Tree Languages Definable with One Quantifier Alternation

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The following problem is decidable:

Input: A regular tree language L, given by a tree automaton.

Question: Is L definable by a formula with quantifier prefix $\exists^* \forall^*$ and also by a formula with quantifier prefix $\forall^*\exists^*$

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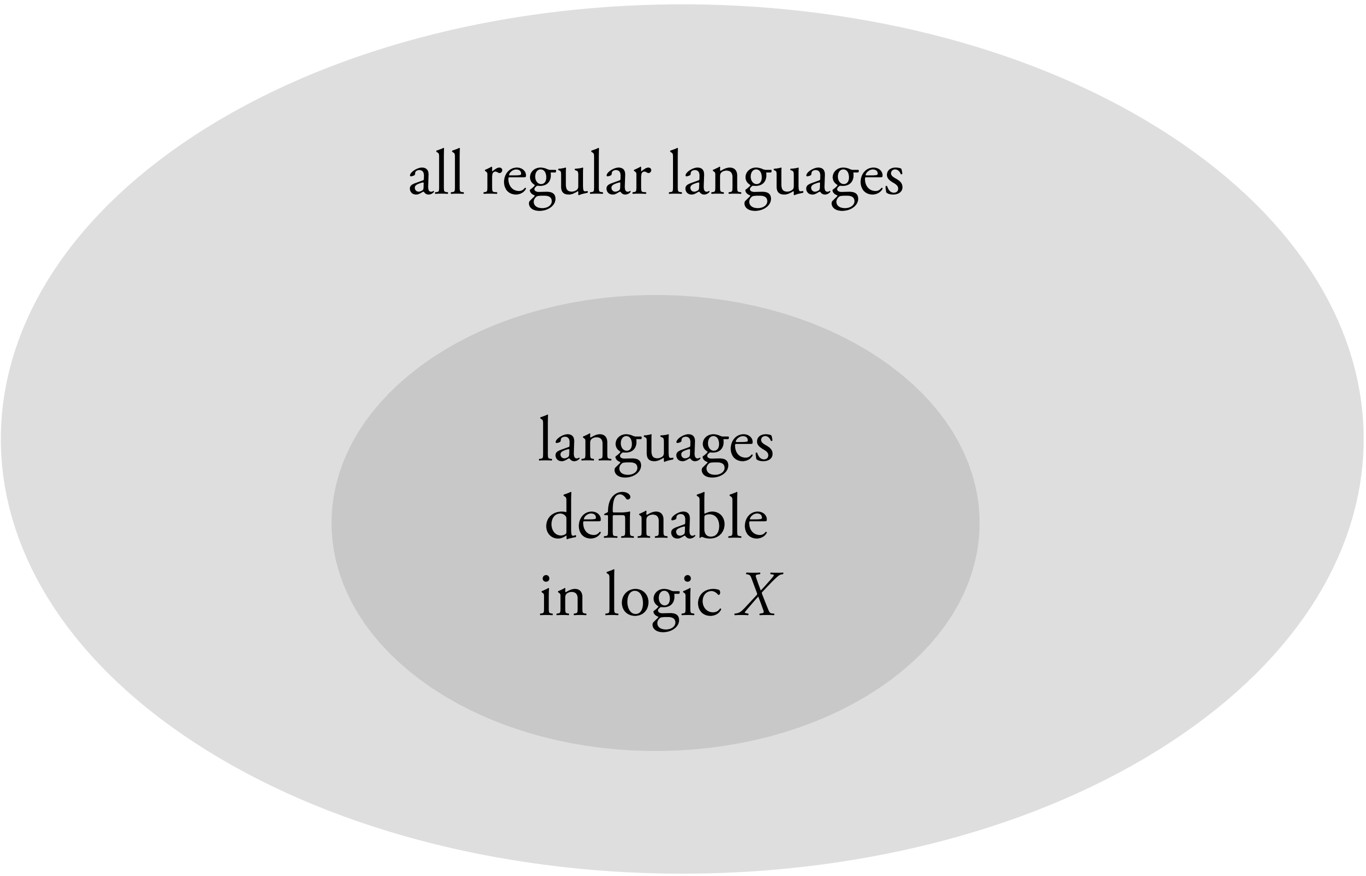
Understand logic X =
give na algorithm to decide if a language L is definable in X



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the quantifier alternation hierarchy, etc.

This paper is part of a program investigating the algebra-logic connection for trees. Eventually, we want to answer questions such as:

– what is the expressive power of first-order logic on trees?
– what is a tree group?
– is there a Krohn-Rhodes decomposition theory?

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$$\exists x.\ a(x) \ \wedge \ \big(\forall y < x.\ b(y)\big)$$

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$$\mathsf{F}\big(a \ \wedge\ \neg(\mathsf{F}^{-1}c)\big)$$

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$$\forall x \exists y.\ c(x) \Rightarrow \big(y < x \wedge a(y)\big)$$

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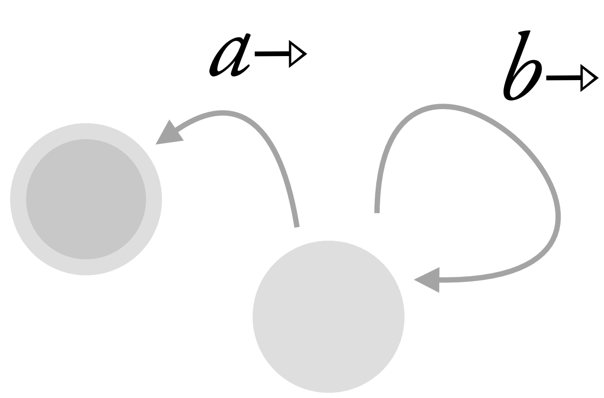

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   "go right to first *a;* go left to first *c*" fails
   "go right to first *a*" works

