First Order and Chain Definability of Regular Tree Languages

Igor Walukiewicz (LaBRI); Mikolaj Bojanczyk (Warszawa)

Summary

- Quick reminder of logic and languages

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- Overview of FOL definable word languages

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- FOL definable tree languages and some characterisations

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- Chain logic and some conjectures
- Conclusion

Regular languages and logic

Let Σ be an alphabet and $w = a_0 \dots a_n$ a word over Σ. This word is represented as a relational structure

$$\underline{w} = (\mathrm{dom}(w), S^w, <^w, (Q_a^w)_{a \in \Sigma})$$

called the *word model* for w, where $\mathrm{dom}(w) = \{0, \dots, n\}$, S^w is the successor relation on $\mathrm{dom}(w)$, $<^w$ is the natural order and $Q_a^w = \{i : a_i = a\}$.

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Thm: A language is MSOL definable iff it is regular

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The language $(aa)^*$ is not FOL definable

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Cor:[of 2,3] It is decidable whether a given regular language is FOL definable.

The tree case

For a finite binary tree t a similar structure $\underline{t}$ is considered:

$$\underline{t} = (\mathrm{dom}(t), S_0^t, S_1^t, <^t, (Q_a^t)_{a\in\Sigma})$$

where $\mathrm{dom}(t) \subseteq \{0,1\}^*$ is the set of nodes of the tree, S_i^t denotes the i-th successor relation

$$S_i^t = \{(v, v\cdot i) : v, v\cdot i \in \mathrm{dom}(t)\}$$

and $<^t$, Q_a^t are defined as in the word case.

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$$\exists X.\forall x.[\text{root}(x) \vee \text{leaf}(x)] \Rightarrow X(x)) \wedge$$
$$(\forall x, x_0, x_1.[S_0(x, x_0) \wedge S_1(x, x_1)] \Rightarrow [X(x) \Leftrightarrow \neg(X(x_0) \Leftrightarrow X(x_1))])$$

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Fact: The property (1) is not FOL definable

Main question

Our unattained goal is two answer the question:

> Given a regular tree language L decide whether L is FOL definable.

CTL*

CTL* formulas over the alphabet $\Sigma = \{a_0, \ldots, a_n\}$ are defined by the following grammar:

$$F := \exists F | F \mathrm{U} F | F \wedge F | \neg F | a_0 | \ldots | a_n$$

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Each CTL* formula ψ is translated to a two-variable FOL formula $[\![\psi]\!](x, y)$:

- $[\![a_i]\!](x, y) = Q_{a_i}(x)$
- $[\![\psi \wedge \varphi]\!](x, y) = [\![\psi]\!](x, y) \wedge [\![\varphi]\!](x, y)$
- $[\![\neg\psi]\!](x, y) = \neg[\![\psi]\!](x, y)$
- $[\![\psi \mathrm{U} \varphi]\!](x, y) = \exists z \leq y.[[\![\varphi]\!](z, y) \wedge \forall z' \in (x; z].[\![\psi]\!](z', z))]$
- $[\![\exists\psi]\!](x, y) = \exists y.[[\![\psi]\!](x, y)]$

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$$\exists x.Q_c(x) \wedge \forall y < x.\exists z > y.(Q_a(z) \wedge \forall(x' \in [y;z).Q_b(x')$$

$$\psi \mathrm{U}^* \varphi := \psi \wedge (\psi \mathrm{U} \varphi)$$

$$\exists[(\exists b \mathrm{U}^* a) \mathrm{U}^* c]$$

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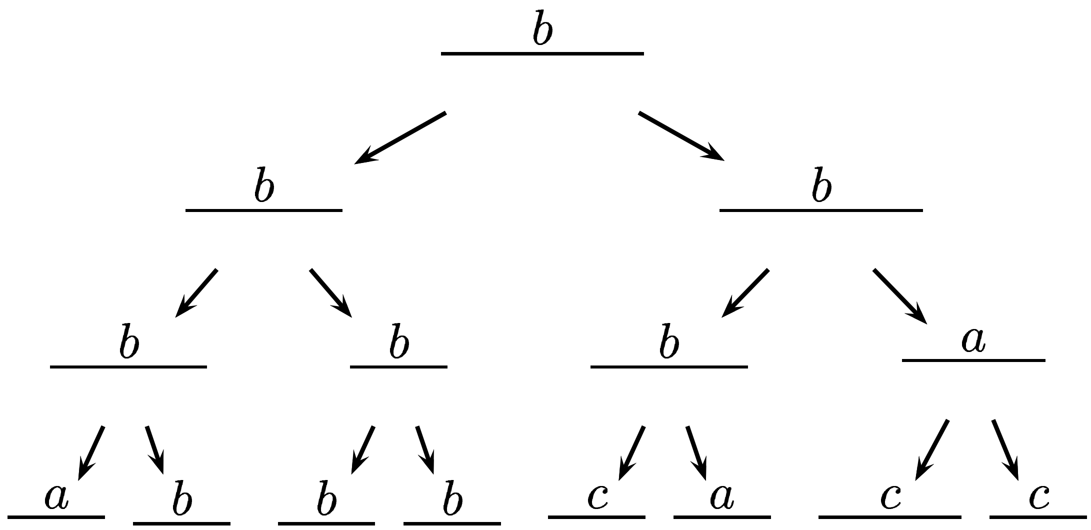

