Piecewise Testable Tree Languages

Mikołaj Bojańczyk, Luc Segoufin, Howard Straubing

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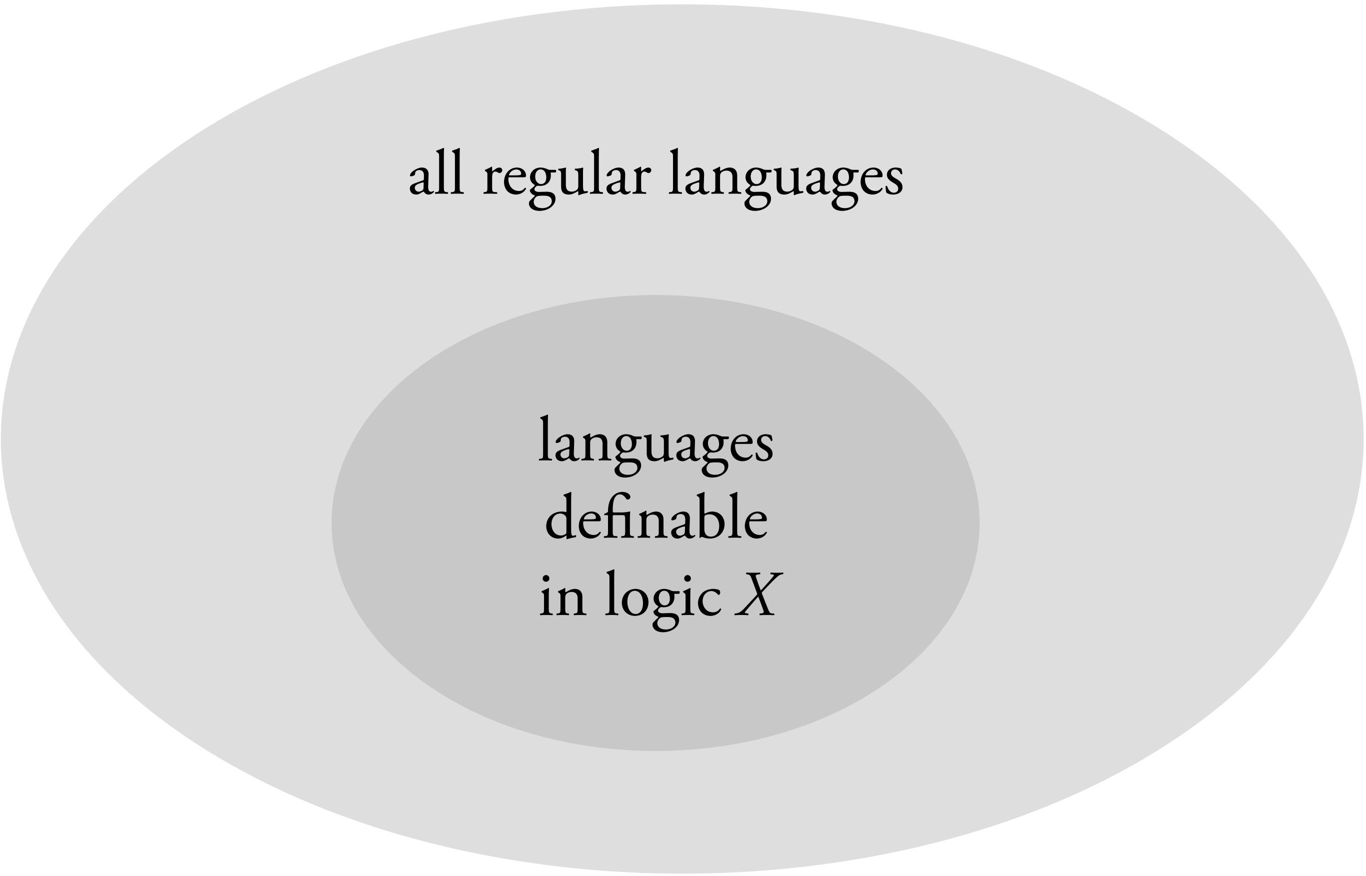

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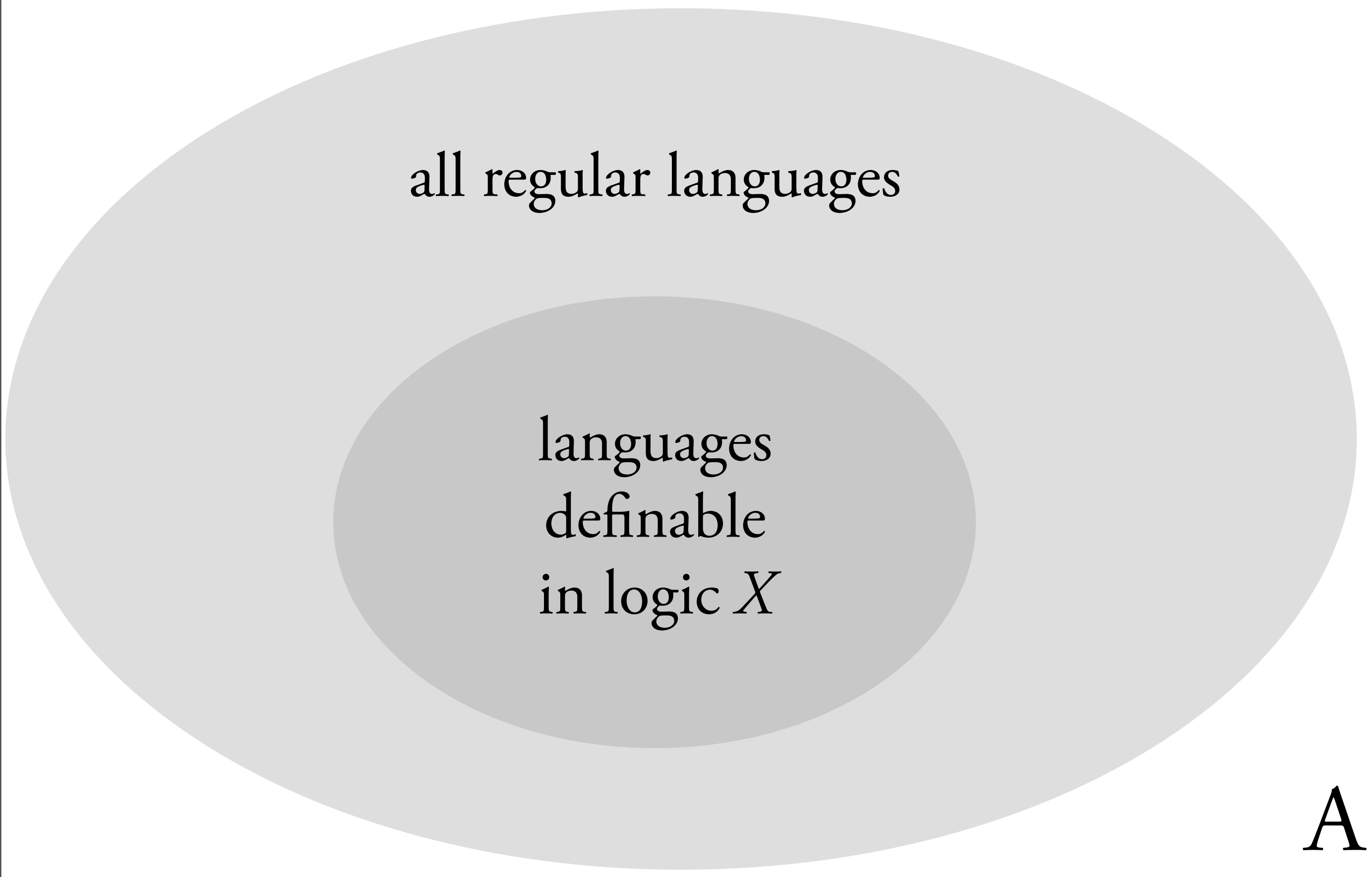

Theorem. (I. Simon, 1975)
A word language is piecewise testable
iff
its syntactic monoid is J-trivial.

a c b a c

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A word language is called *piecewise testable* if it is a boolean combination of languages “words that contain *w* as a piece”

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Fact. A language is piecewise testable iff it can be defined by a boolean combination of $\Sigma_1(\leq)$ formulas.

$$\exists x \exists y\ a(x) \wedge b(y) \wedge x \leq y$$

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$$w \sim_L w'$$

holds if for every $u,v \in \Sigma^*$

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Language	Its syntactic monoid		
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a^*ba^*	a^*	a^*ba^*	$a^*ba^*b(a+b)^*$

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Infix relation in a monoid

For $s,t,u \in M$, we say s is an infix of tsu

We say $s,t \in M$ are in the same *J*-class if they are mutual infixes

Example. The syntactic monoid of $(aa)^*$ has two elements, $(aa)^*$ and $a(aa)^*$, which are in the same *J*-class.

A monoid is *J*-trivial if each *J*-class has one element.

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If s and t are in the same J-class, then for any n one can find representatives of s and t with the same pieces of size n.

w	uwv	$u'uwvv'$	$uu'uwvv'v$	$u'uu'uwvv'vv'v$	…
s	t	s	t	s	

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Several arguments, all difficult.

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There is a rich theory connecting logic, regular languages, and algebra.

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What about trees?