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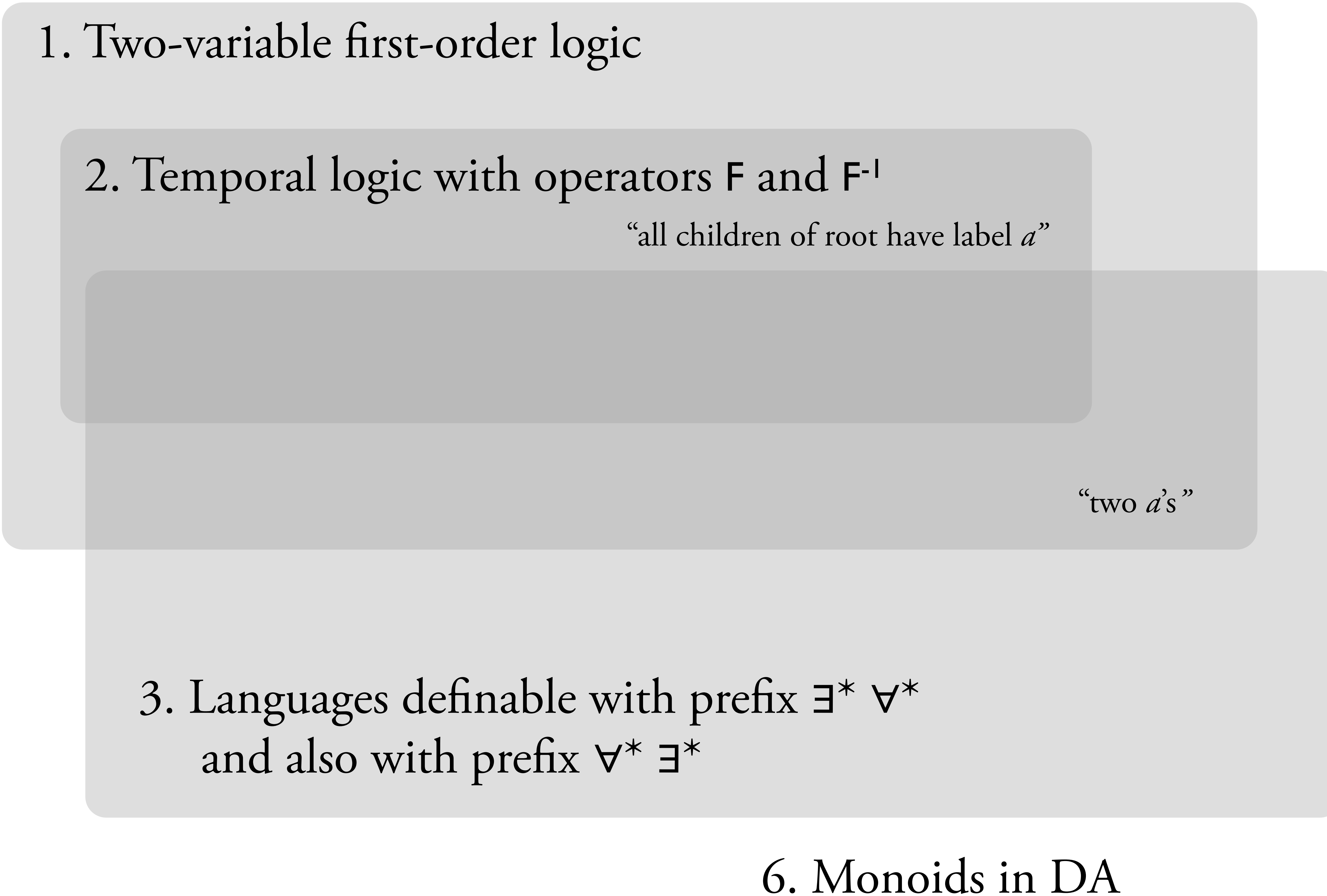
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"all children of root have label *a*"
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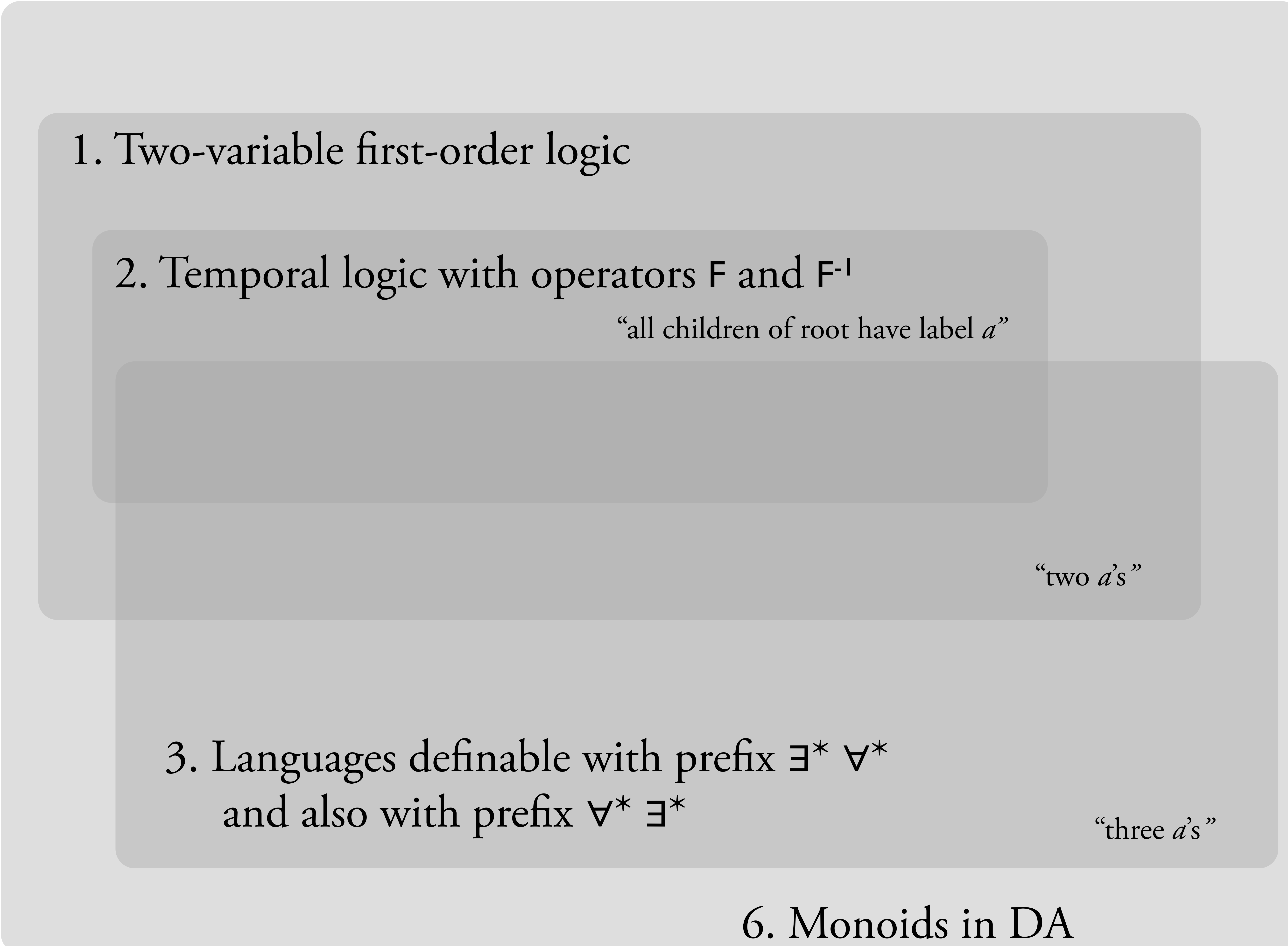
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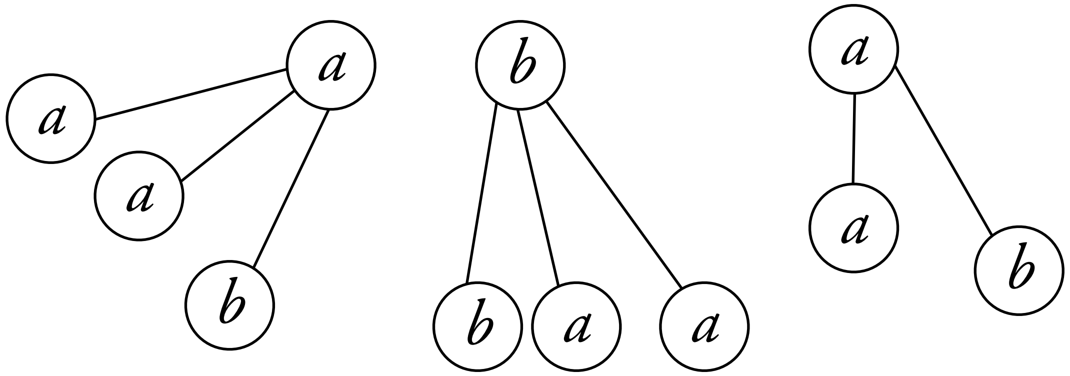
We consider forest languages
instead of tree languages

(a *forest* is a sequence of trees)



We use first-order formulas to describe properties of forests.
Variables quantify over nodes. Predicates allowed are:

"*x* ancestor of *y*" "*x* lexicographically before *y*" "label of *x* is *a*"

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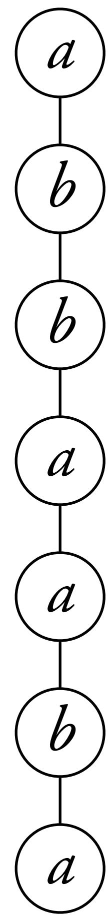
Question:

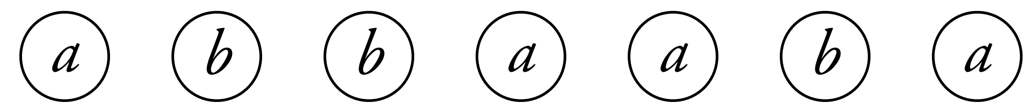
What forest languages can be defined in Δ_2?

Preferably, give an algorithm that decides if $L \in \Delta_2$.

a word w can have two encodings:



ver(w)

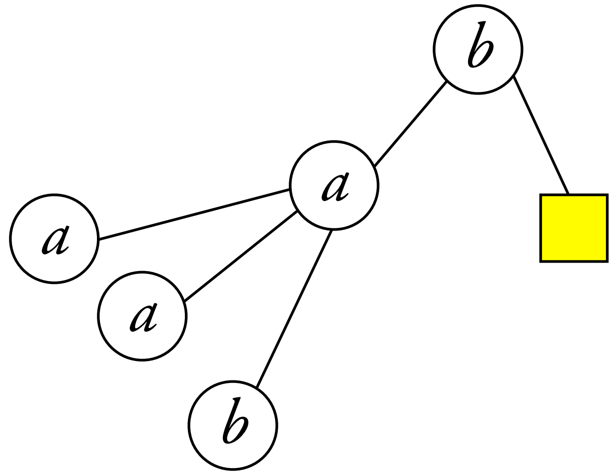


hor(w)

Fact. if L is a word language definable in Δ_2, then both hor(L) and ver(L) are forest languages definable in Δ_2.

Our characterization is stated as an identity. Intuitively, a forest language is definable in Δ_2 iff it admits a certain pumping lemma.

