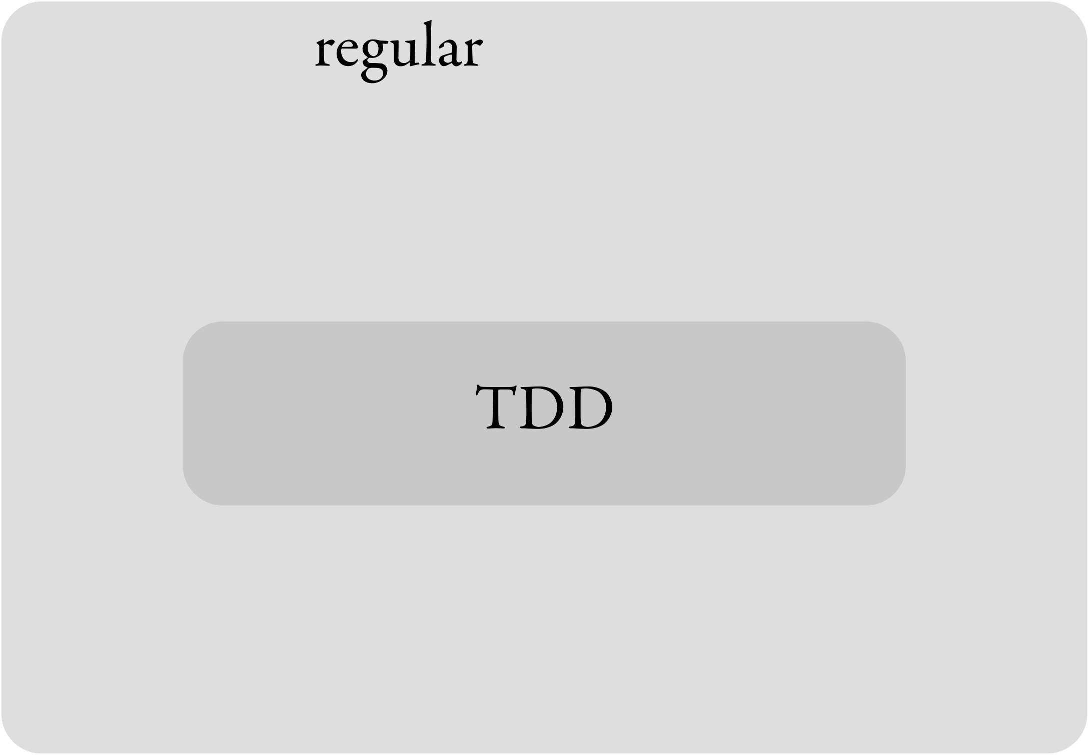
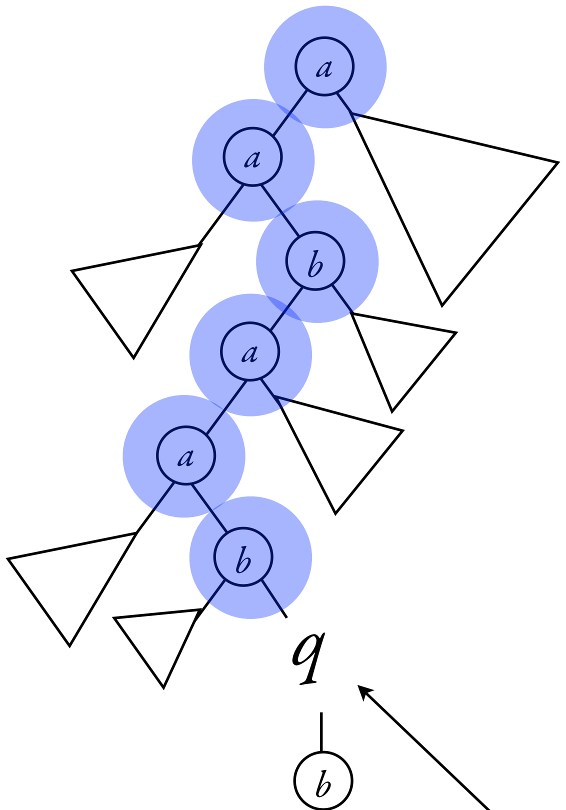


Fact. The following are equivalent for a tree language L:

– L is recognized by a deterministic top-down tree automaton

– L is equivalent to "all paths in K", for a regular word language $K \subseteq (\Sigma\times\{0,1\})^*$

Top-down determinism, continued.



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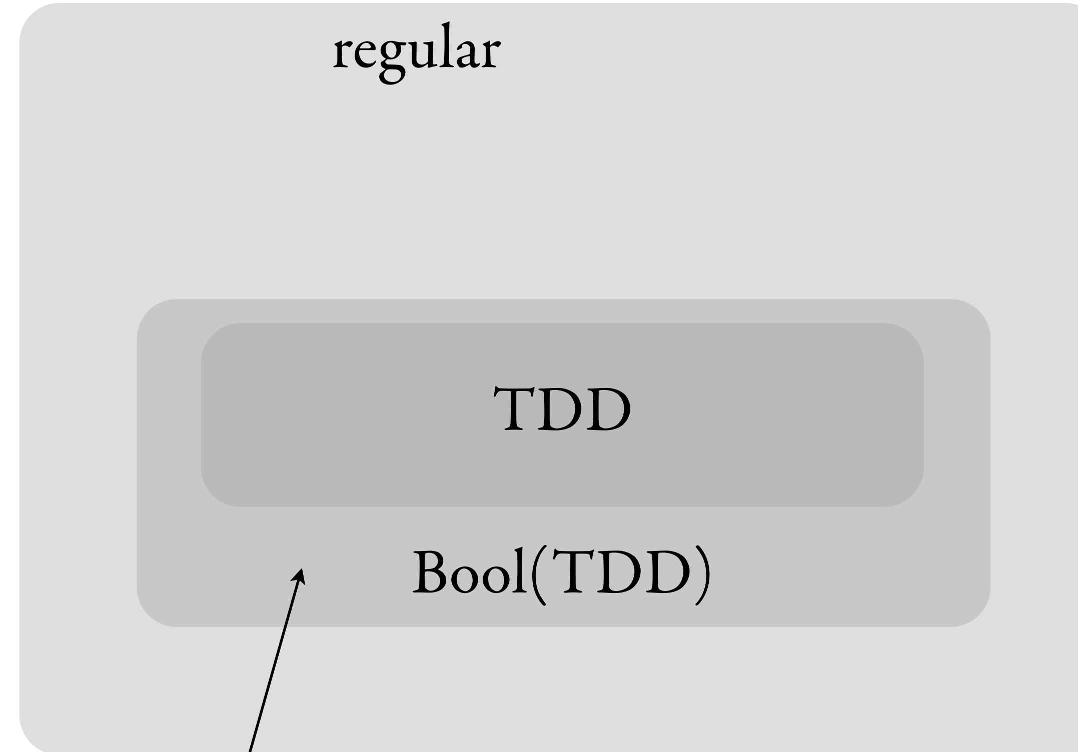

languages of the form: all paths .., some path ...

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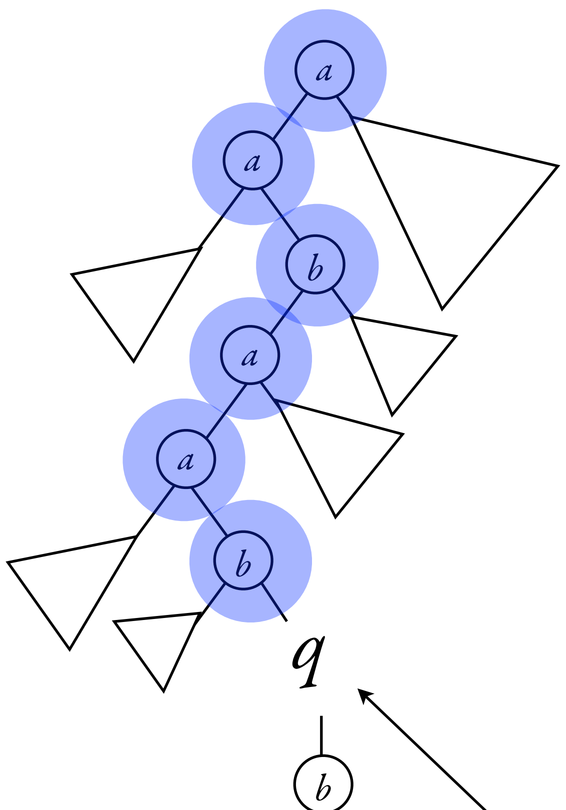