This paper is part of a program to extend the algebra-logic connection to trees

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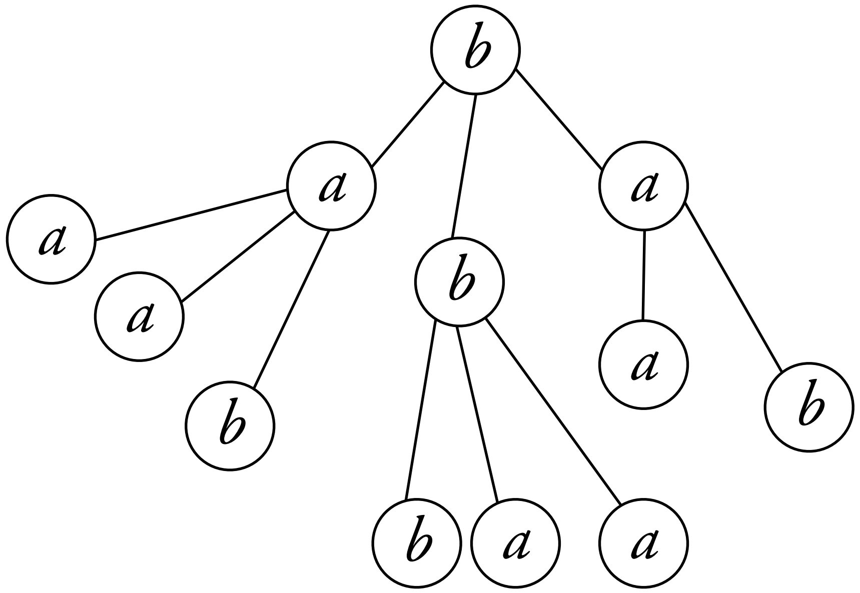
A *tree* is finite, unranked and labeled



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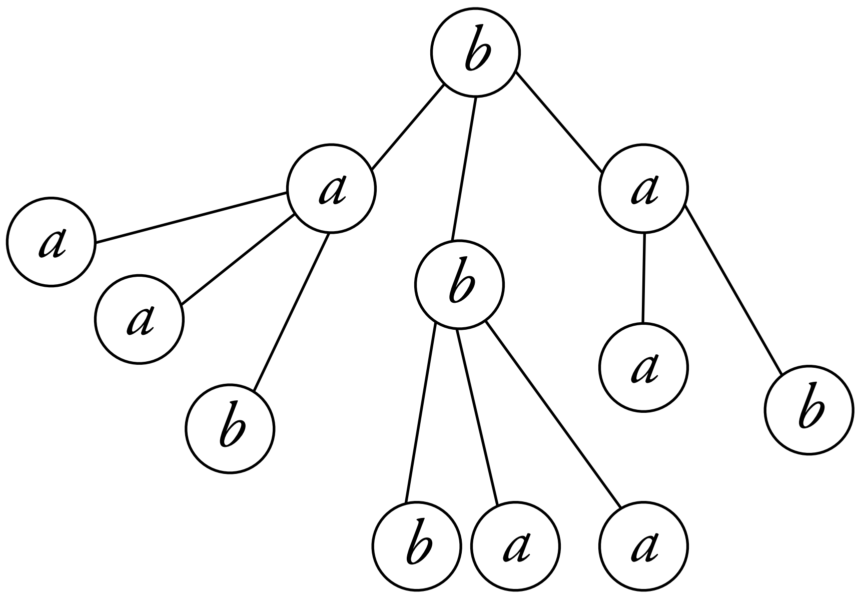
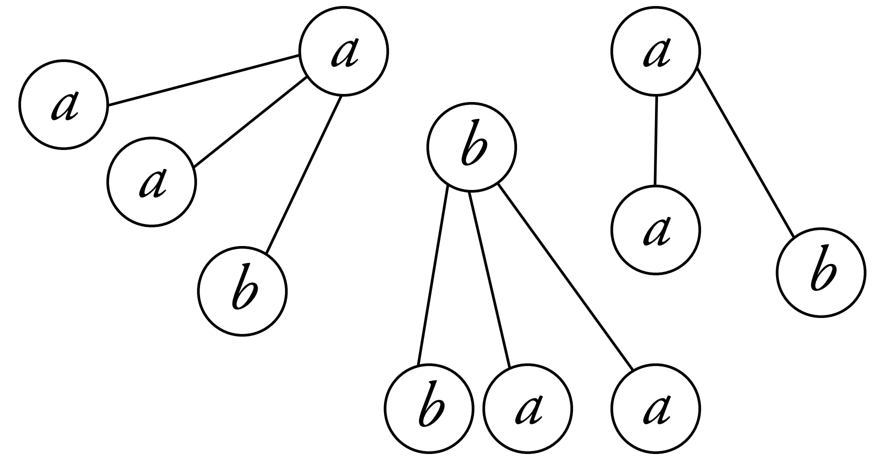


A *forest* is a sequence of trees



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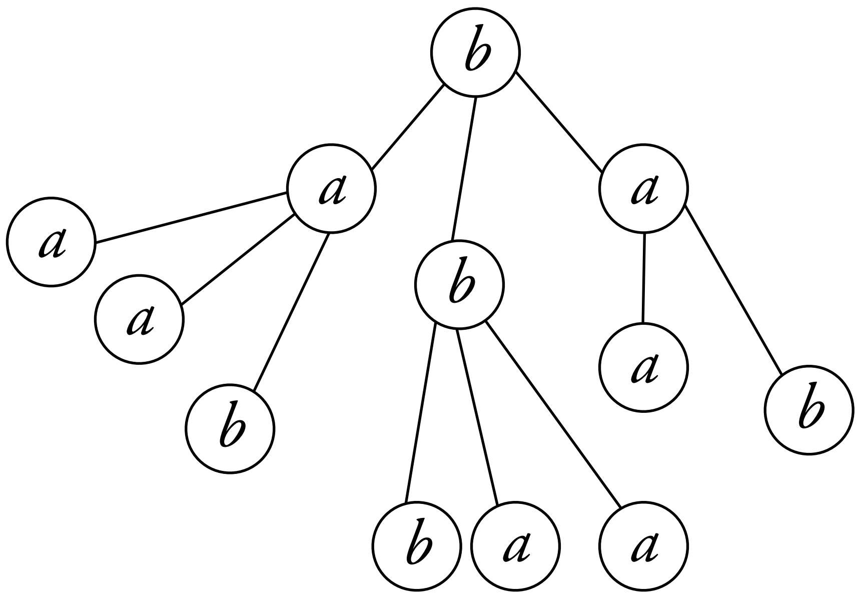

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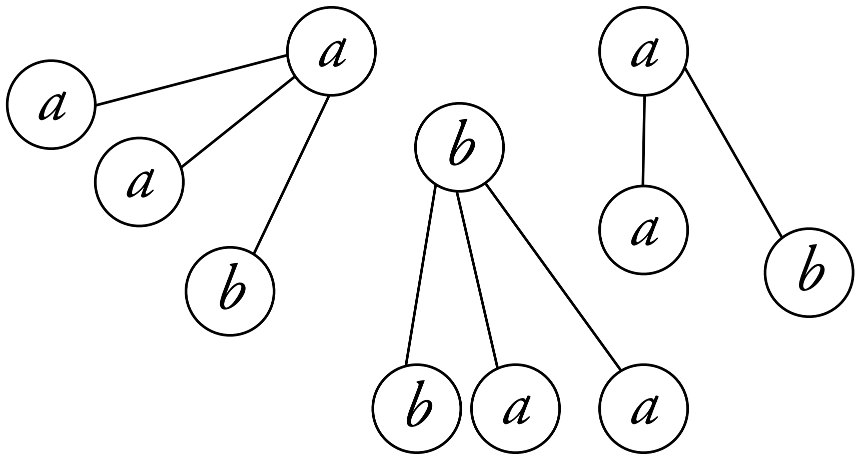
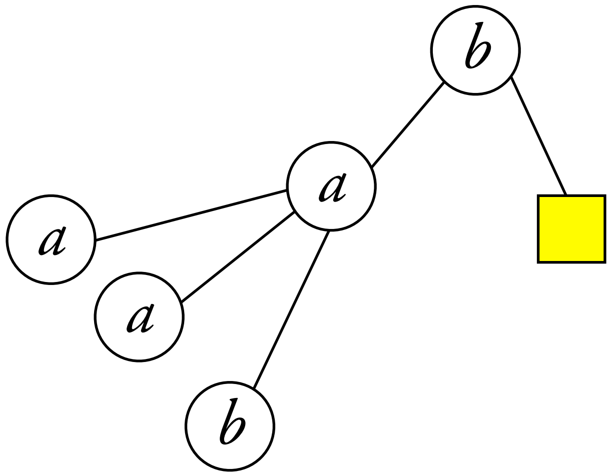


A *context* is a forest with a hole in a leaf



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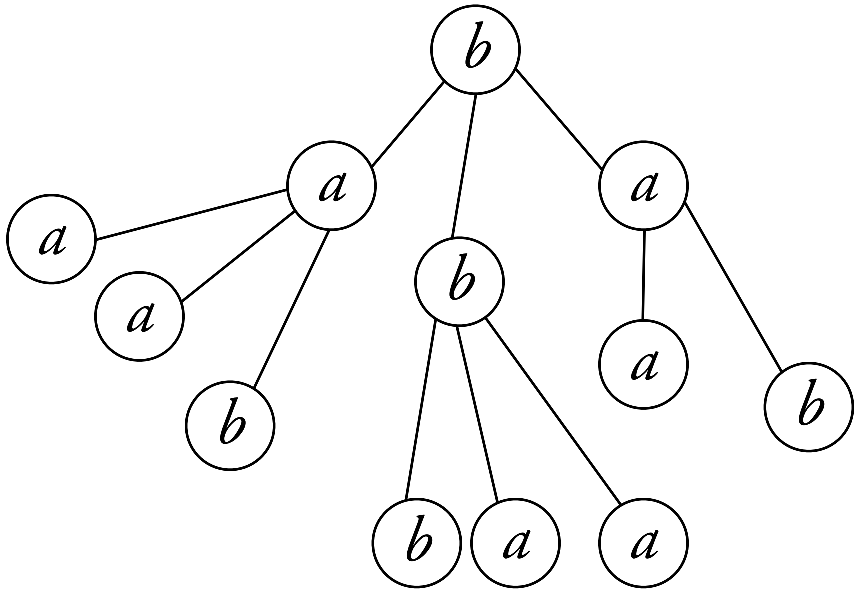