A *context* is a forest with a hole in a leaf



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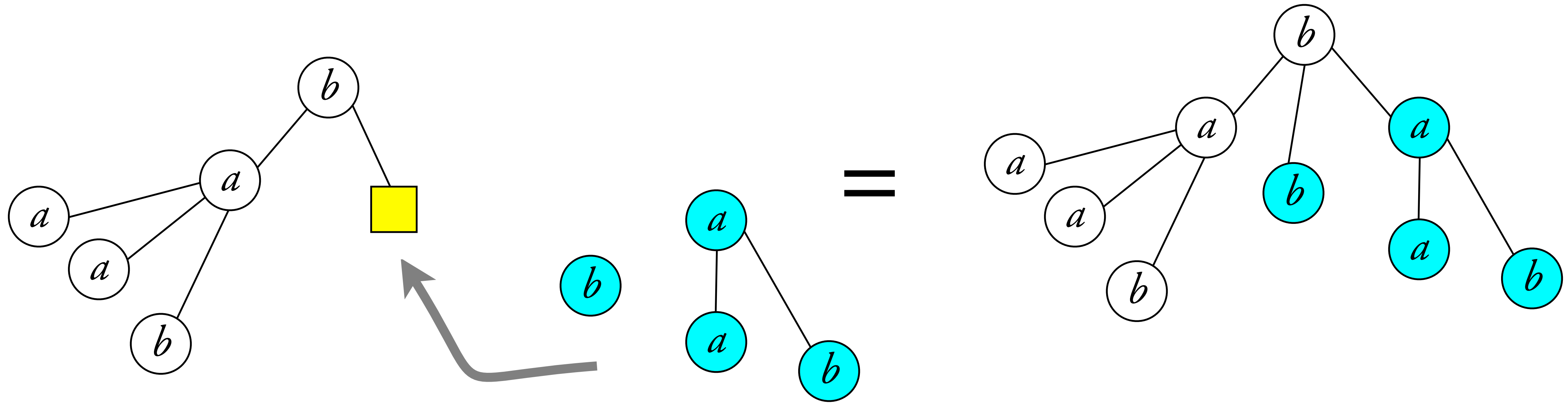

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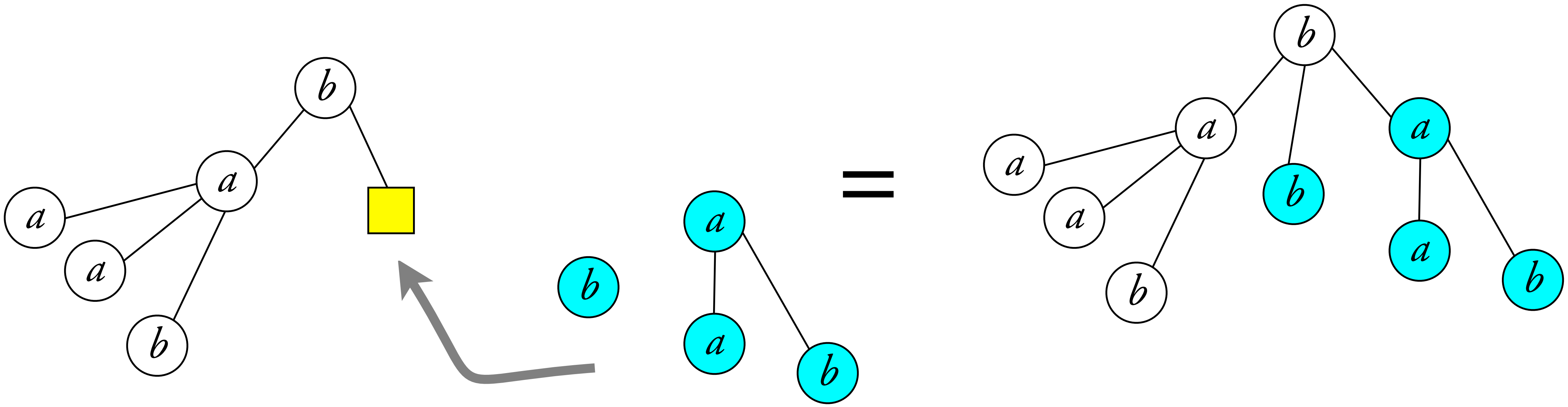

Notion of piece for contexts.

Myhill-Nerode congruence for a forest language L.

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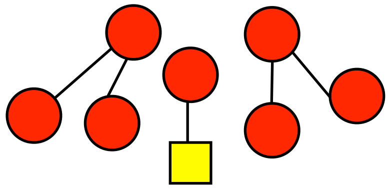
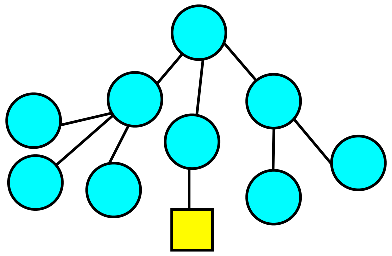
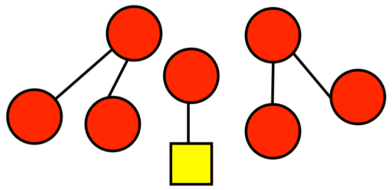
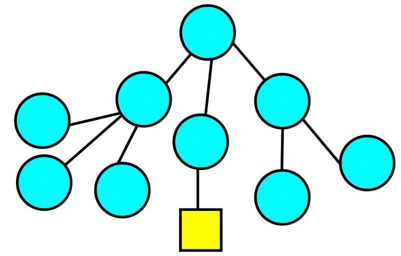
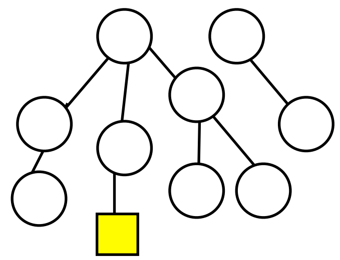
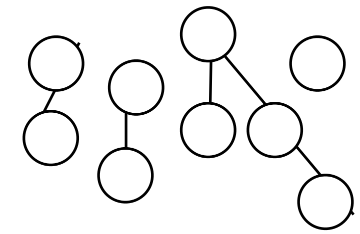
Two contexts  and  are called L-equivalent if

Myhill-Nerode congruence for a forest language L.

Two contexts  and  are called L-equivalent if

for every context  and every forest 

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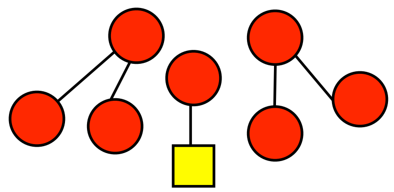
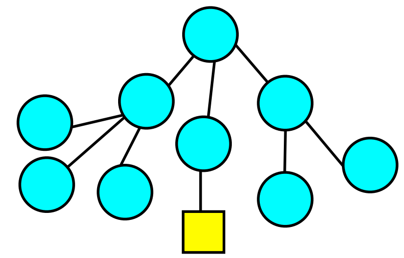
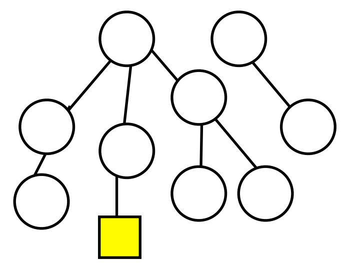
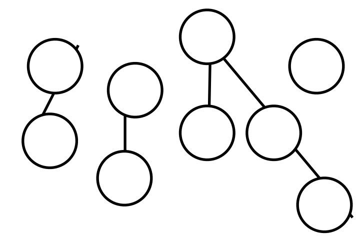
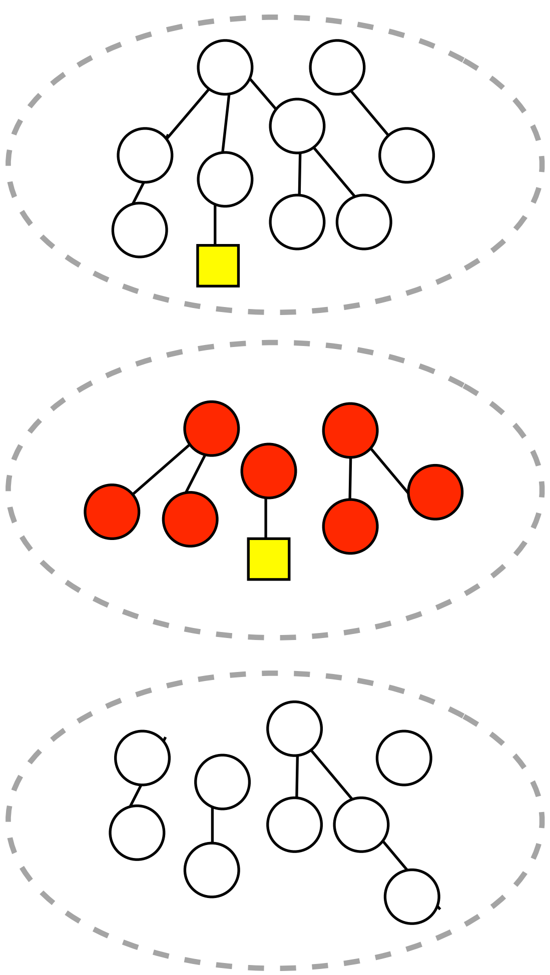
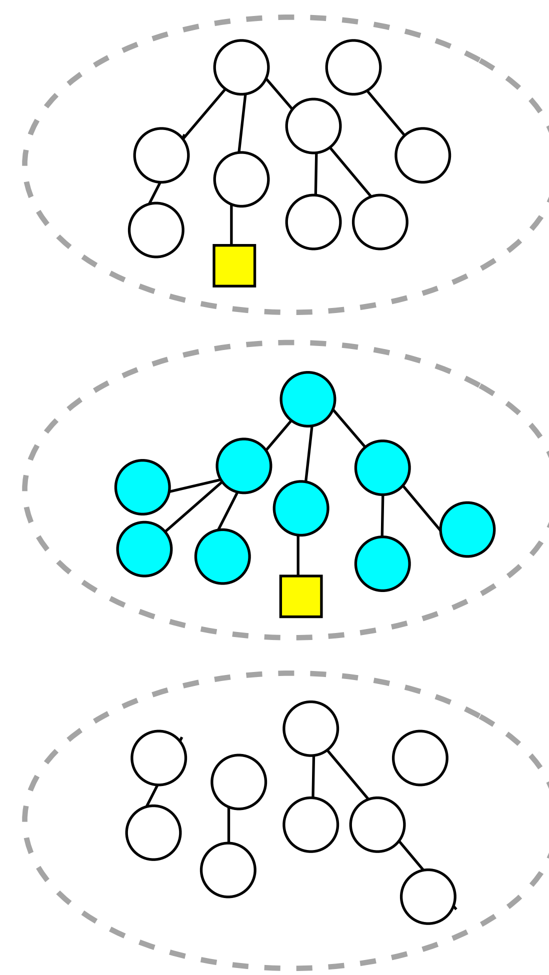
Two contexts  and  are called L-equivalent if