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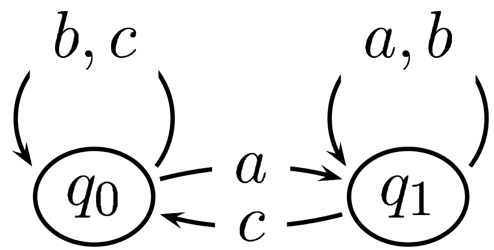

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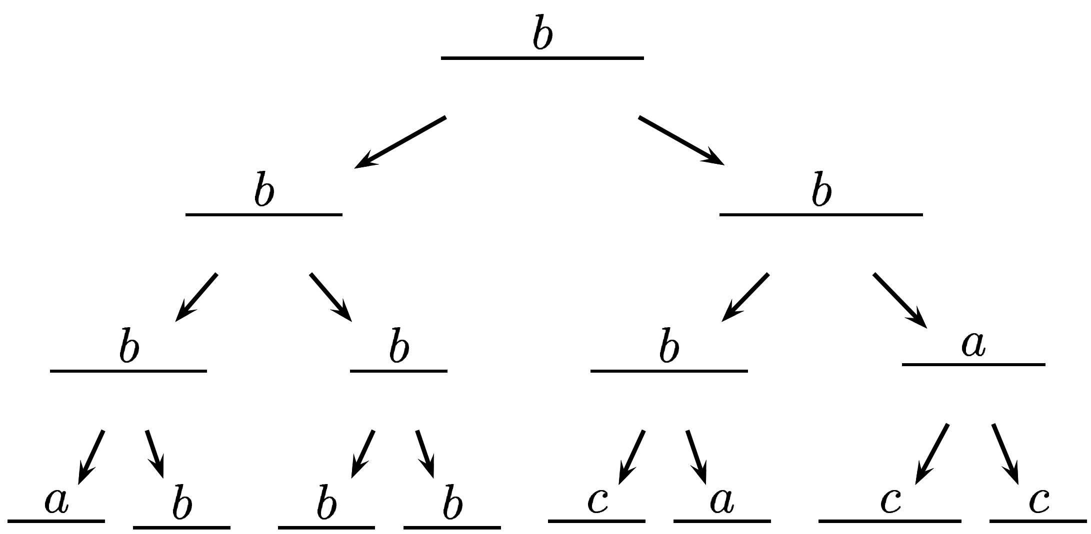

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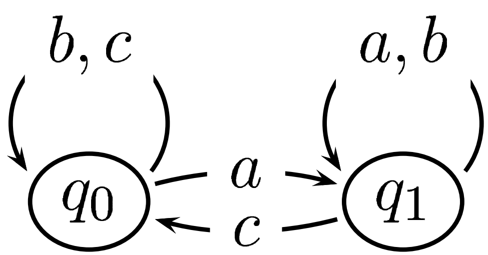

$c \quad a \quad b \quad b \quad c \quad b \quad c \quad a \quad c \leftarrow$

$q_0 \quad q_1 \quad q_1 \quad q_1 \quad q_0 \quad q_0 \quad q_0 \quad q_1 \quad q_0 \quad q_0$