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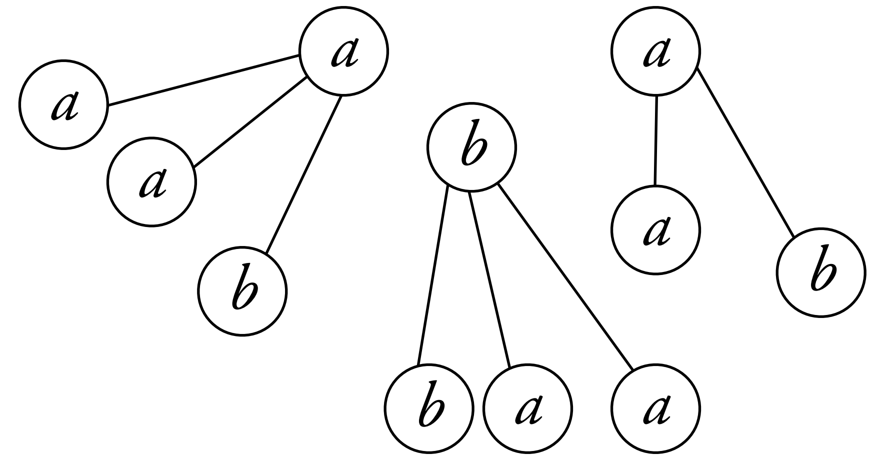

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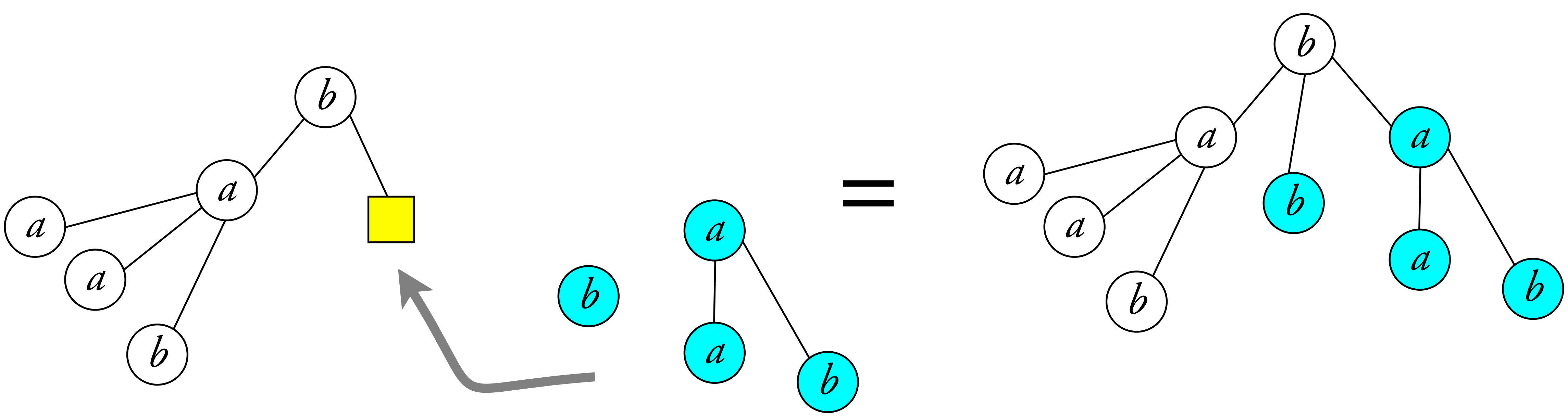
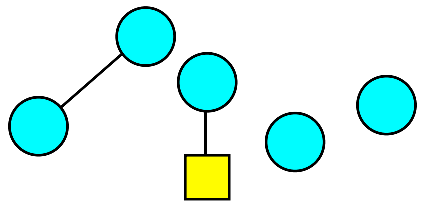
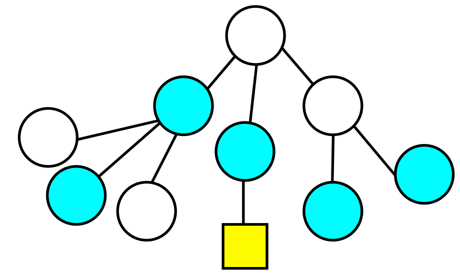
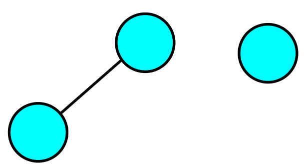
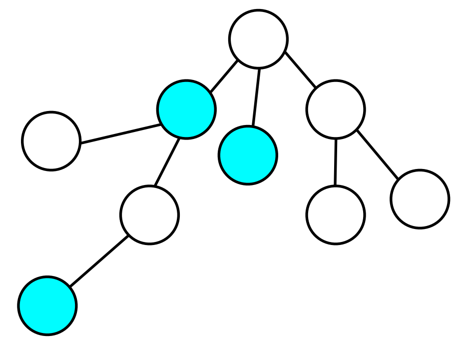


Notion of piece for forests and contexts.

 is a piece of 

 is a piece of 

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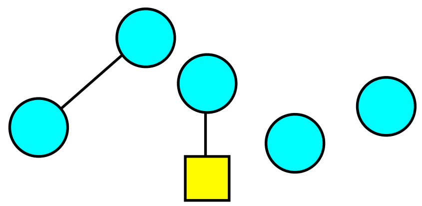
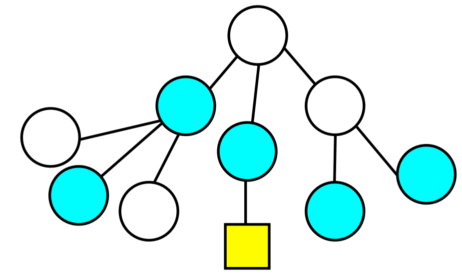
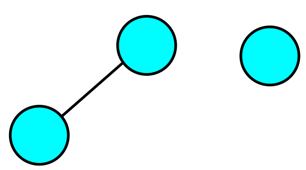
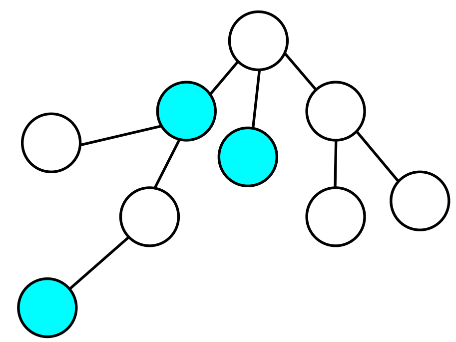
 is a piece of 

 is a piece of 

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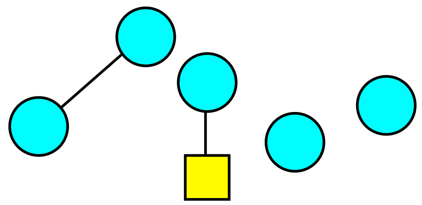
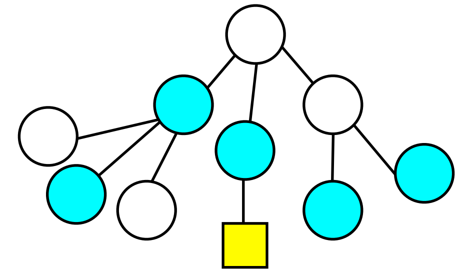
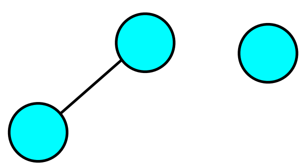
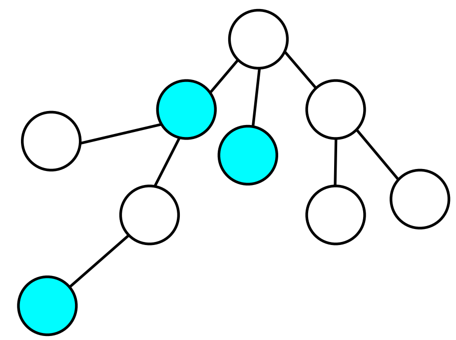
 is a piece of 

 is a piece of 

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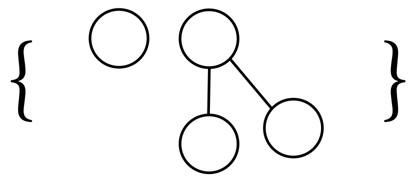
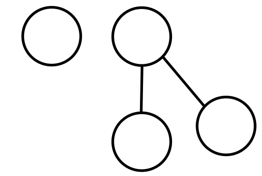
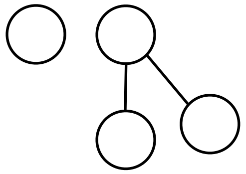
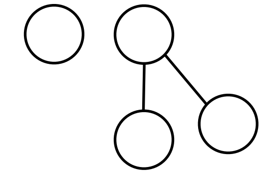


contains piece 

contains no piece with 5 nodes

{  }

contains piece 

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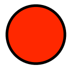
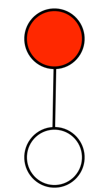
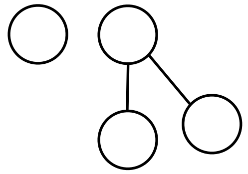
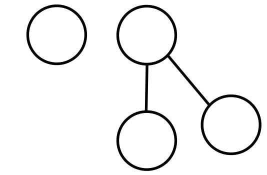
all leaves are 

contains no piece 

{  }

contains piece 

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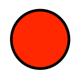
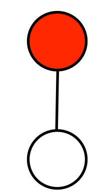
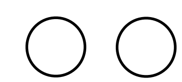
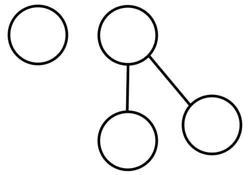
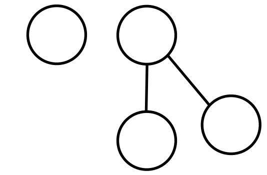
all leaves are 

contains no piece 

forest is a word (vertically)

contains no piece 

{  }

contains piece 

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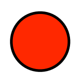
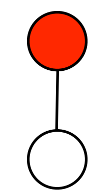
all leaves are 

contains no piece 

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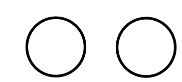
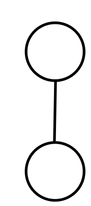
contains no piece 

forest is a word (horizontally)

contains no piece 

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We want the forest extension of:

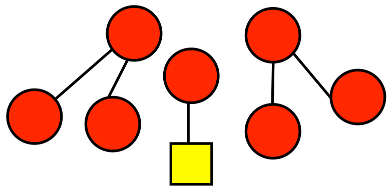
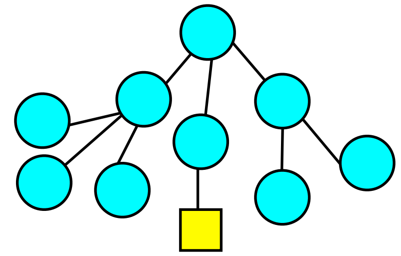
Theorem. (I. Simon, 1975)
A word language is piecewise testable
iff
its syntactic monoid is J-trivial.