for every context  and every forest 

 $\in L$ iff  $\in L$

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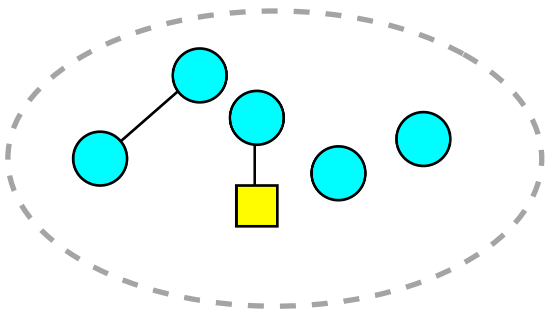
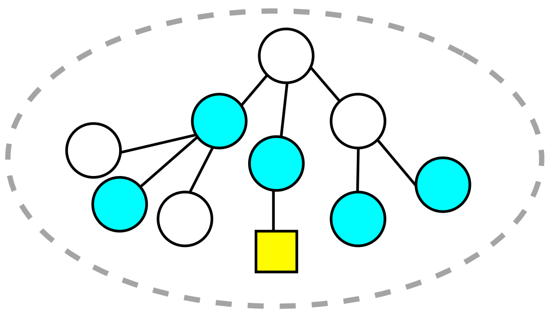
if  is a piece of  , then

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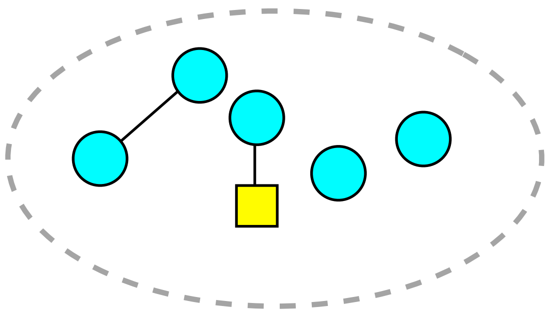
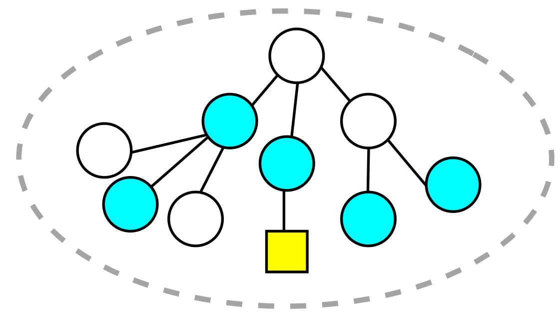
if  is a piece of  , then

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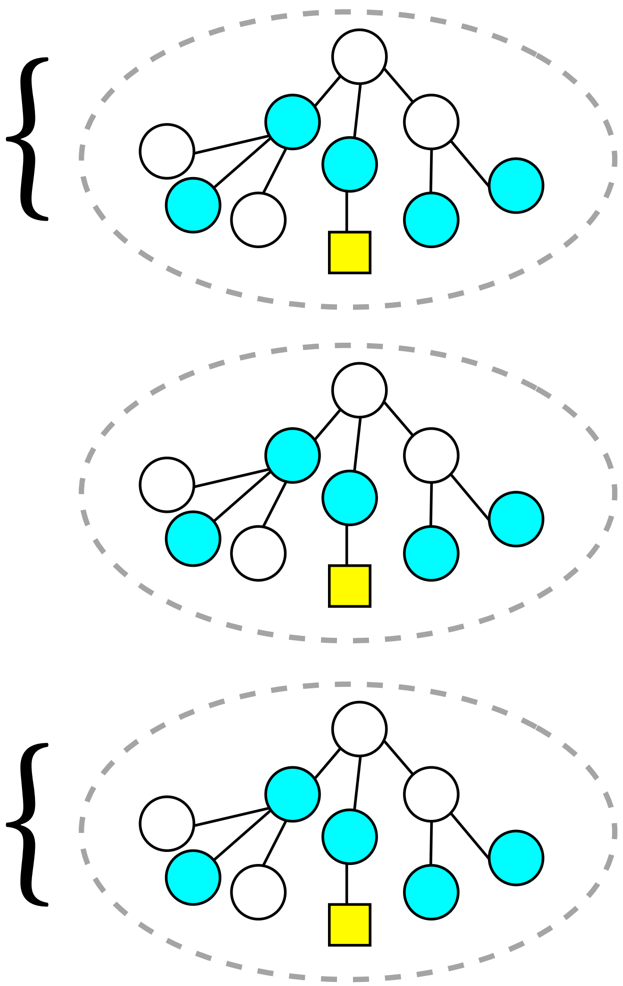

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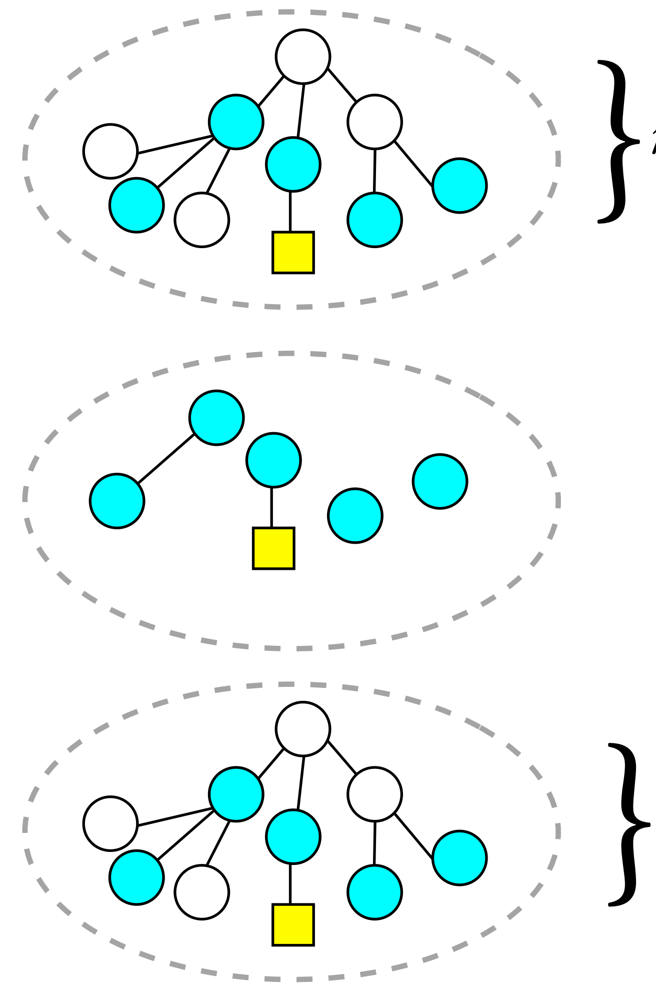

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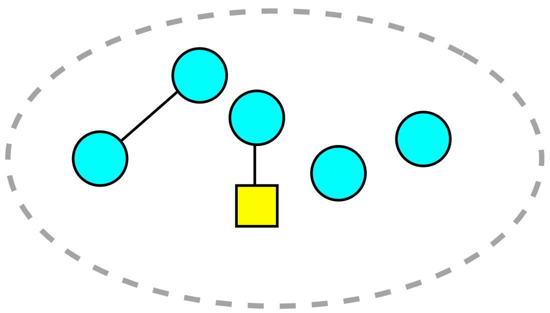
This criterion is decidable.
We also have variants of the theorem for unordered trees / forests.