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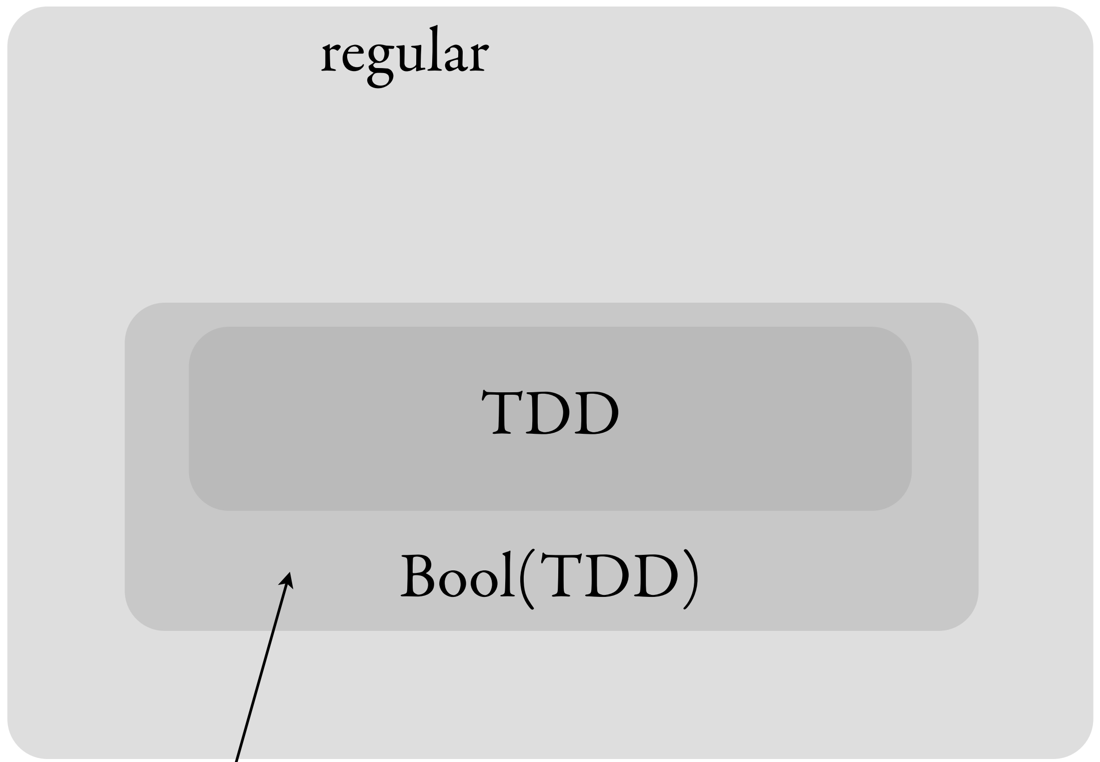

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It is not difficult to give an algorithm deciding $L \in$ TDD.
Open problem: give an algorithm deciding $L \in$ Bool(TDD).

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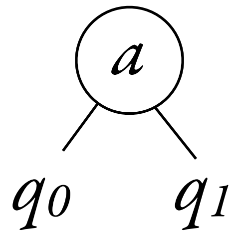
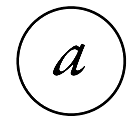
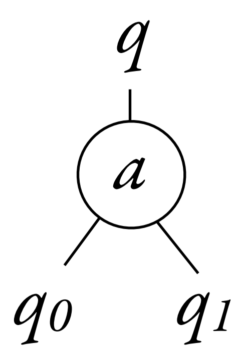
Minimal automata

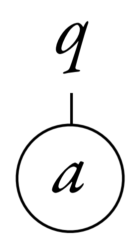
for the purpose of minimal automata, we change the definition.

We remove transitions 



and add accepting states. A run must have an accepting state at the root.

(These transitions 



work if the node is the root or not)

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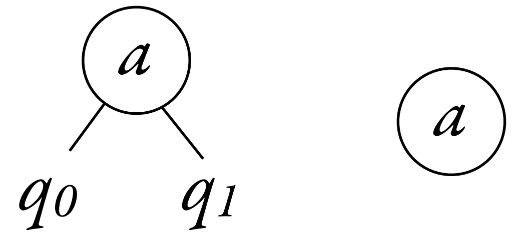

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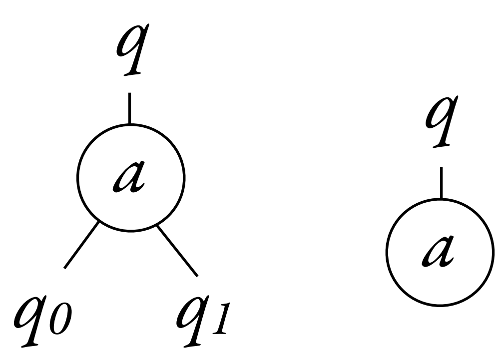

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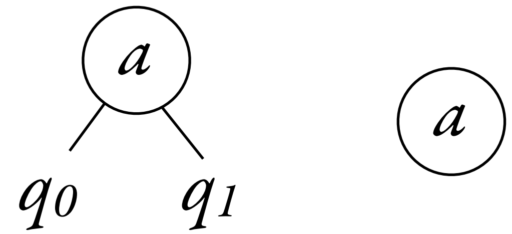

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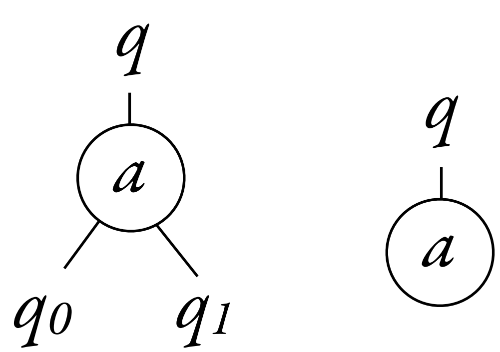

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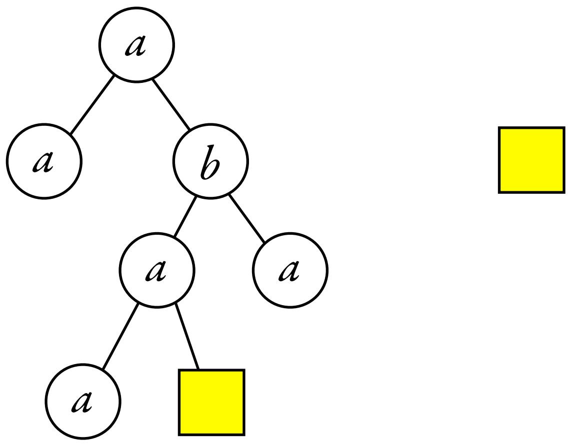
Tuple definition: $(Q, \Sigma, \delta : Q\times\Sigma\times Q \to Q,\ I : \Sigma \to Q, F \subseteq Q)$

Minimalization

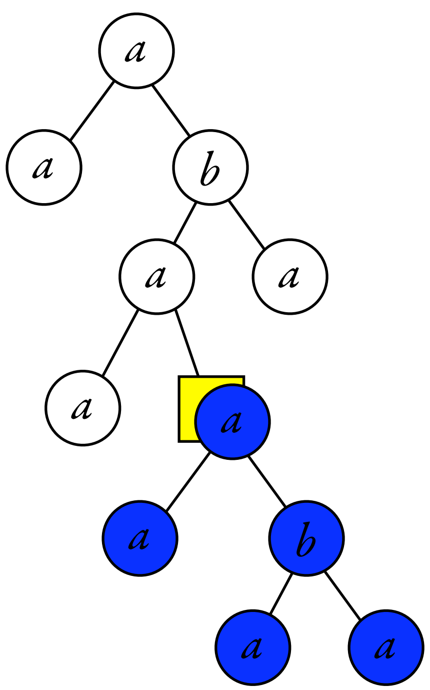
Proposition. For every regular tree language, there is a unique (up to isomorphism) minimal deterministic bottom-up tree automaton.
The minimal automaton can be obtained from any other automaton in polynomial time.

States of the minimal automaton: equivalence classes of the Myhill-Nerode congruence.

Myhill-Nerode congruence for tree languages

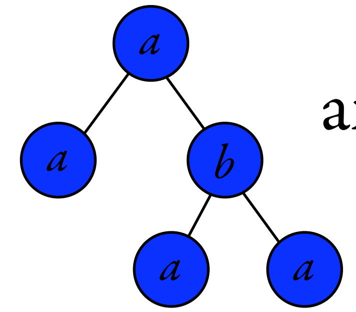


context

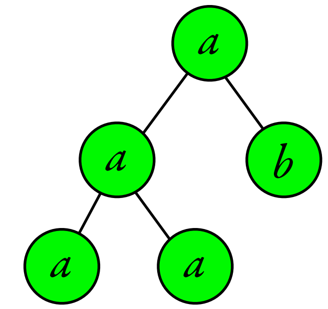


substitution

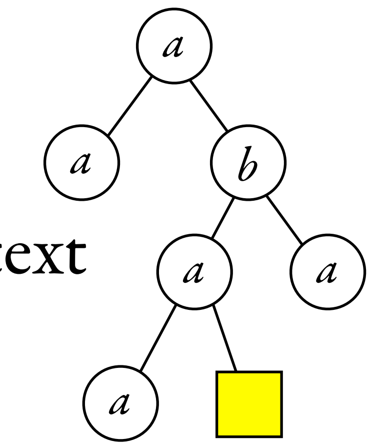
Two trees

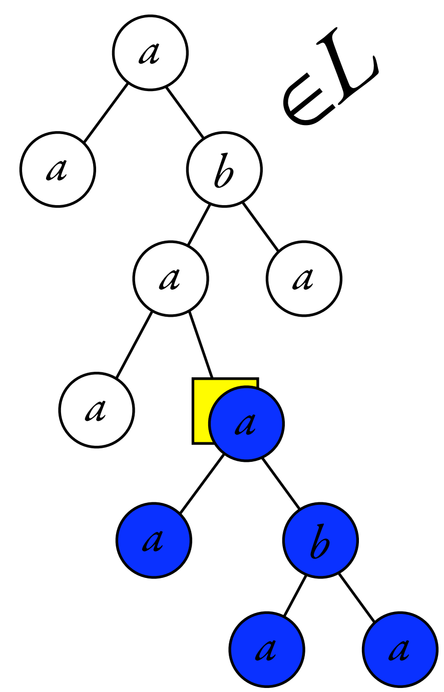


and



are L-equivalent if for every context





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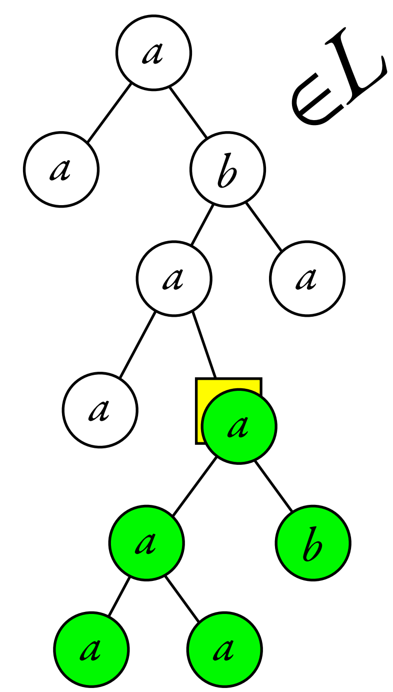