What is a syntactic monoid for forest languages?

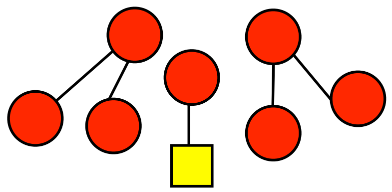
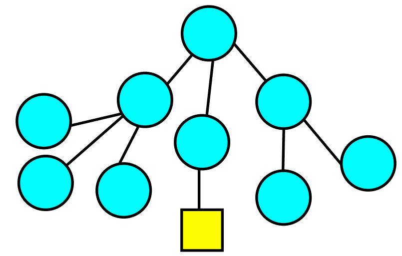
Although a definition exists (forest algebra), here we will only talk about Myhill-Nerode equivalence.

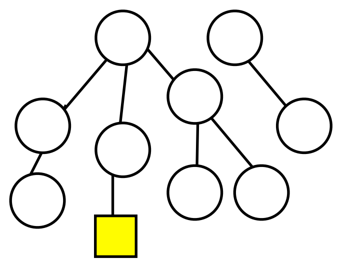
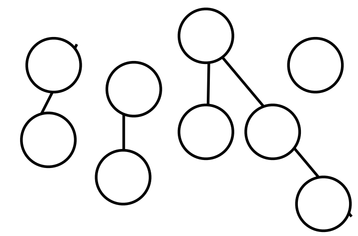
Myhill-Nerode congruence for a forest language L.

Myhill-Nerode congruence for a forest language *L*.

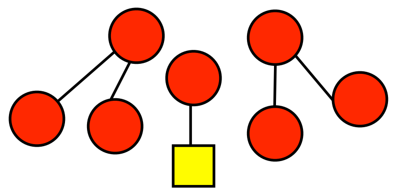
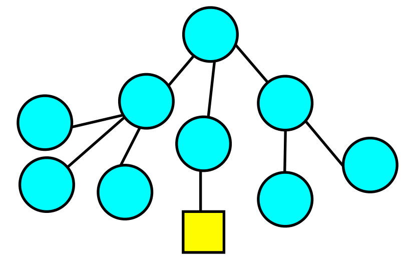
Two contexts  and  are called *L*-equivalent if

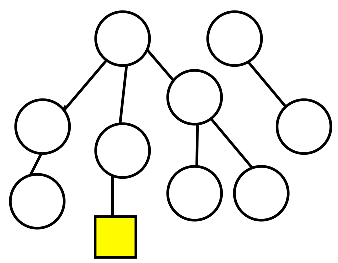
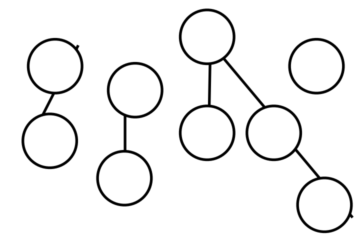
Myhill-Nerode congruence for a forest language L.

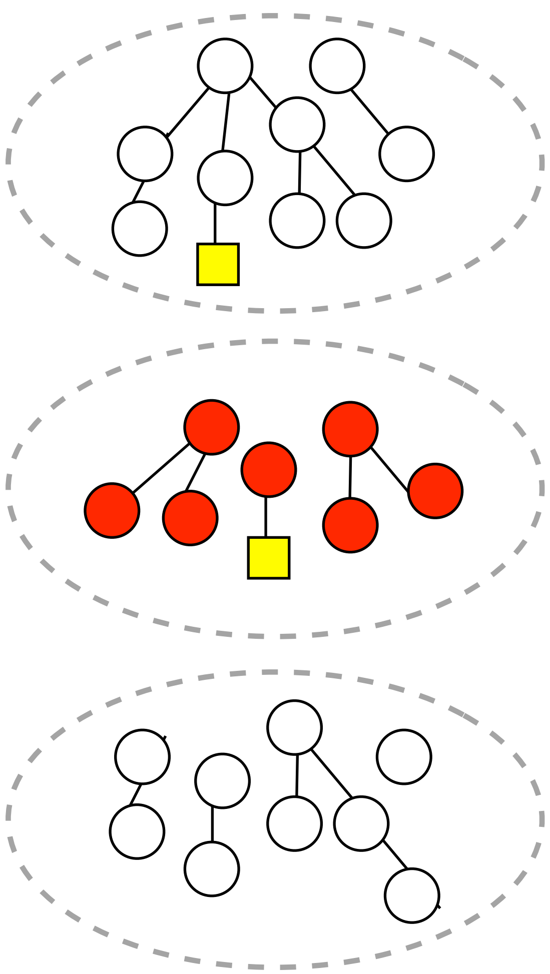
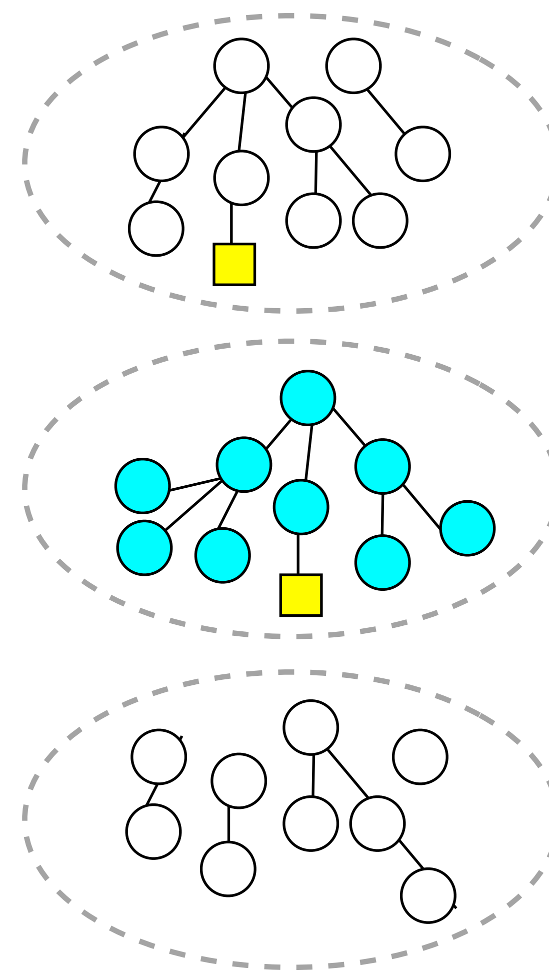
Two contexts  and  are called L-equivalent if

for every context  and every forest 

Myhill-Nerode congruence for a forest language L.

Two contexts  and  are called L-equivalent if

for every context  and every forest 

 $\in L$ iff  $\in L$

Main Theorem.