if  is a piece of , then

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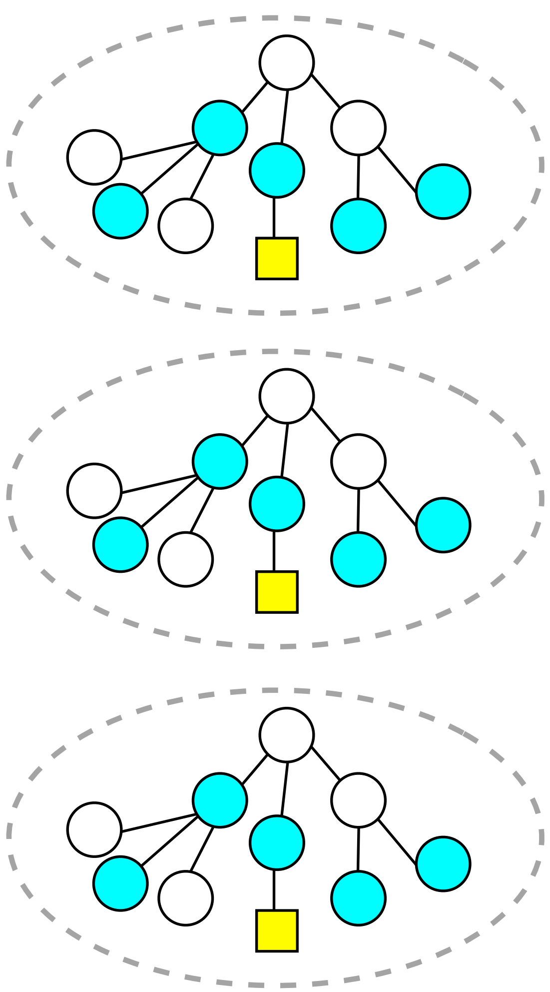

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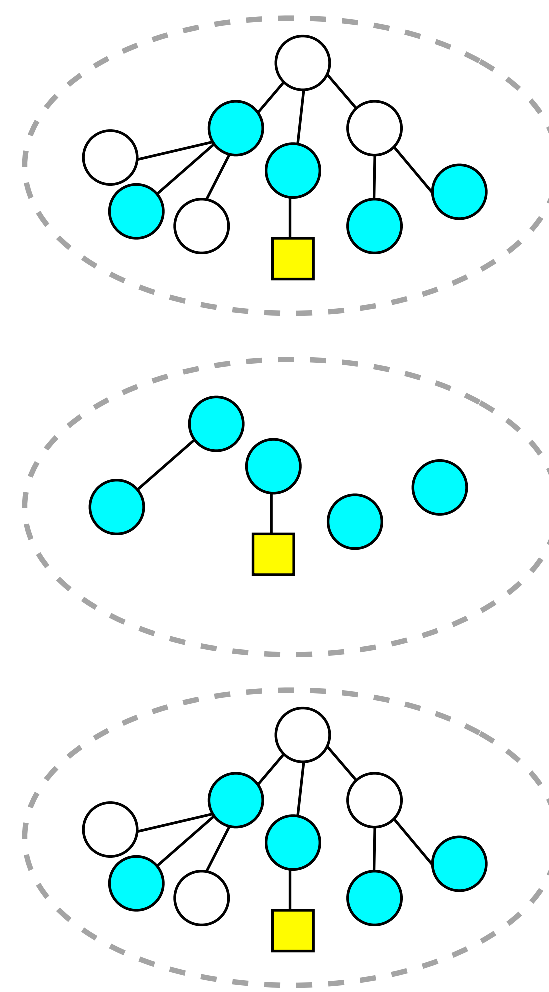

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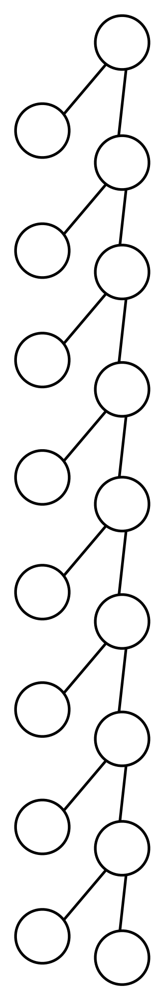
Application.

The set of binary trees (every node has zero or two children) is not definable in Δ_2



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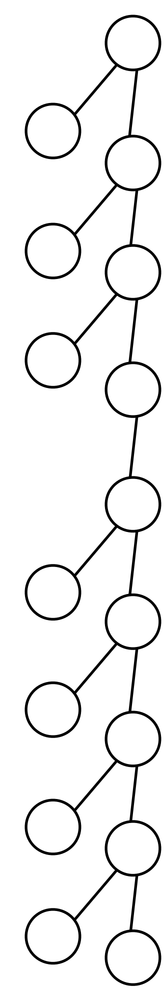

Big project: understand the expressive power of first-order logic on trees.

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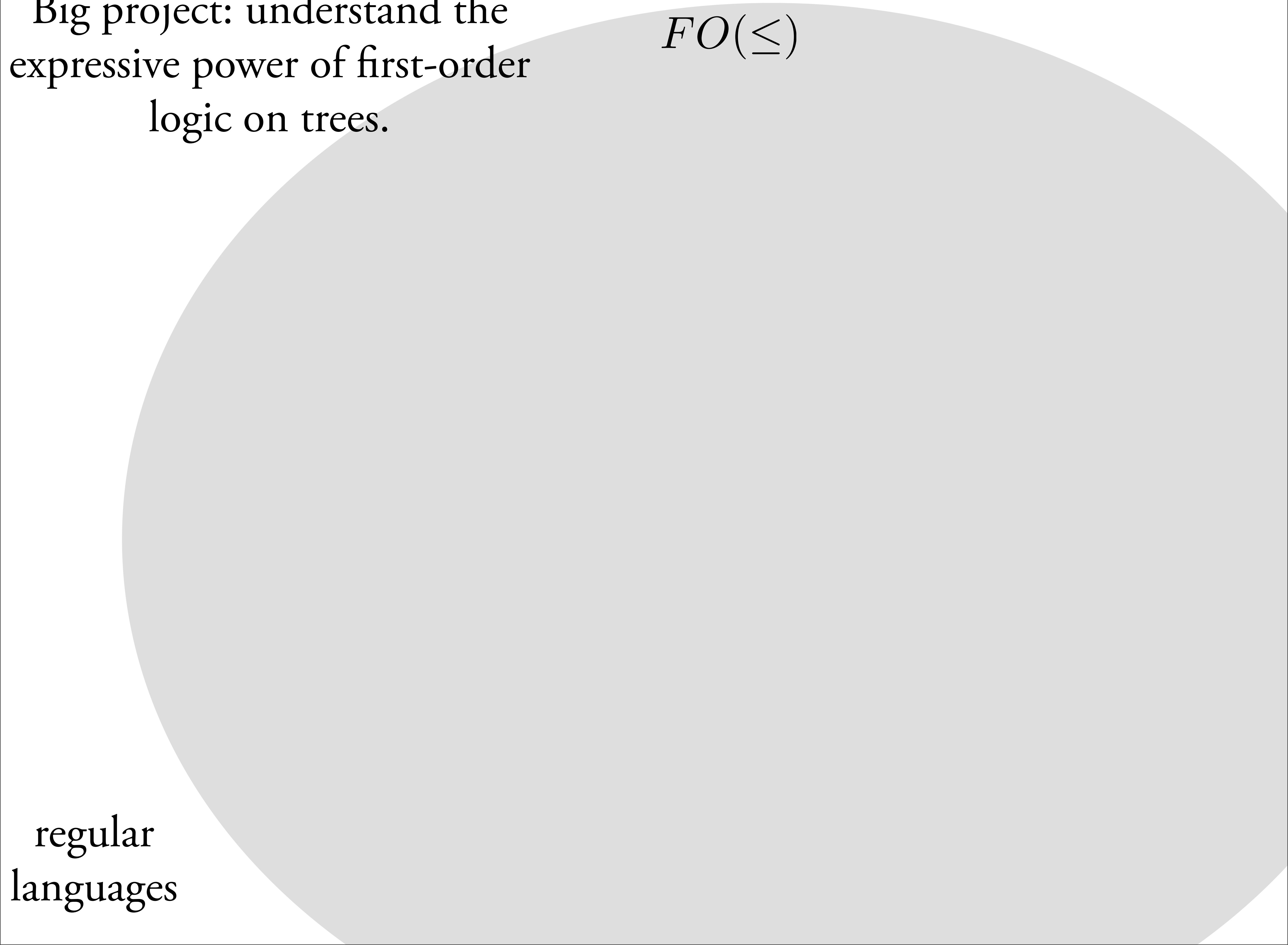