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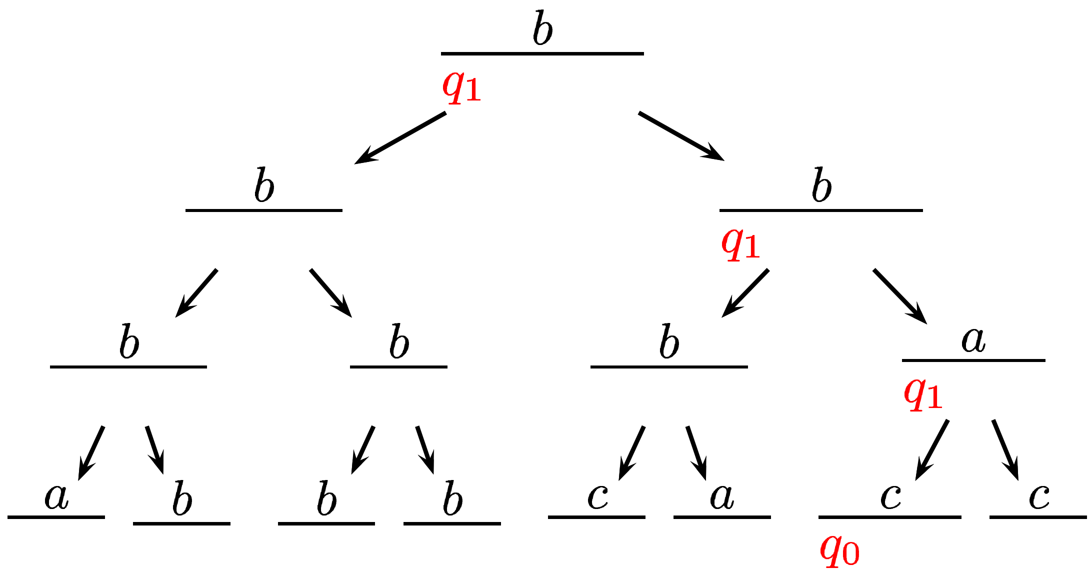

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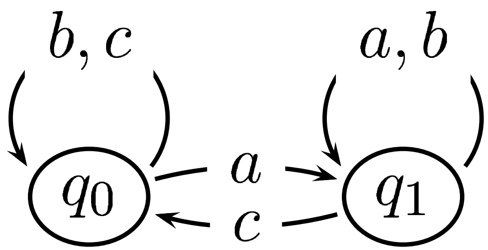

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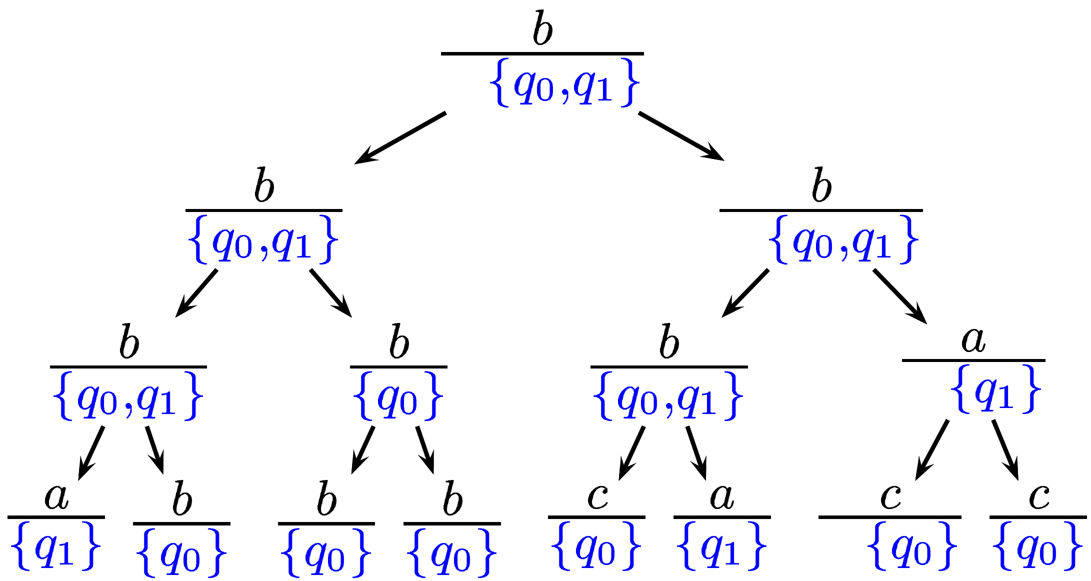

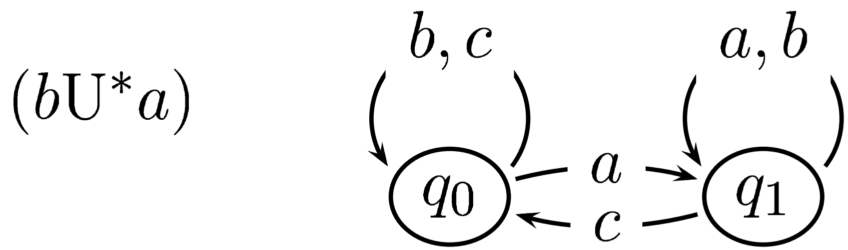

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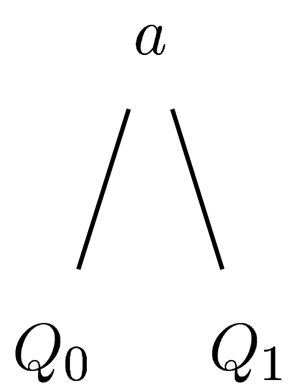
Word-sum automata