A forest language is piecewise testable iff

the following holds for all sufficiently large n

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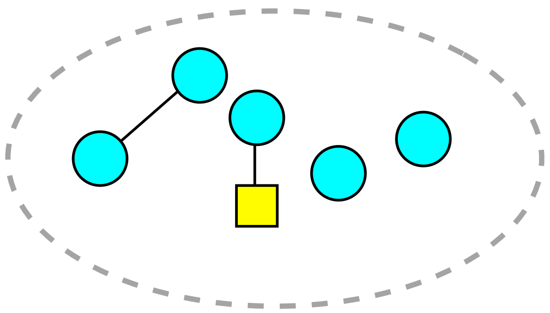
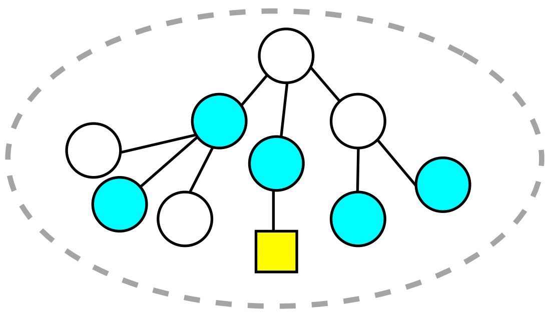
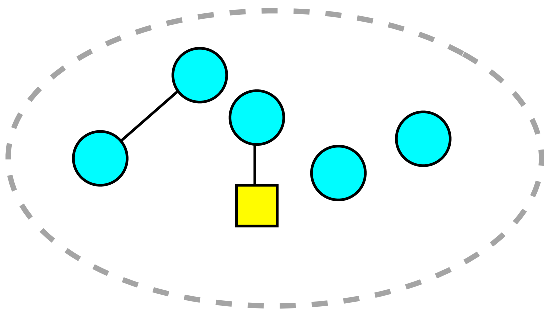
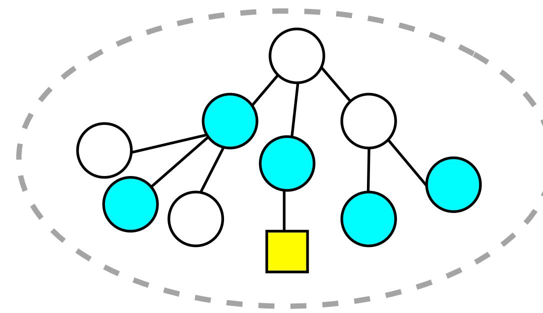
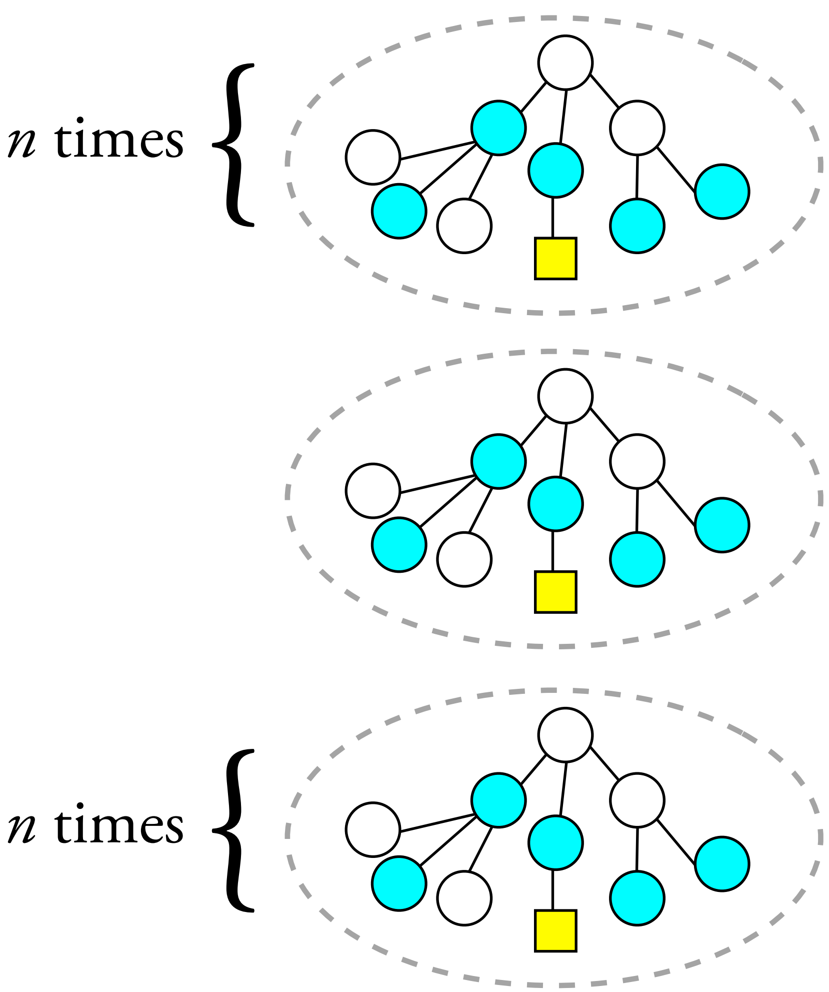
if  is a piece of  , then

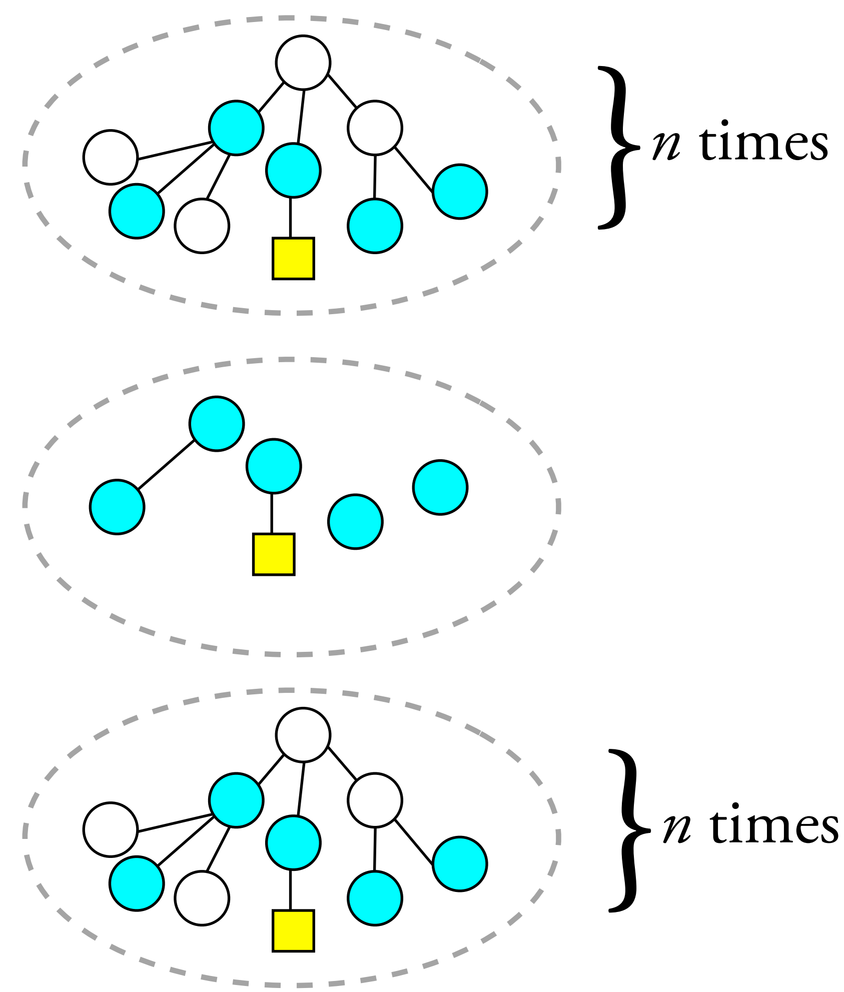
Main Theorem.

A forest language is piecewise testable iff the following holds for all sufficiently large n

if  is a piece of  , then



is equivalent to



Main Theorem.

A forest language is piecewise testable iff
the following holds for all sufficiently large n

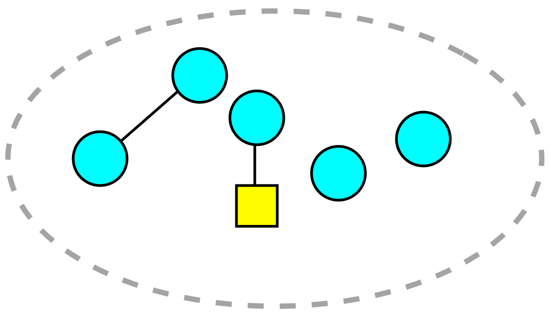
This criterion is decidable.