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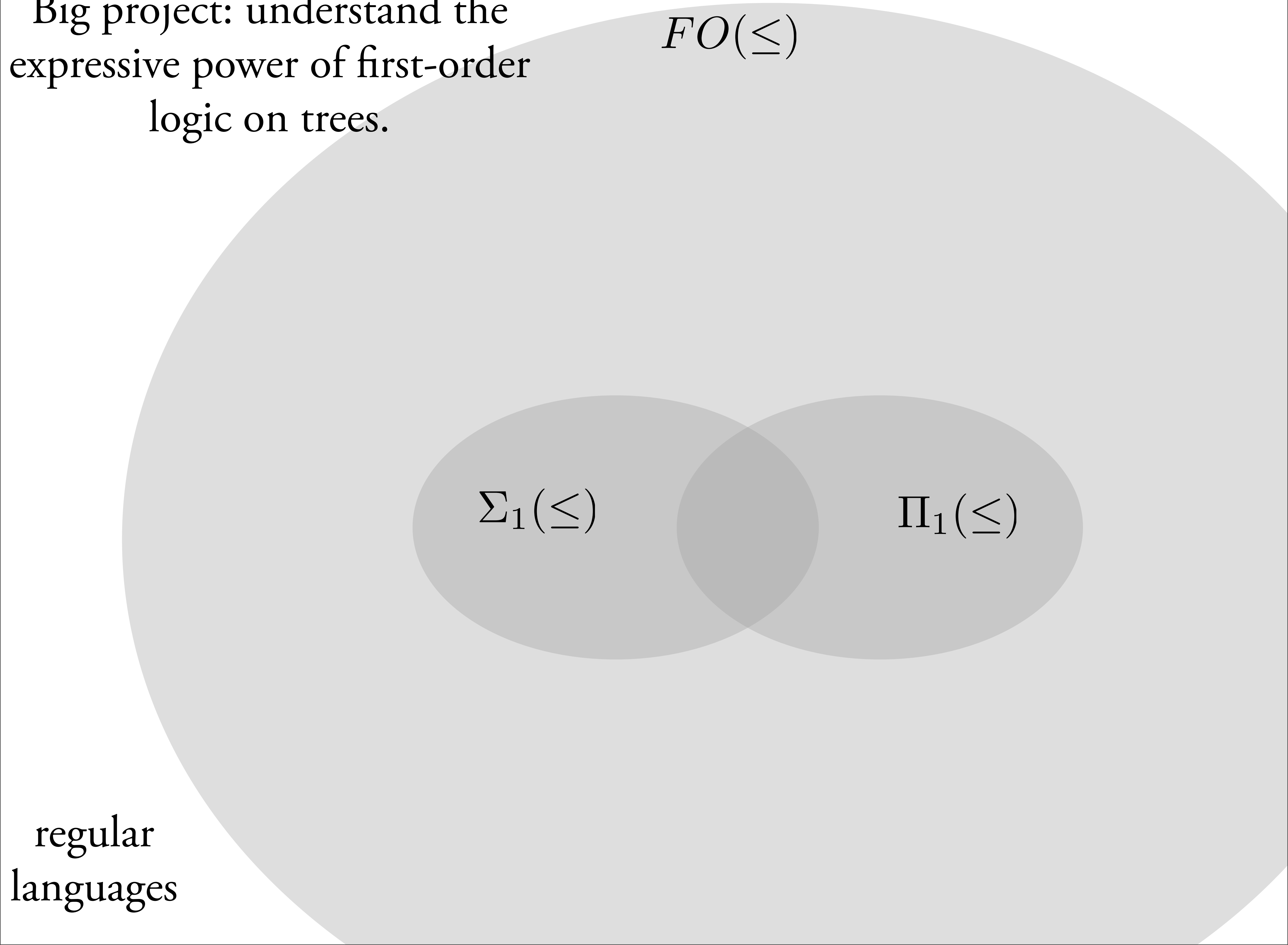

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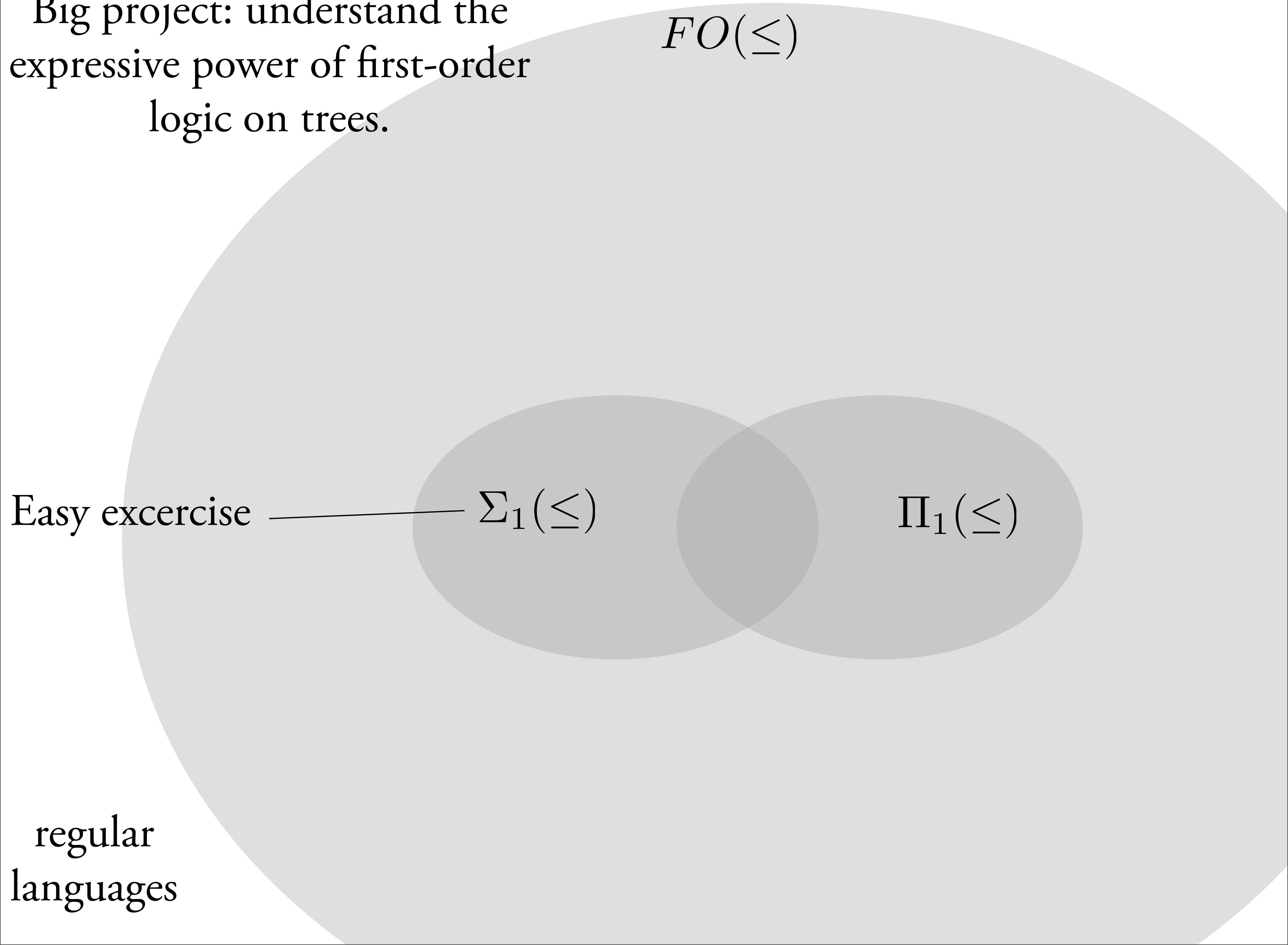