Myhill-Nerode congruence for tree languages, examples

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“two a’s”

classes of the Myhill-Nerode congruence:

0=“zero a’s”

1=“one a”

2=“two a’s”

3=“three or more a’s”

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balanced binary tree

infinitely many classes:

"unbalanced tree"

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Myhill-Nerode congruence for tree languages, examples

“two a’s”

classes of the Myhill-Nerode congruence:

0=“zero a’s”

1=“one a”

2=“two a’s”

3=“three or more a’s”

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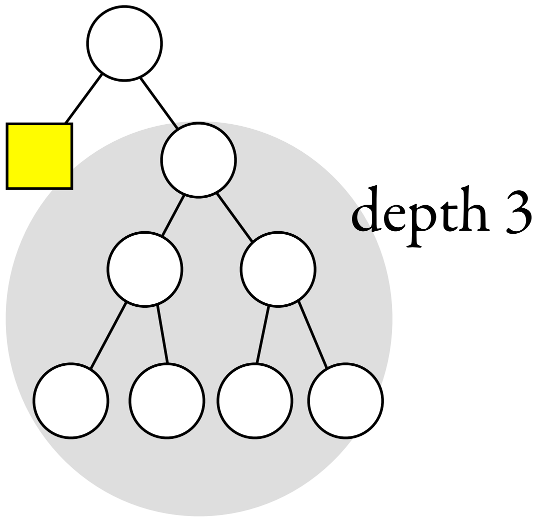

this context distinguishes a balanced tree of depth 3 from any other tree.

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Yield of a tree: word with leaf labels, left-to-right.

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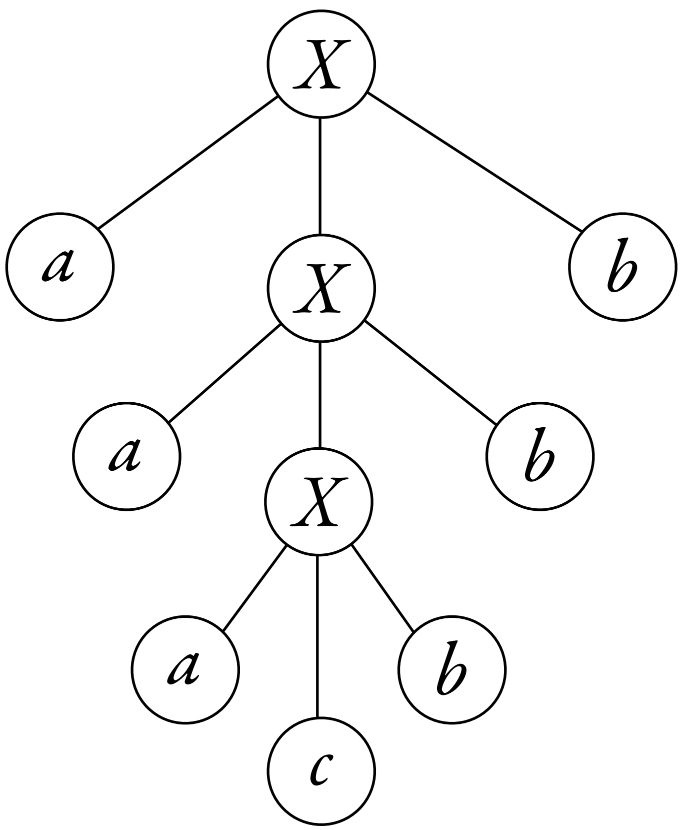

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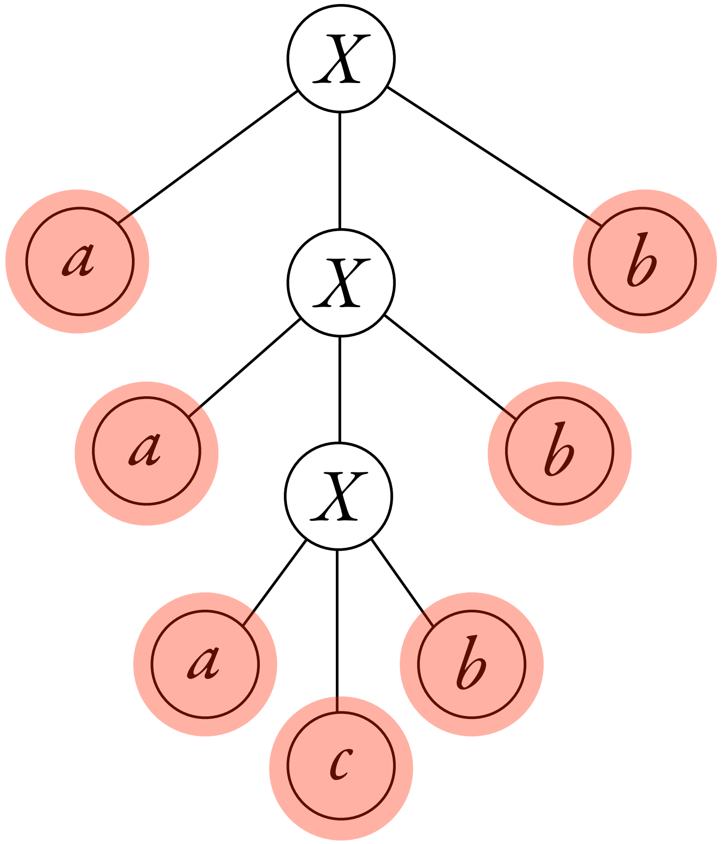

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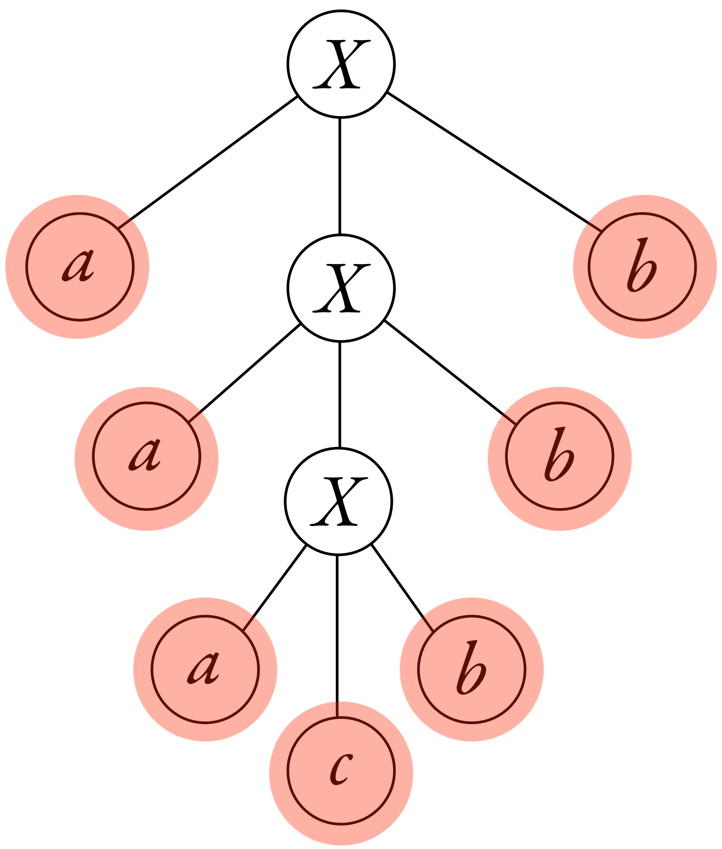	Let A be a nondeterministic, top-down tree automaton. The grammar $G(A)$ has states of A as nonterminals, and rules:

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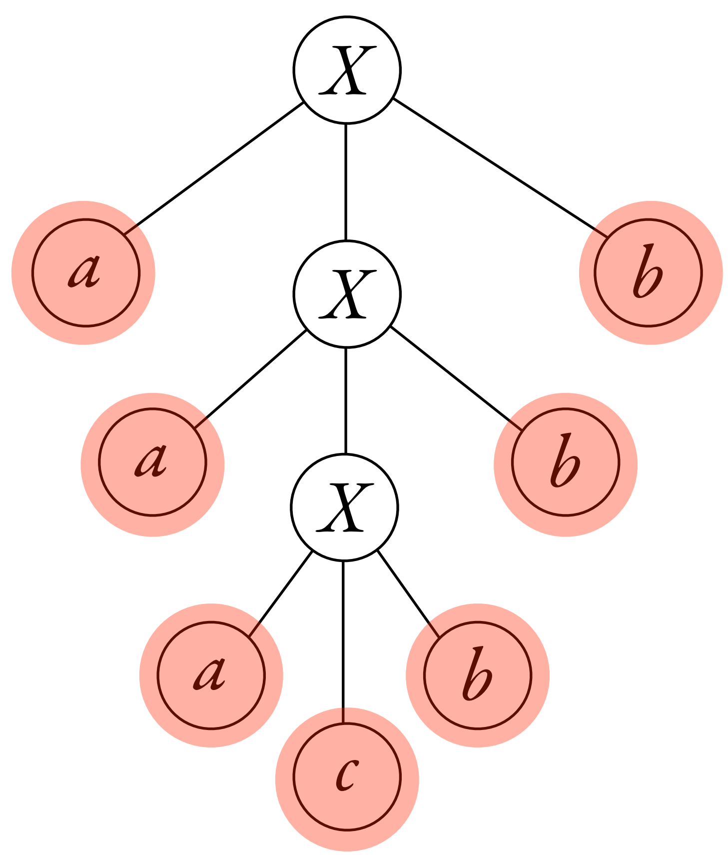

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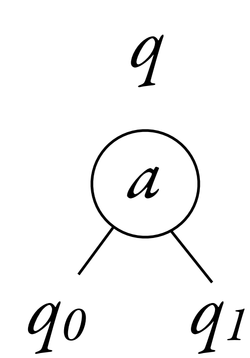
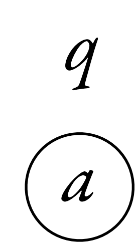
from automata to grammars

Let A be a nondeterministic, top-down tree automaton. The grammar $G(A)$ has states of A as nonterminals, and rules:

automaton transition	grammar rule
	$q \longrightarrow q_0\, q_1$
	$q \longrightarrow a$

What is a Tree Automaton?