We also have variants of the theorem for

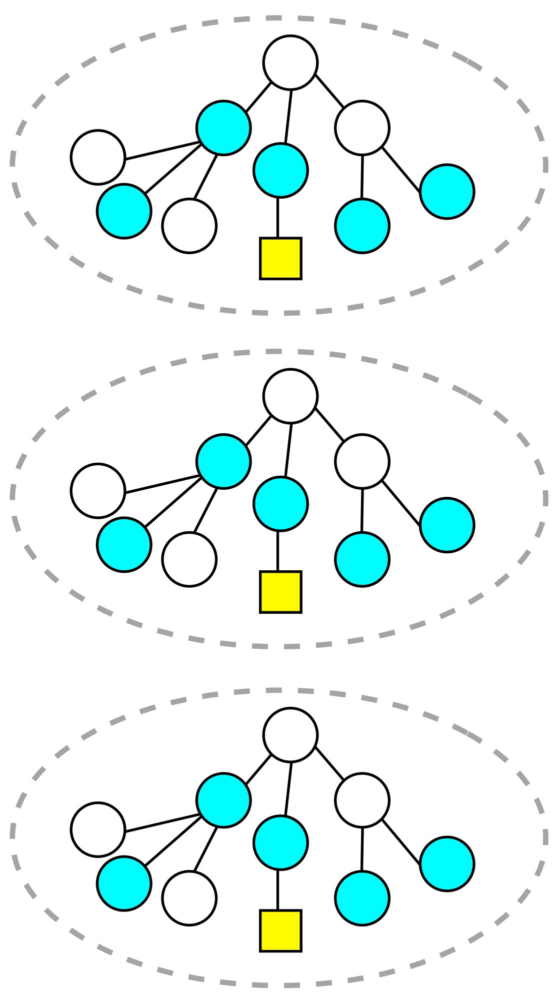
– tree languages

– commutative pieces

– pieces with closest common ancestor

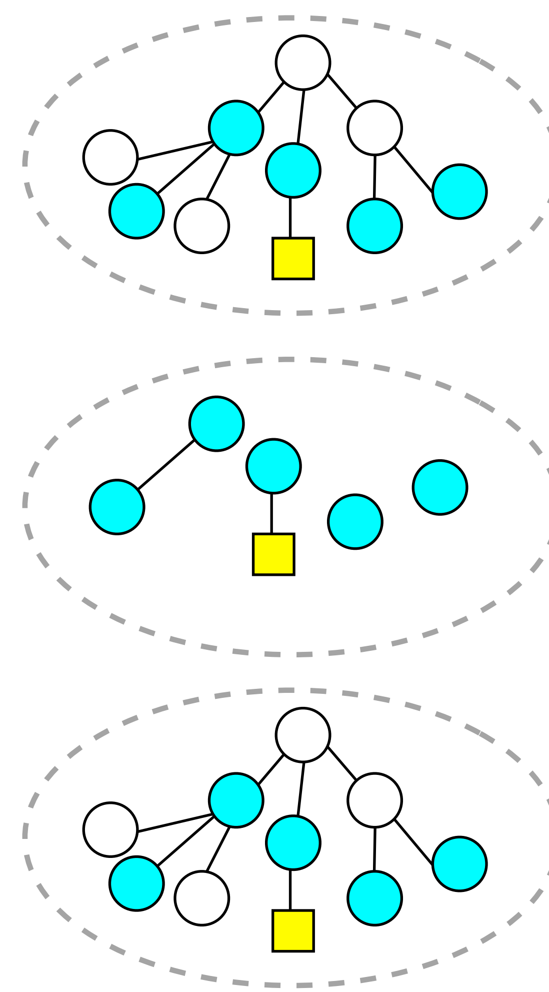
if  is a piece of …, then

n times



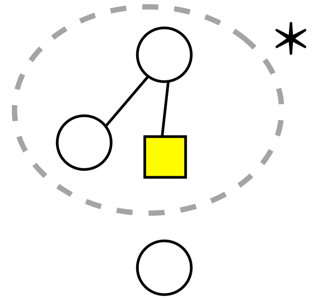
n times

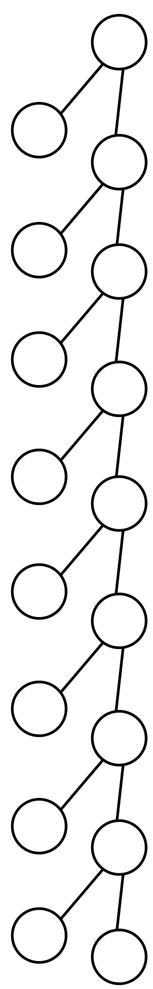
is equivalent to



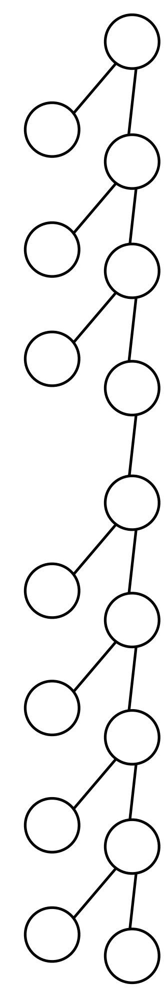
n times

n times

The language  has a J-trivial syntactic monoid, but is not piecewise testable



is confused with



Big project: understand the expressive power of first-order logic on trees.

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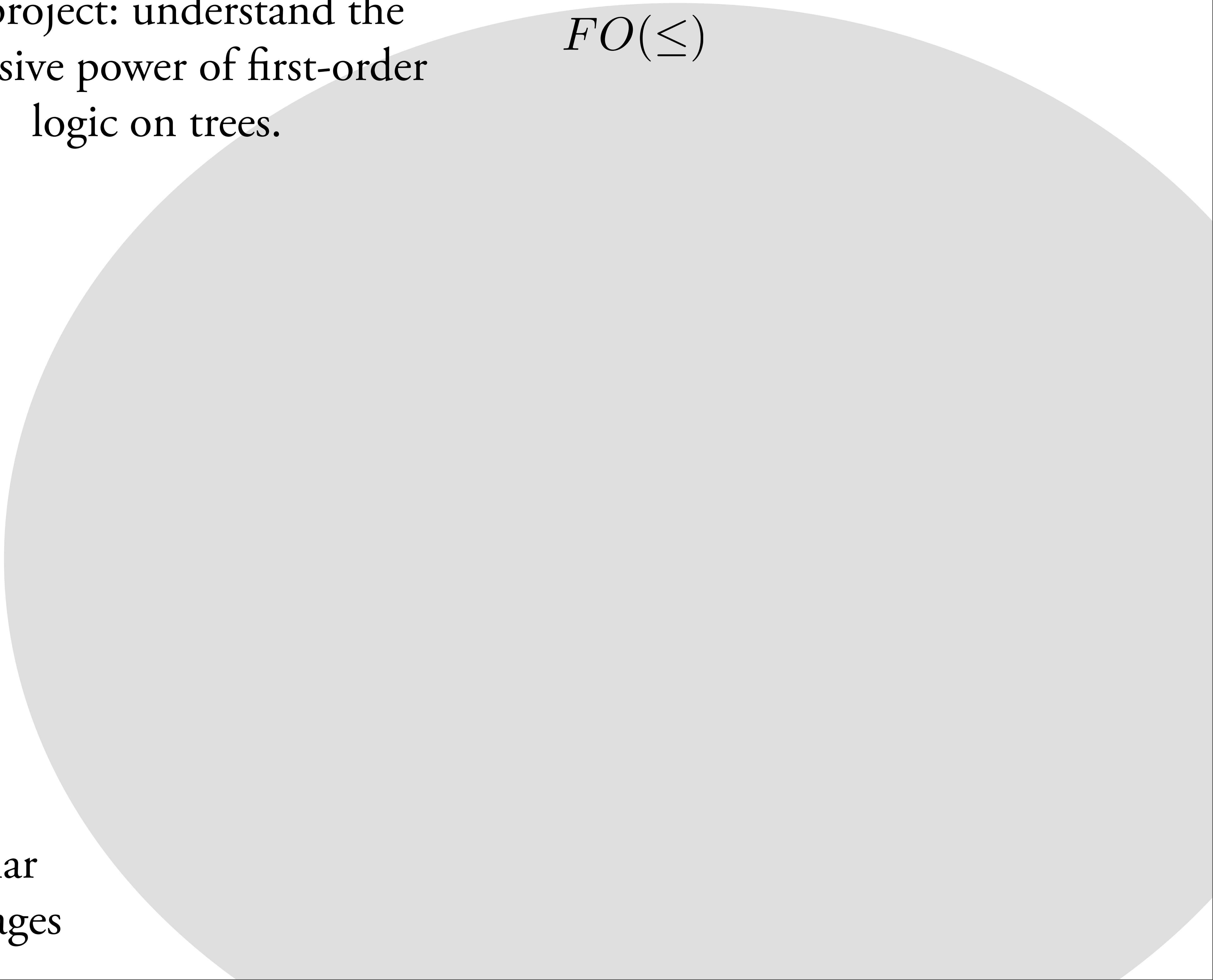
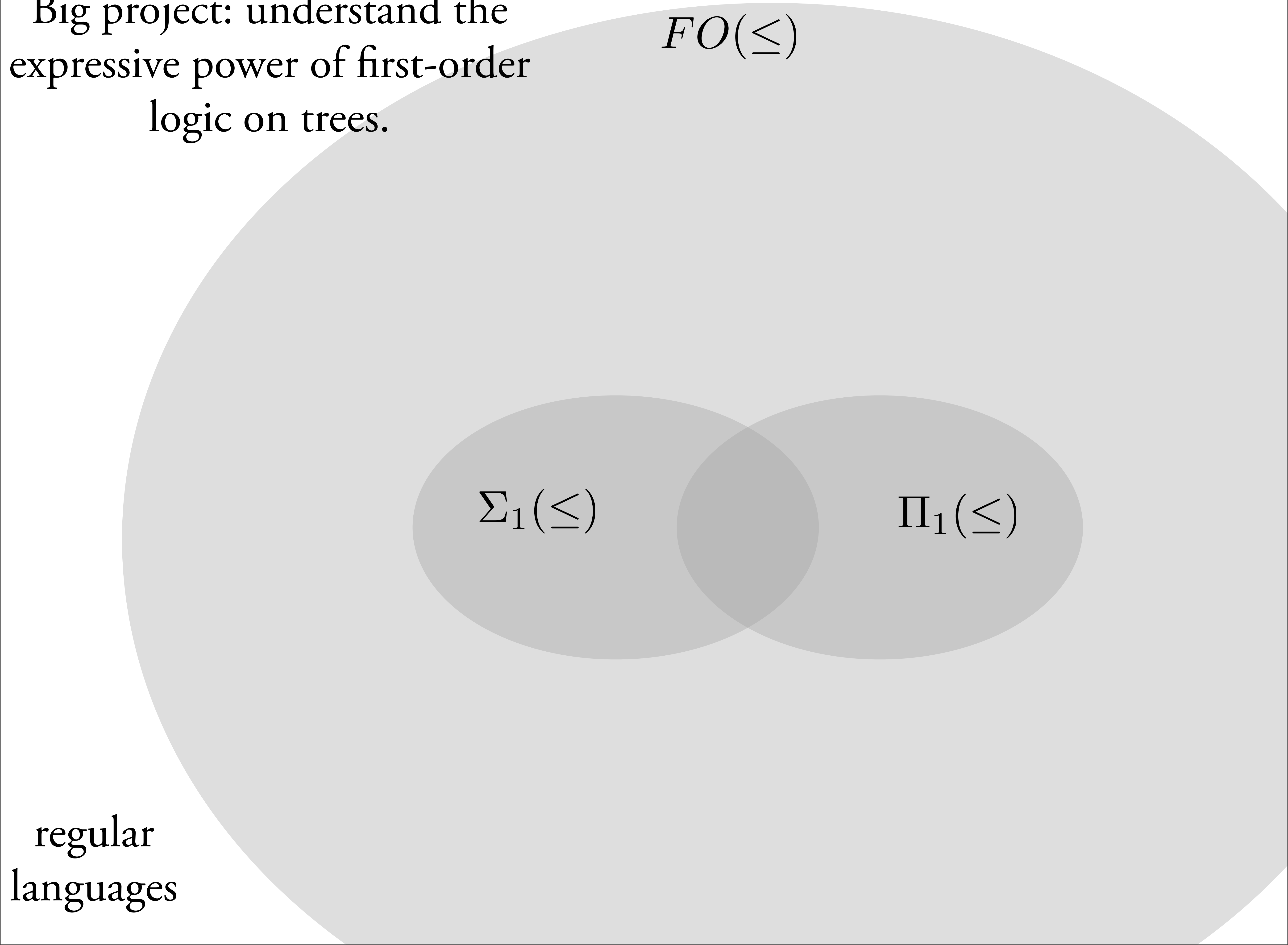


regular languages

Big project: understand the expressive power of first-order logic on trees.