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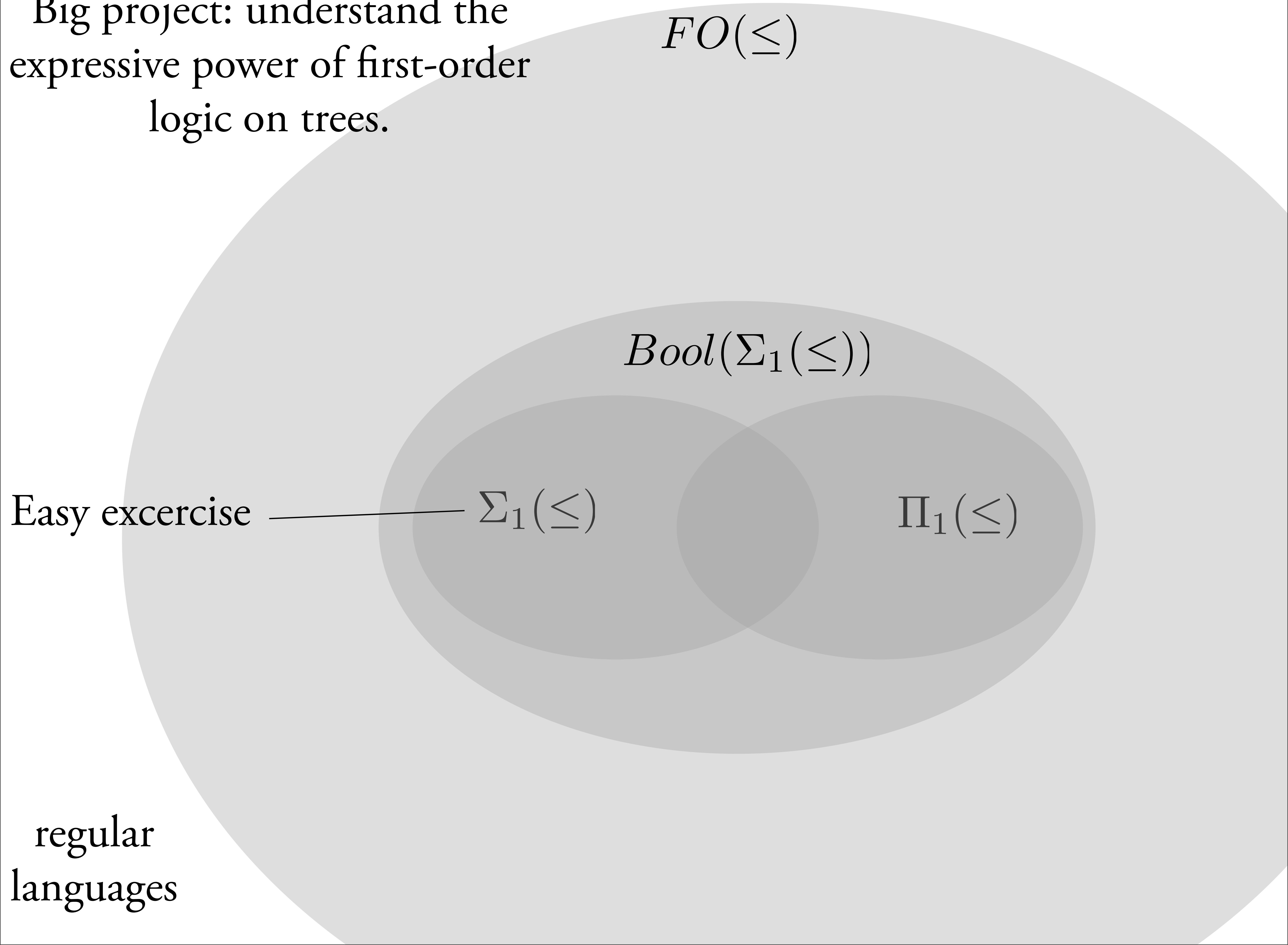

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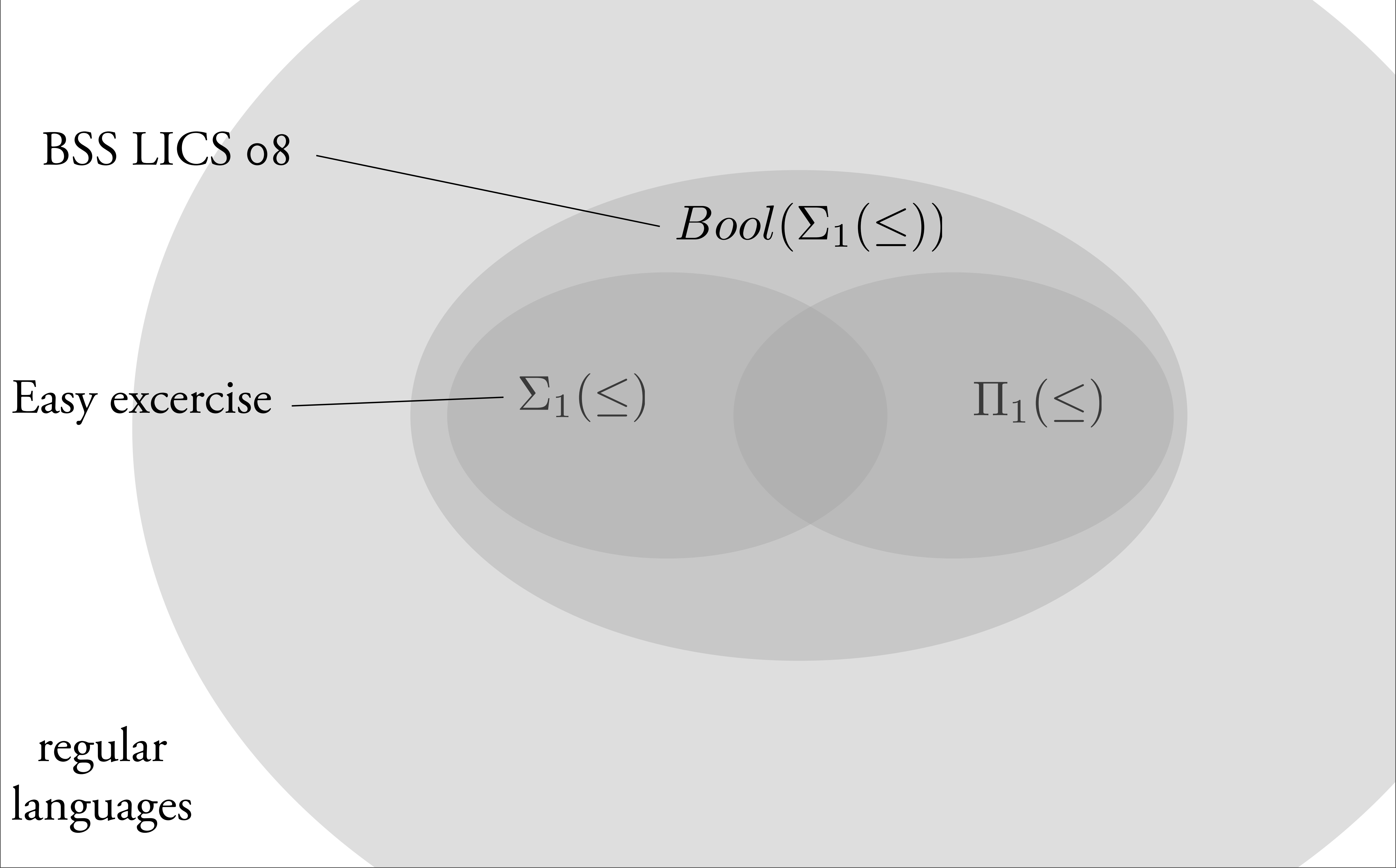

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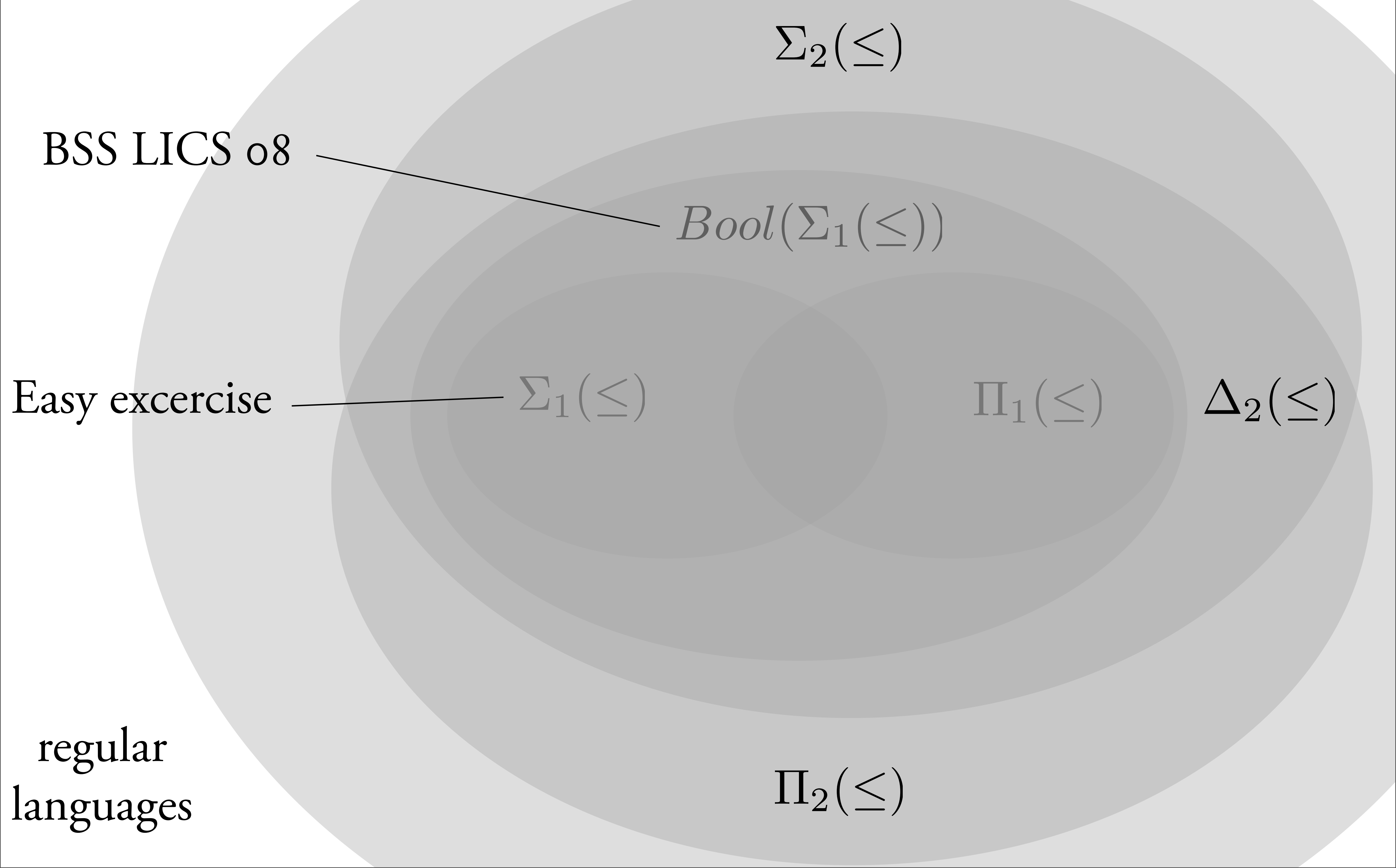