- –definition and examples
- –determinism, bottom-up vs top-down
- –minimization
- –closure properties

Decision Problems

- –emptiness
- –membership
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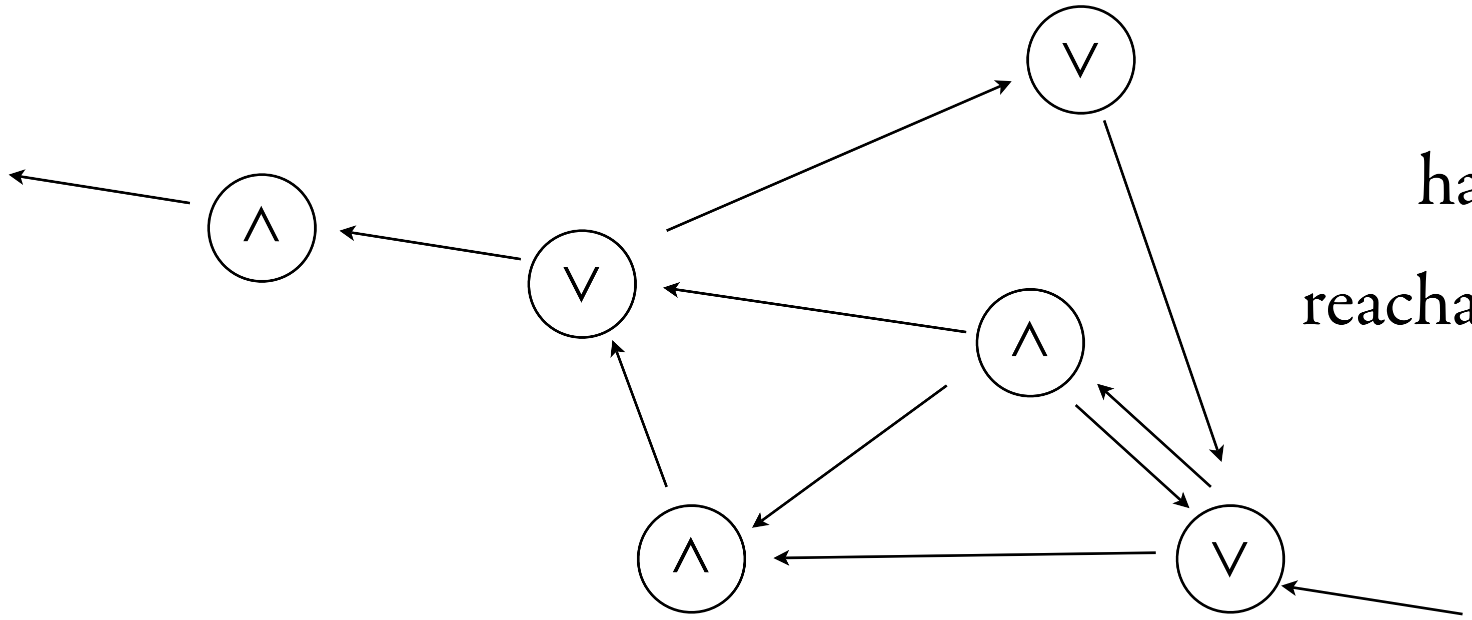

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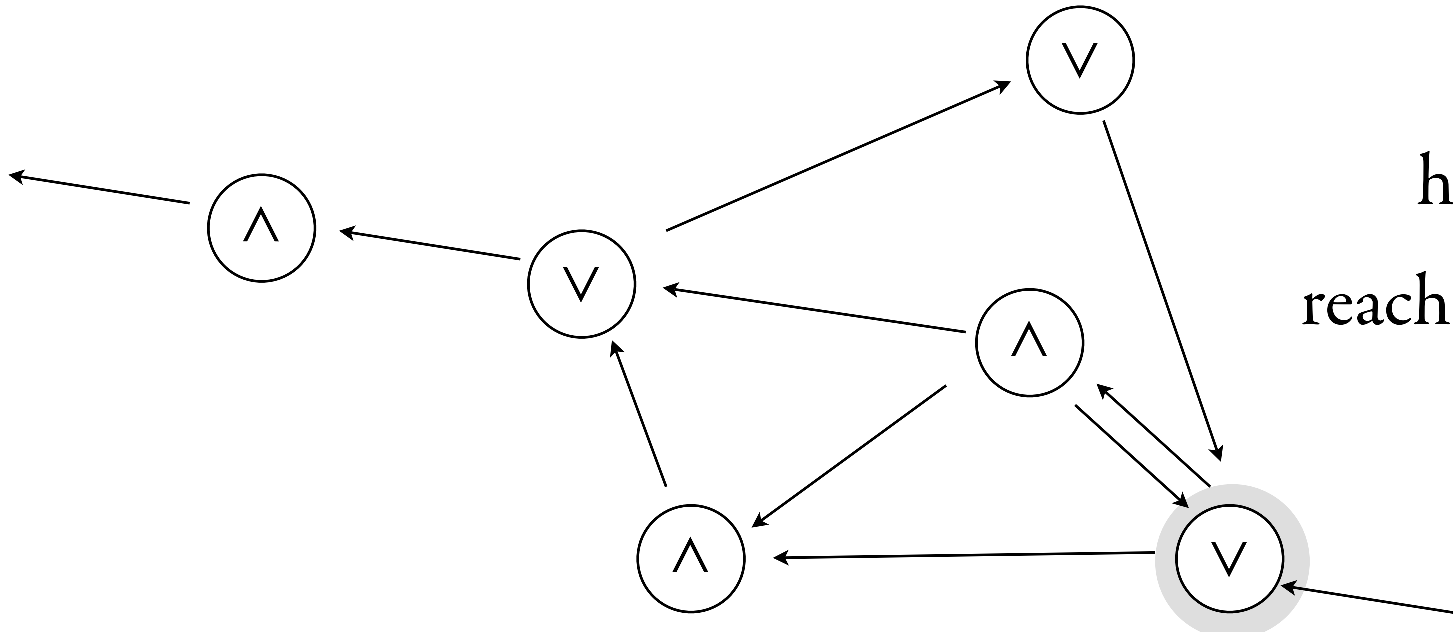

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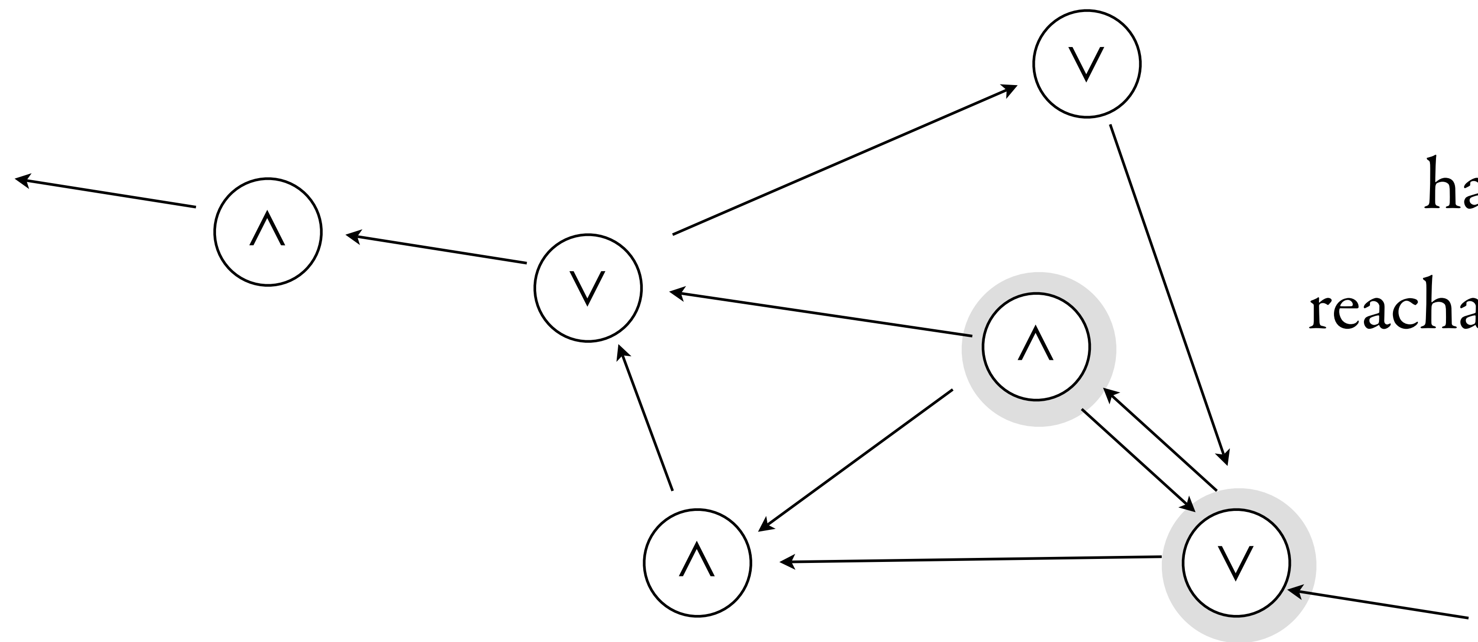