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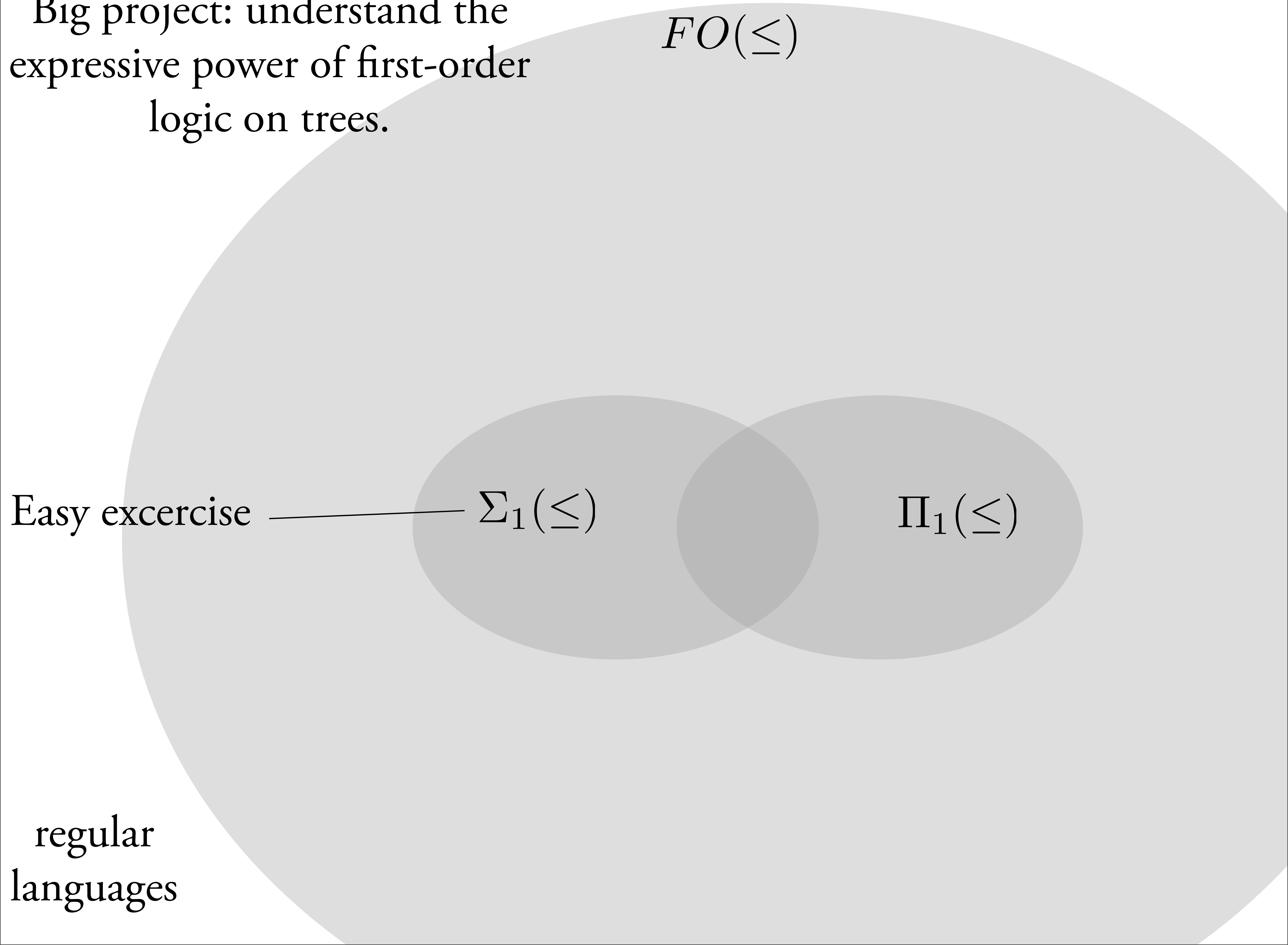

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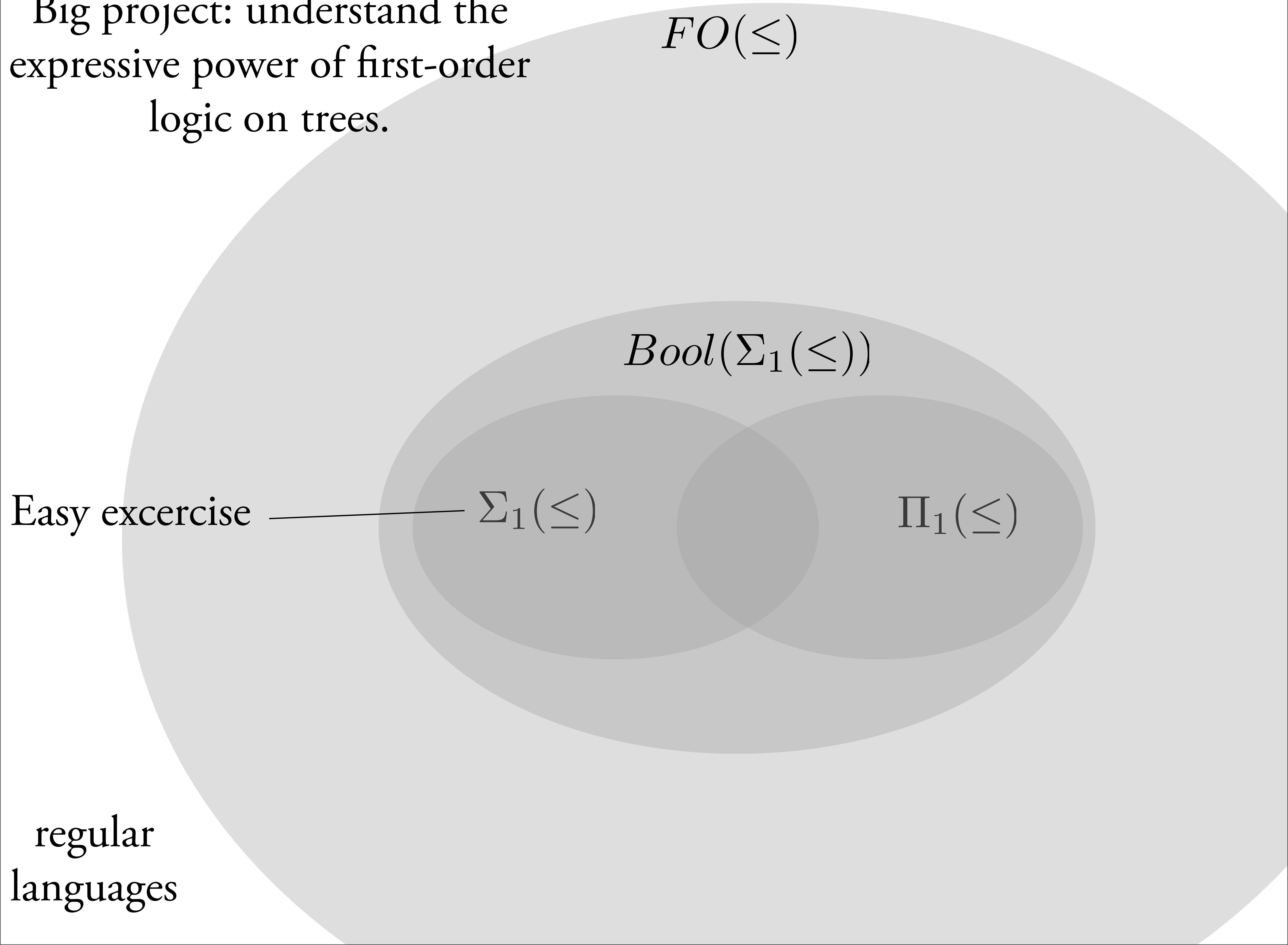