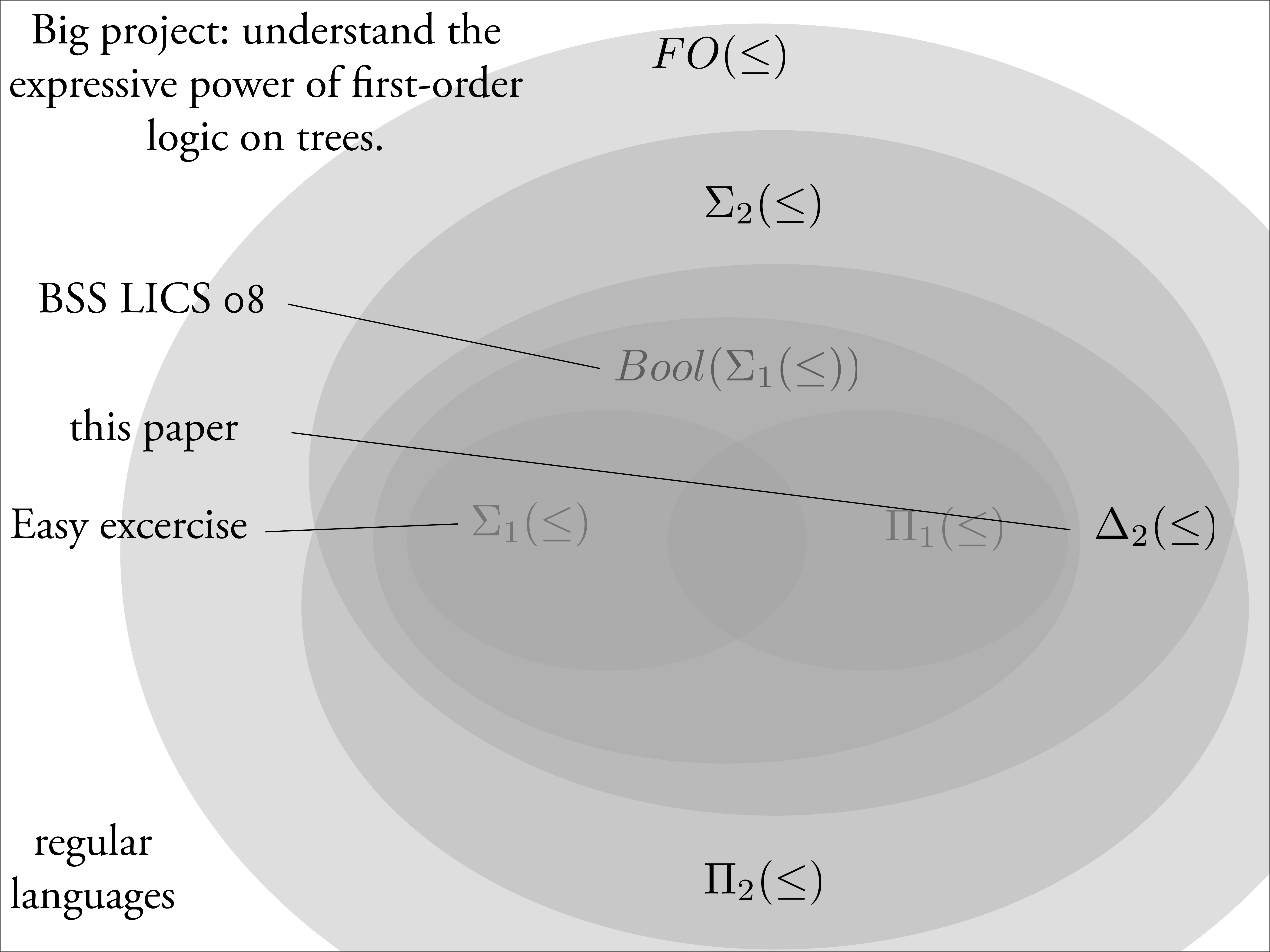
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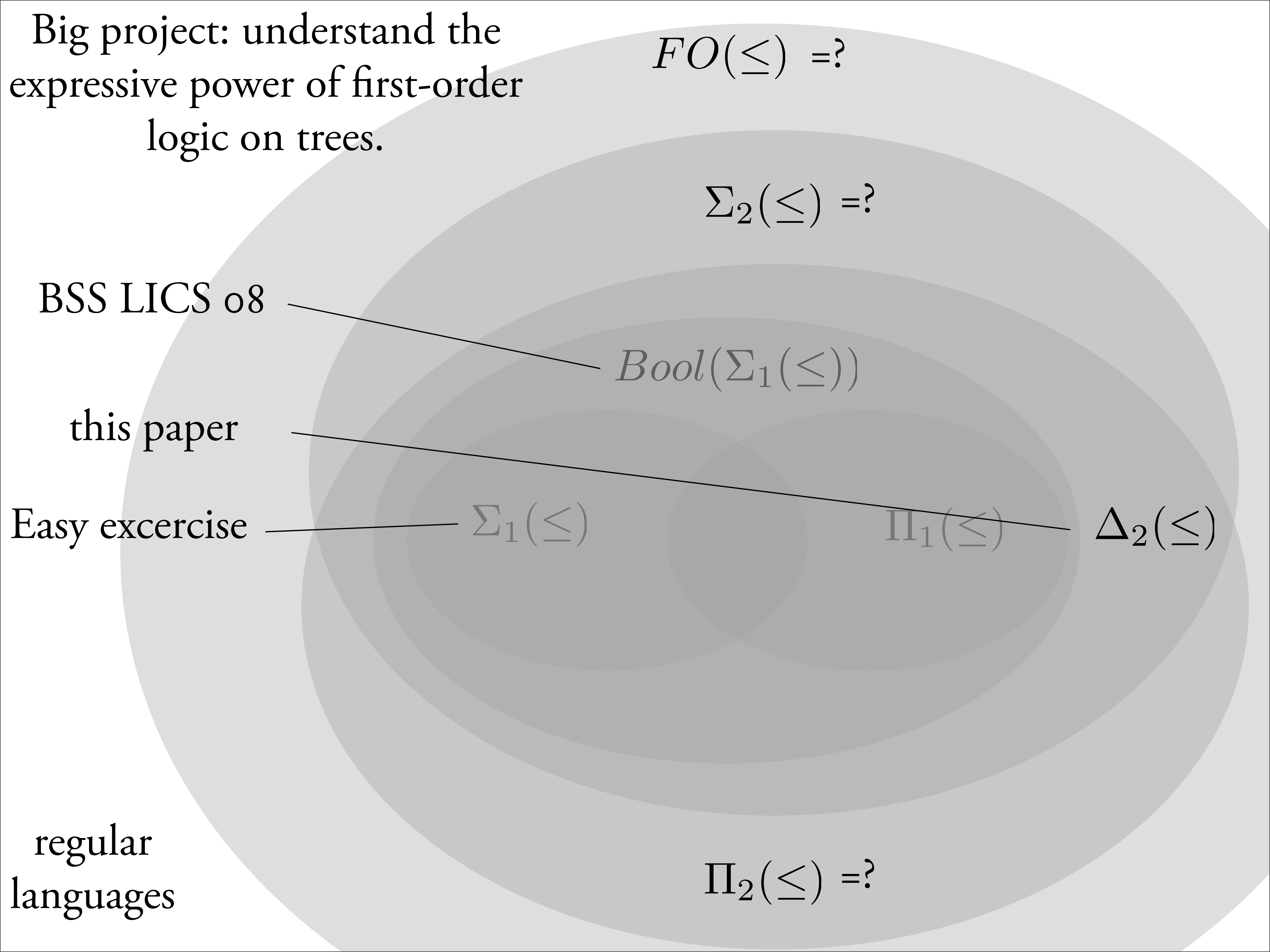
$FO(\leq)$
$\Sigma_2(\leq)$
BSS LICS 08
$Bool(\Sigma_1(\leq))$
this paper
Easy excercise
$\Sigma_1(\leq)$
$\Pi_1(\leq)$
$\Delta_2(\leq)$
regular languages
$\Pi_2(\leq)$



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