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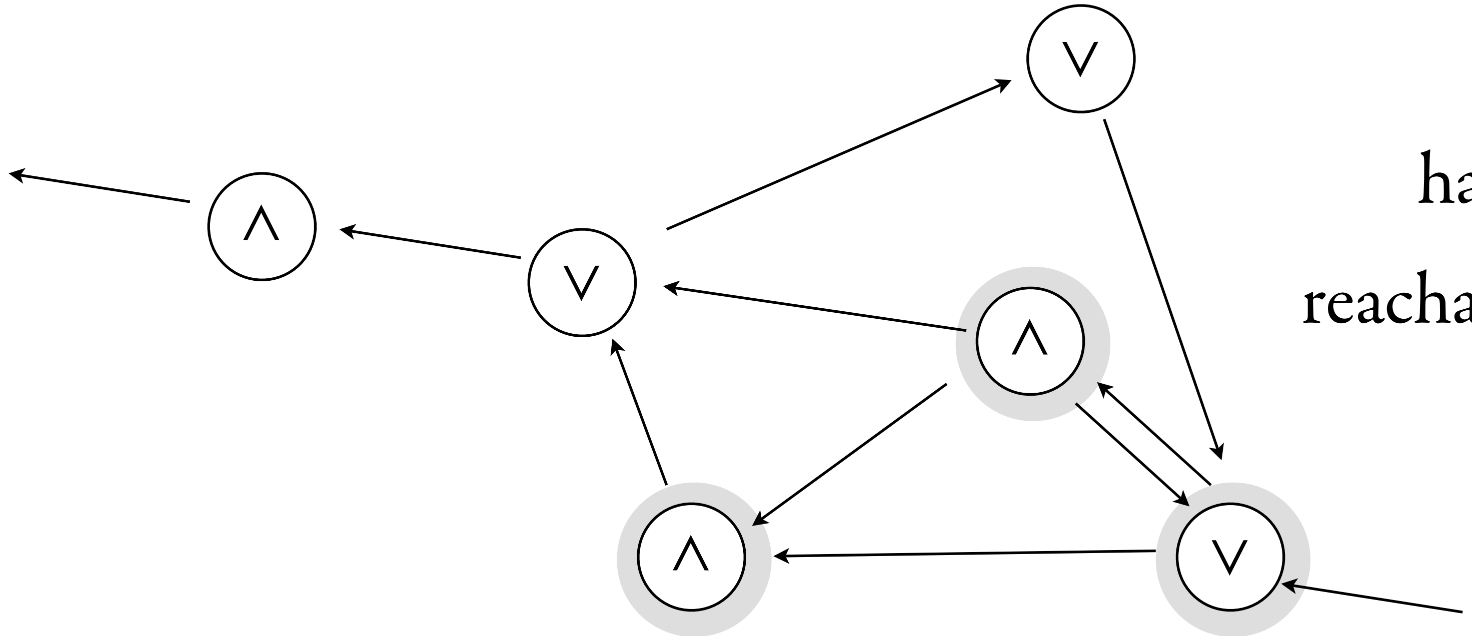

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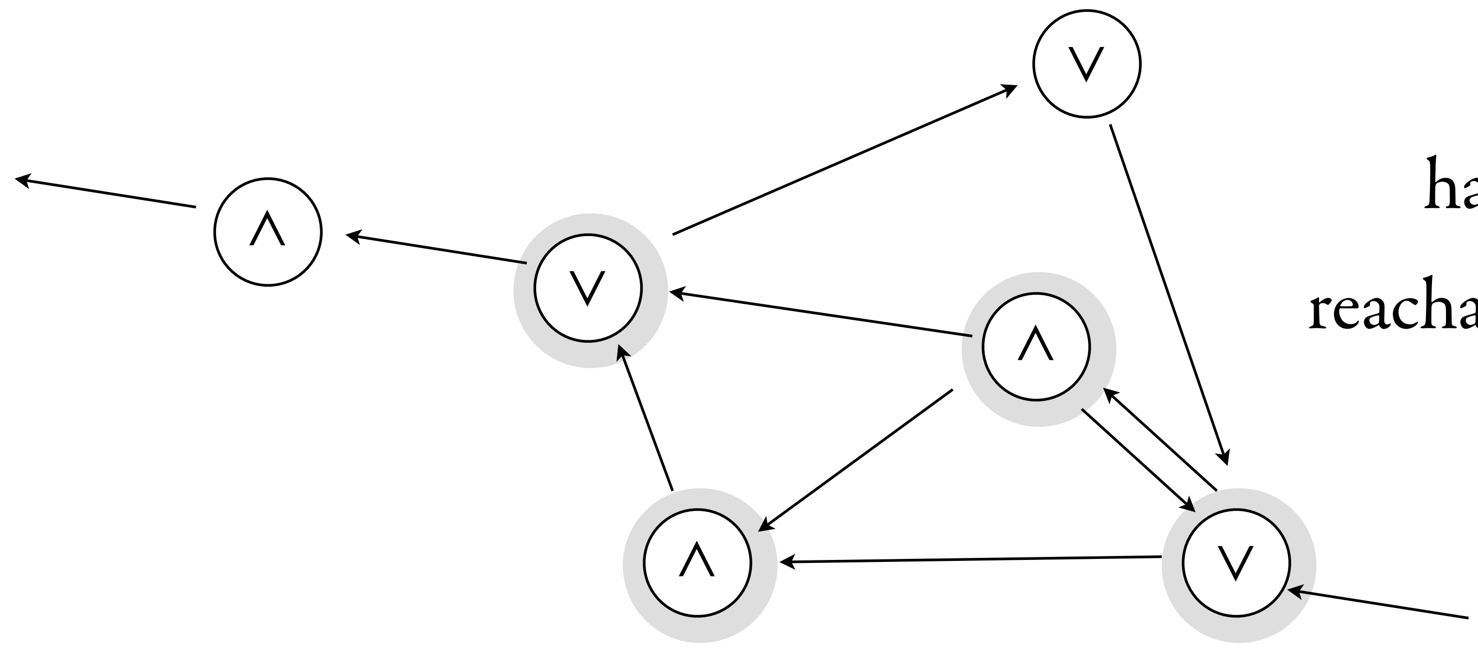

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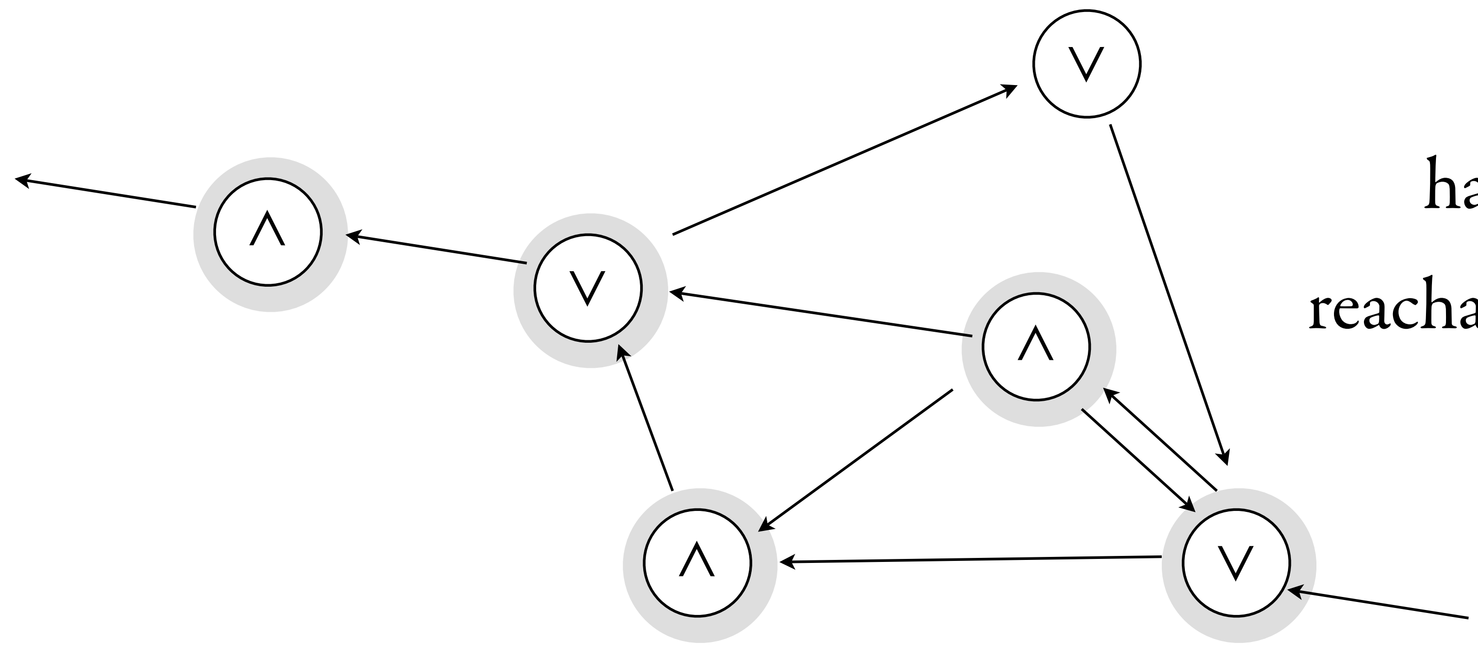