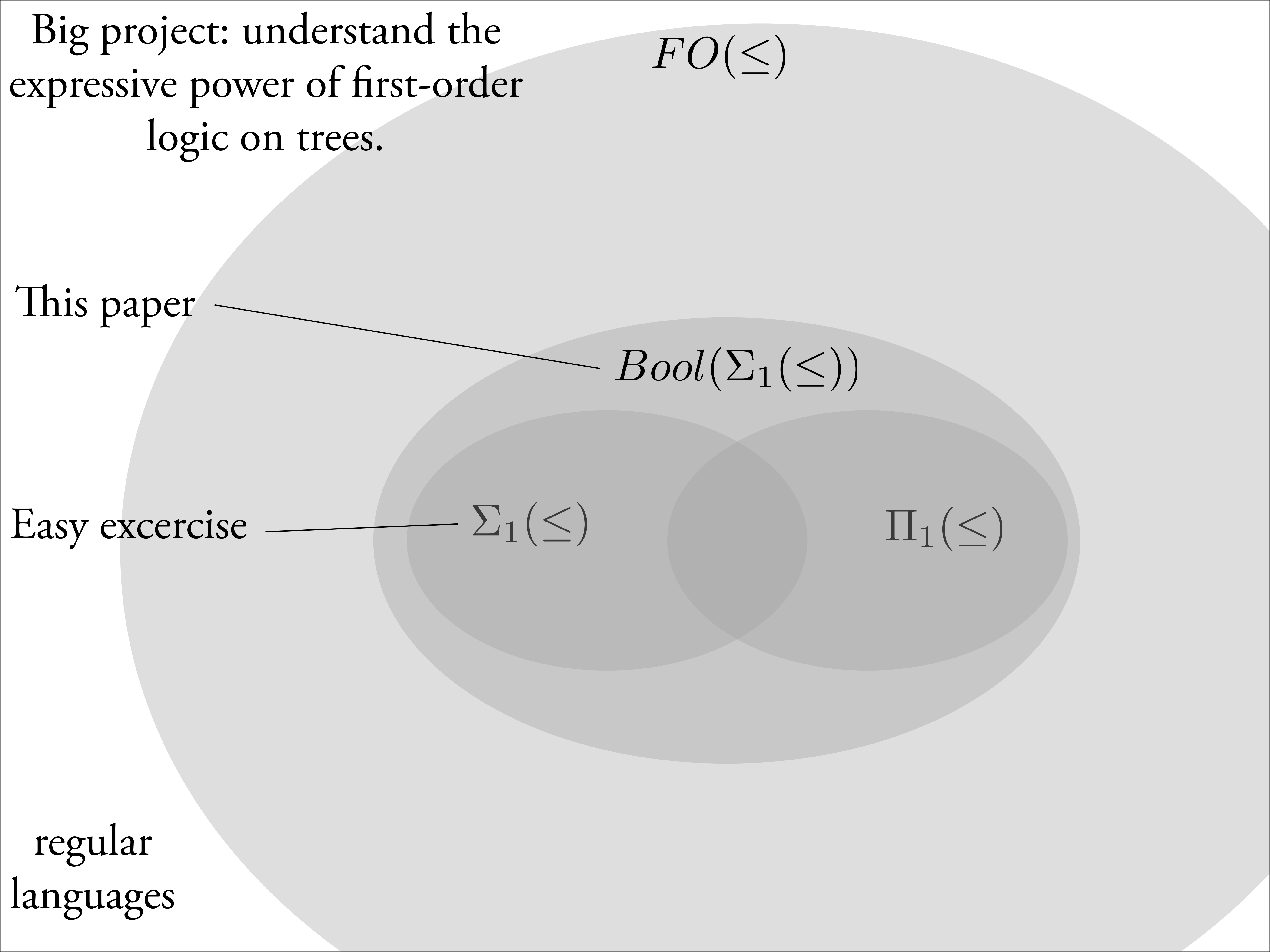
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$FO(\leq)$
This paper
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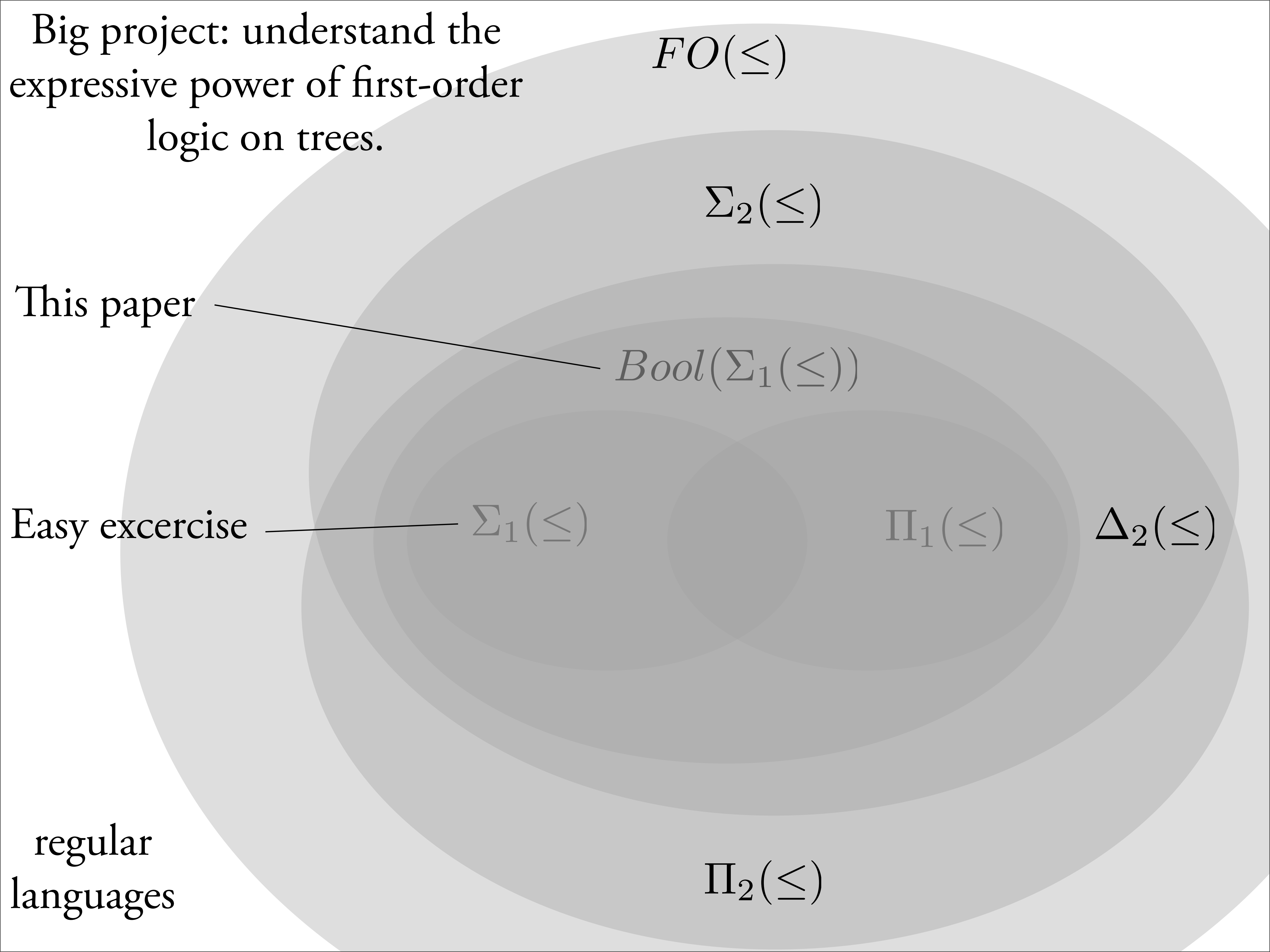
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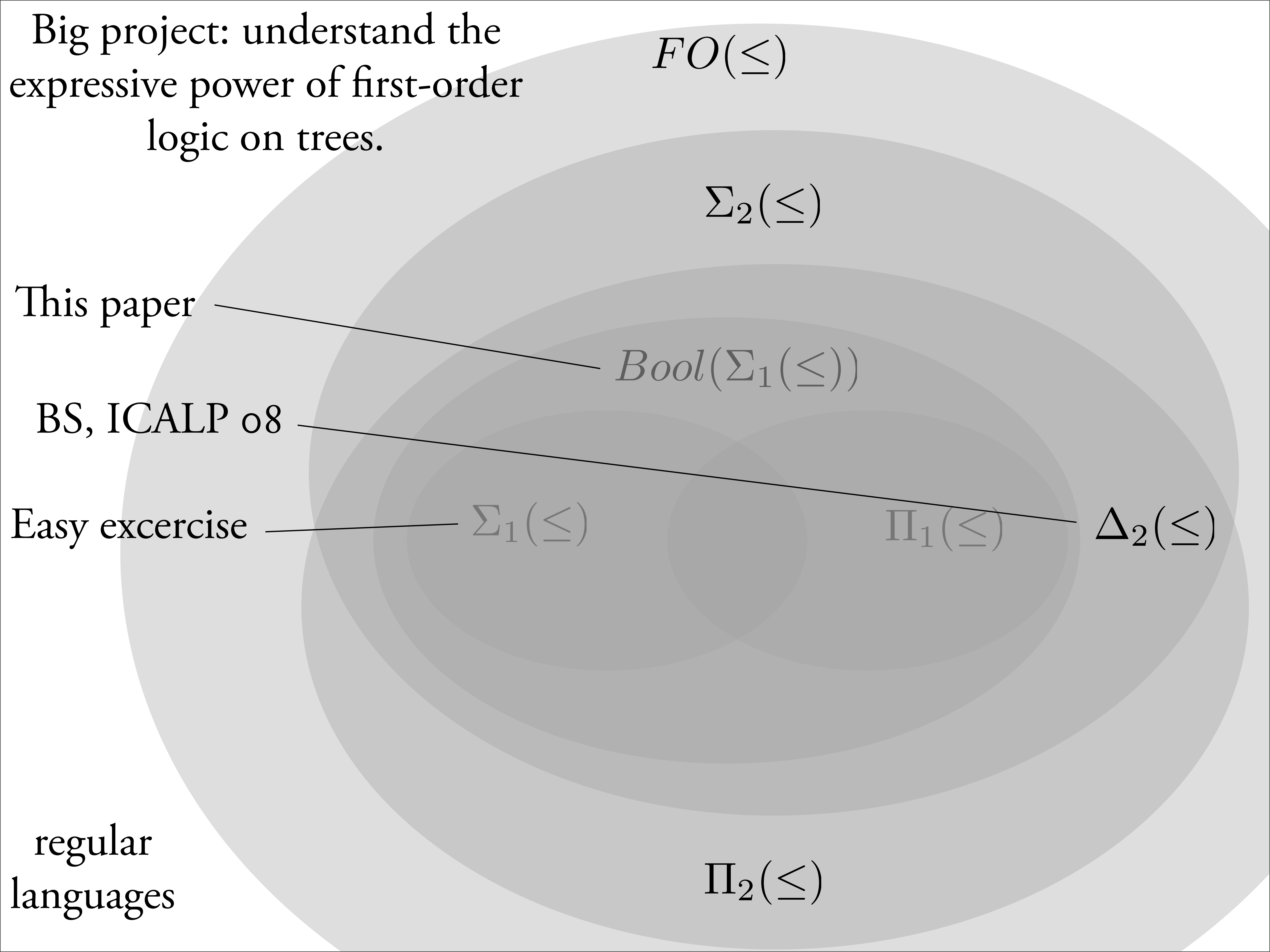

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BS, ICALP 08
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