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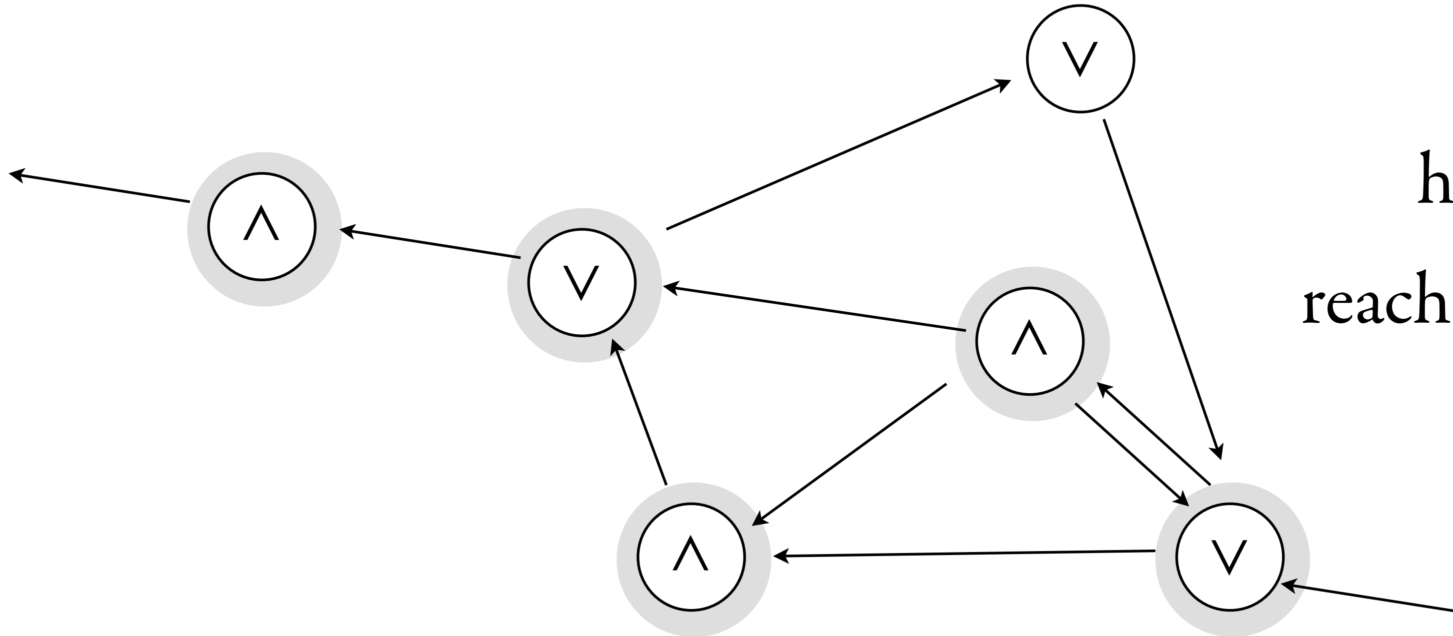

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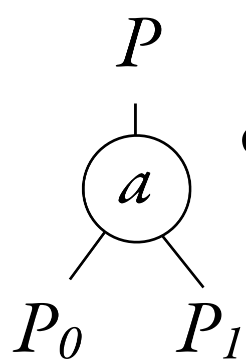

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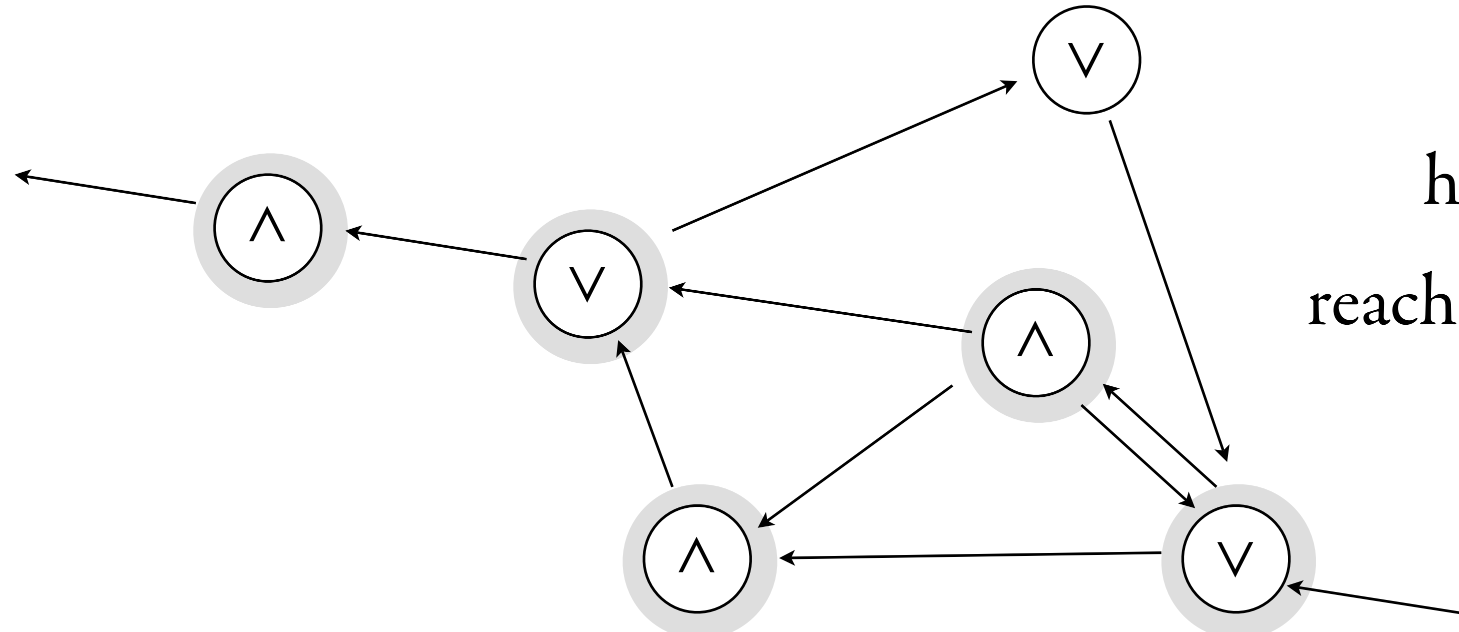

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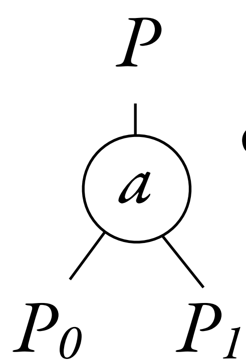
A transition 

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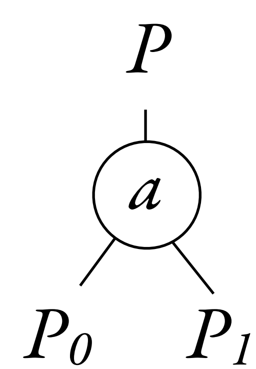
A state p can be reached if some transition 

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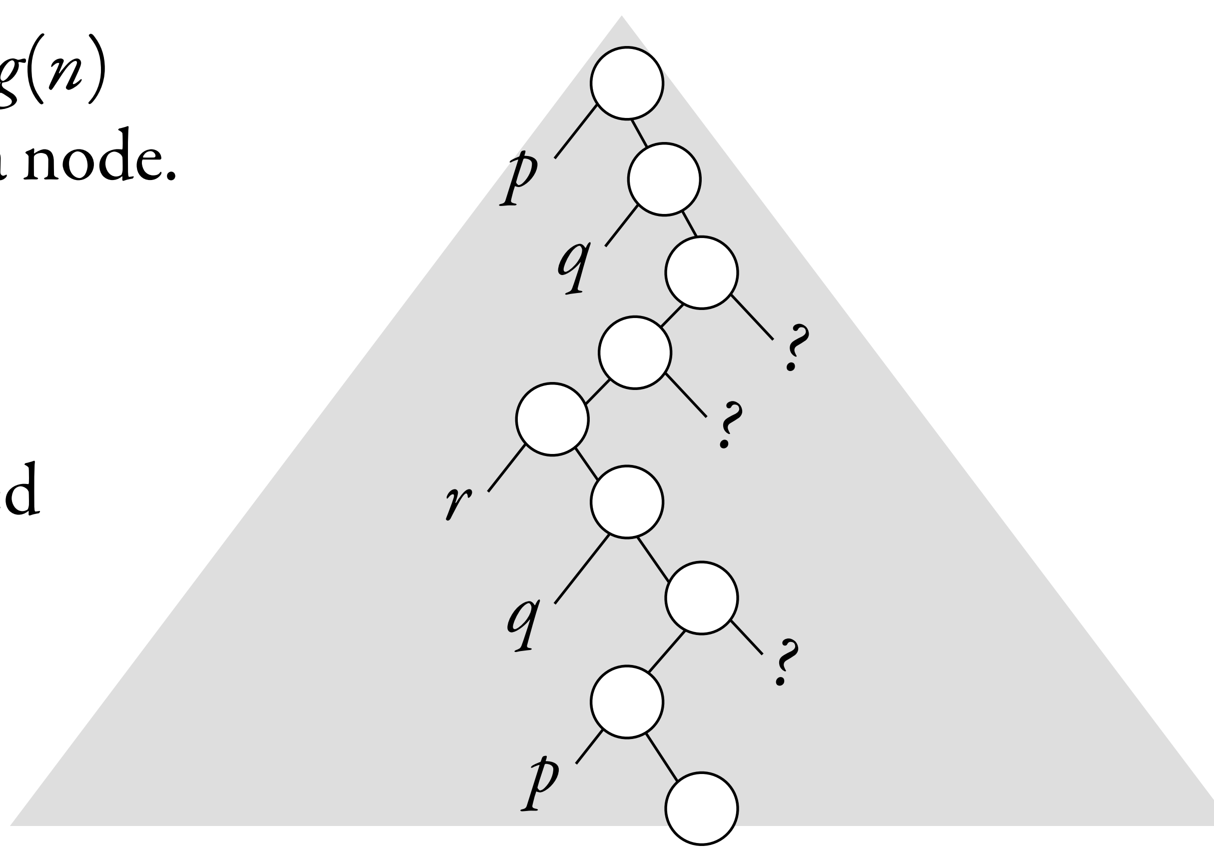

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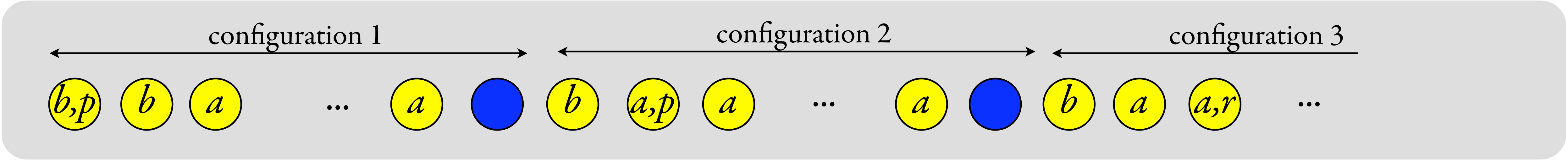

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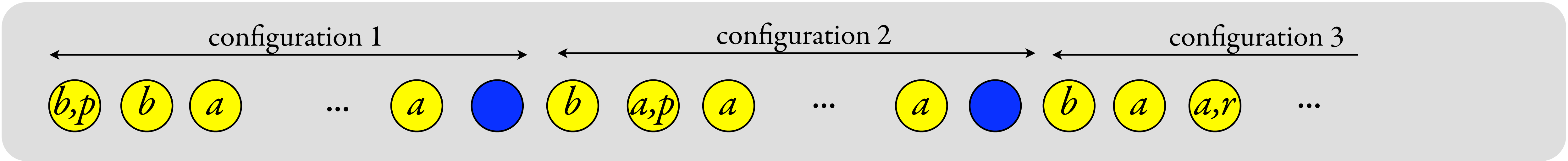

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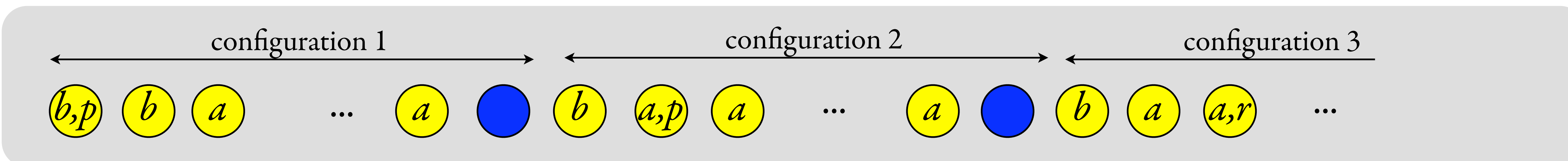

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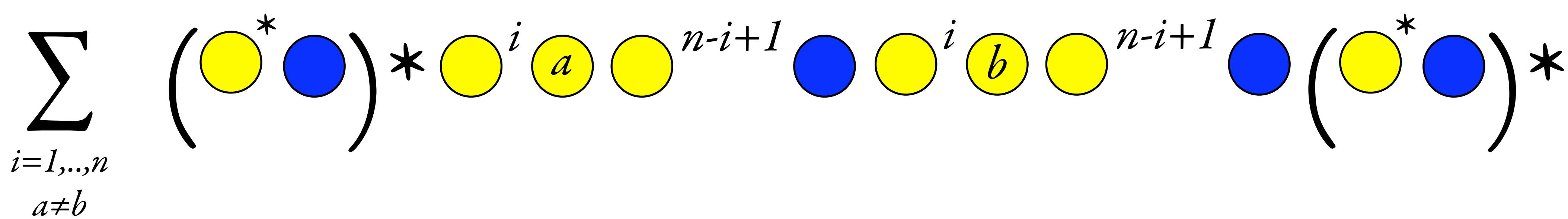

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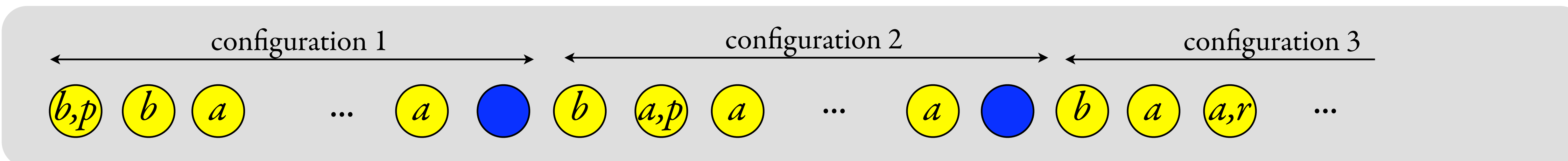

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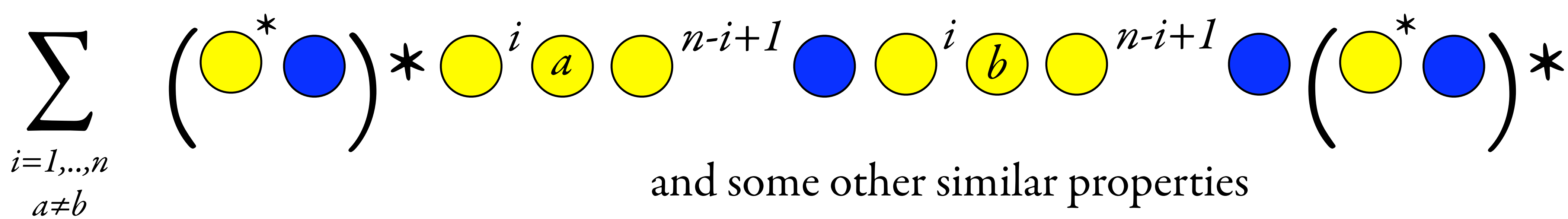

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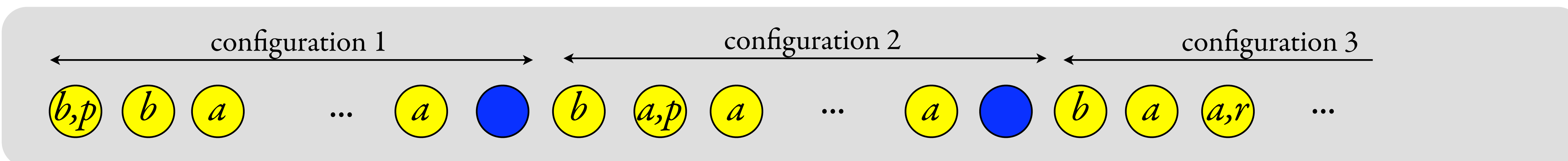

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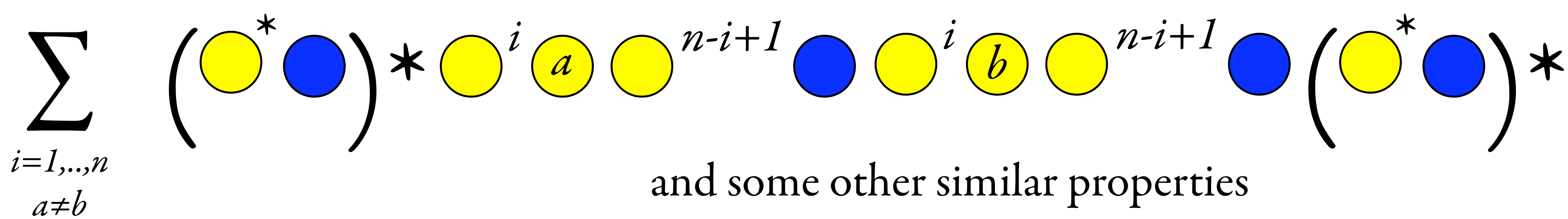

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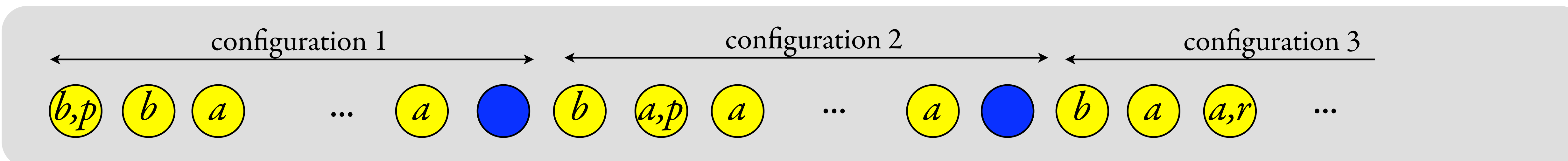

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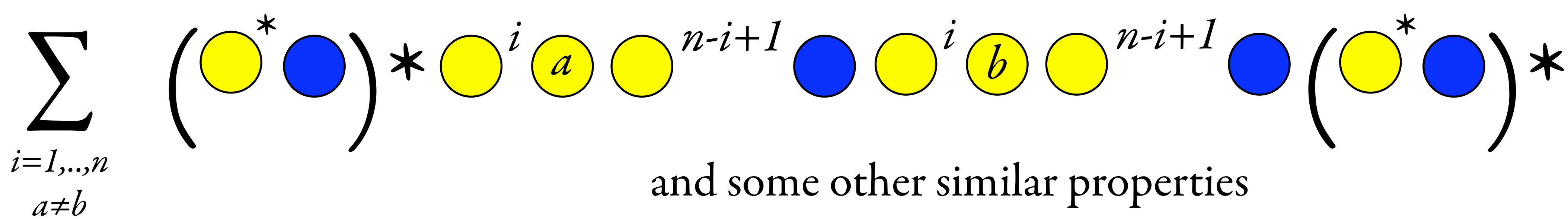

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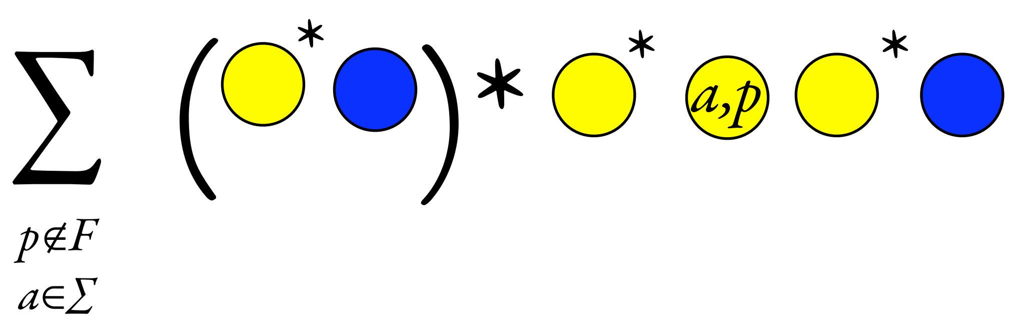

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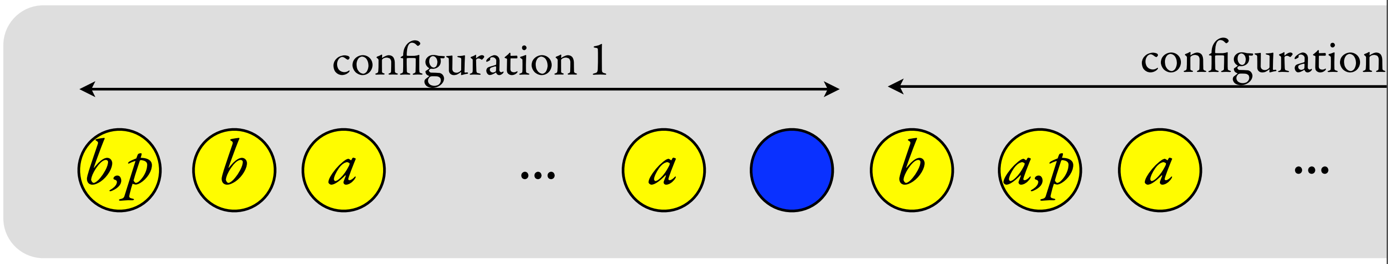

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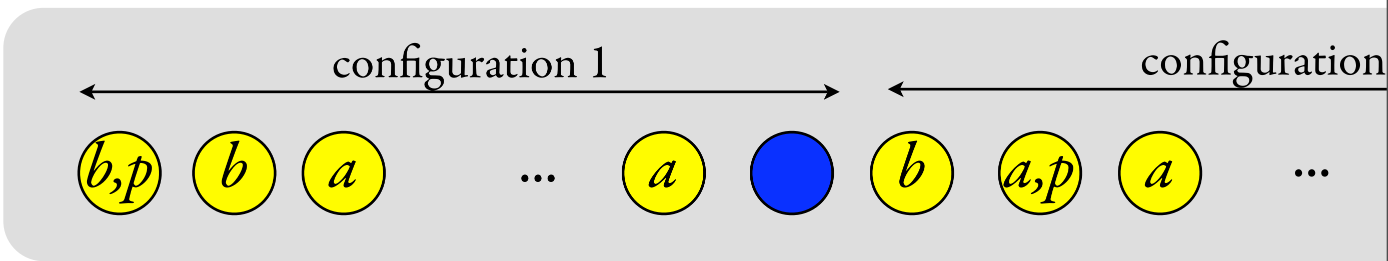

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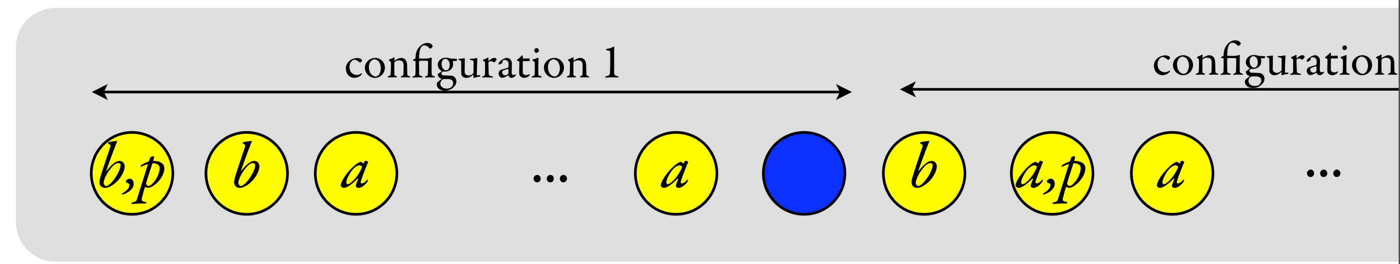

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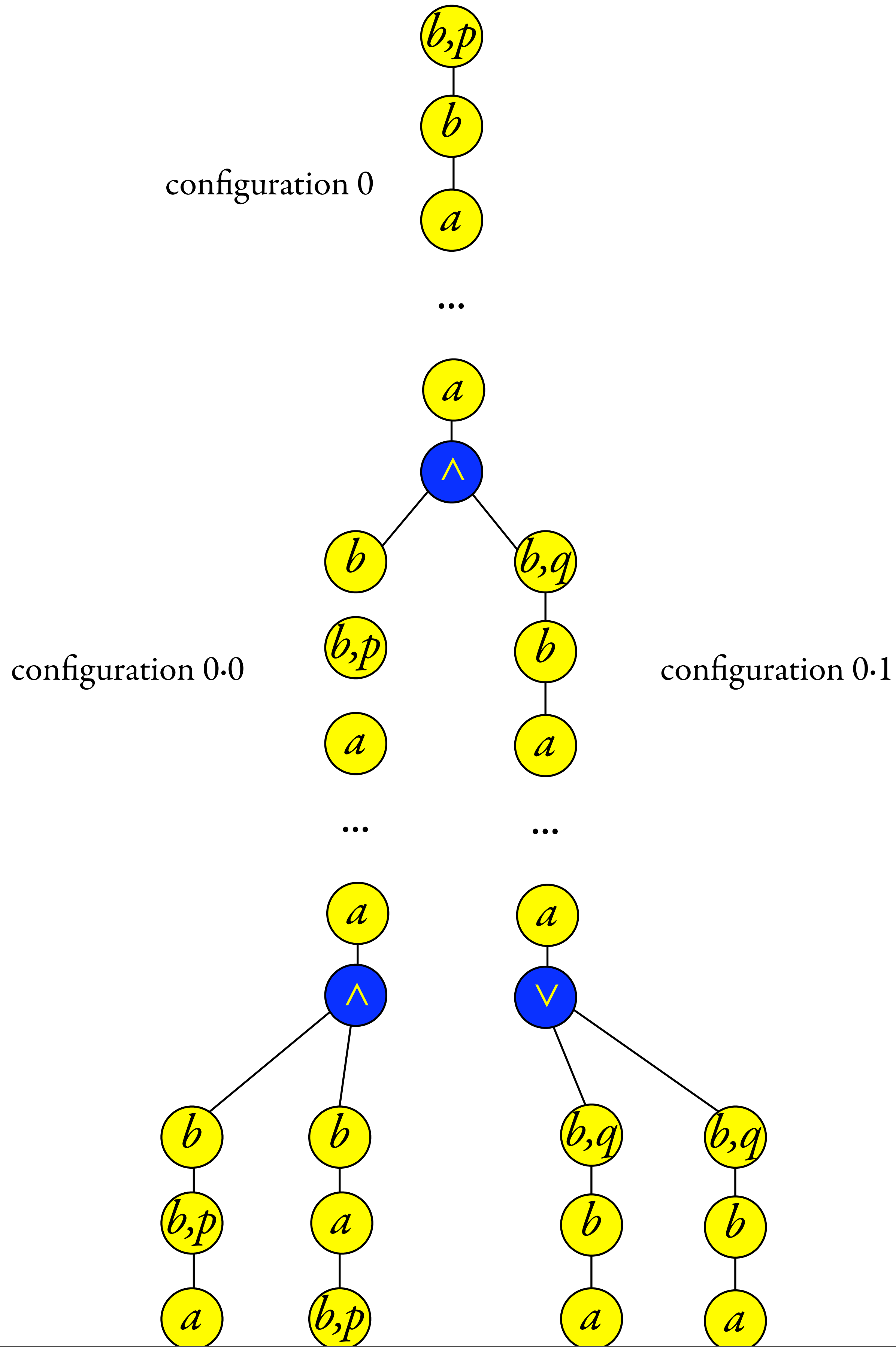

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