Consider a deterministic word automaton $\mathcal{A} = \langle Q, q_0, \delta \rangle$ over the alphabet $\Sigma \times \{0, 1\}$. Let $Q \cdot (a, i) = \{\delta(q, (a, i)) : q \in Q\}$. The automaton $\mathcal{A}_{ws} = \langle \mathrm{P}(Q), \{q_0\}, \delta' \rangle$ is a automaton over Σ-labelled trees whose transition function δ' is defined as follows:

$$Q_0 \cdot (a, 0) \cup Q_1 \cdot (a, 1)$$



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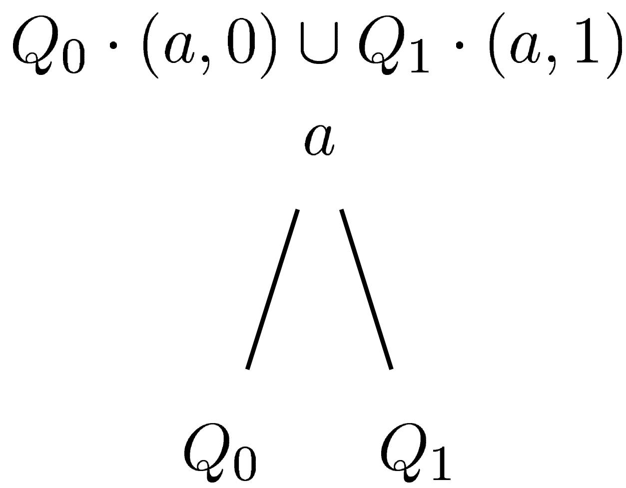

Df: A tree automaton $\mathcal{A}$ is a *word-sum* automaton iff $\mathcal{A} = \mathcal{A}'_{ws}$ for some word automaton $\mathcal{A}'$. The automaton $\mathcal{A}$ is an *aperiodic* word-sum automaton if $\mathcal{A}'$ is aperiodic.

Word-sum automata, continued

For a tree language L, the following are equivalent:

- L is definable by some word-sum automaton.

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Fact: Aperiodic word-sum automata recognize precisely CTL* formulas of $\exists$-depth 1.

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Thm: It is decidable whether a given language is word-sum definable.

Wreath product

Let $\mathcal{A} = \langle Q, q_s, \delta \rangle$ be an automaton over Σ labelled trees and $\mathcal{A}' = \langle Q', q'_s, \delta' \rangle$ an automaton over $\Sigma \times Q$ labelled trees. Assume that both are bottom-up deterministic.

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Df: The *wreath* product of $\mathcal{A}'$ and $\mathcal{A}$ is the automaton $\mathcal{A}' \circ \mathcal{A} = \langle Q \times Q', (q_s, q'_s), \delta_\circ \rangle$ over Σ labelled trees whose transition function is defined as follows:

$$\delta_\circ((q_0, q'_0), a, (q_1, q'_1) = (q, q')$$

where $q = \delta(q_0, q_1)$ and $q' = \delta'(q'_0, (a, q), q'_1))$.

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Thm: A language is chain definable iff it is recognized by a wreath product of word-sum languages.

Chain logic

Df: A set of tree vertices C is a *chain* iff it is totally ordered by the relation $\leq$.
Chain logic (CL) has the same syntax as monadic second order logic, but the semantics for the monadic quantifier $\exists$ are different:

$$t \models \exists X.\psi \quad \text{iff there is a chain } C \text{ such that} \quad t[X := C] \models \psi$$

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- A tree property definable in CL (but not in FOL) is: “there exists a path of even length”.
- A regular tree property not definable in CL is: “the tree has an even number of vertices”.

Plan B

Our unattained plan B is two answer the question:

> Given a regular tree language L decide whether L is chain definable.

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Potthoff example (simplified)

One operator $\otimes$. Leaves labelled with 0, 1. All triples but the below two evaluate to $\perp$, which propagates.

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The Potthoff example contradicts the following conjectures:

- A language is FOL definable iff it is aperiodic
- A chain definable language is FOL definable iff it is aperiodic

Confusion

Let $\mathcal{A} = \langle Q, q_0, \delta \rangle$ be a deterministic bottom-up automaton. Consider a tree t with a designated subsset of leaves V and a function $\sigma : V \to Q$. $t[s] \in Q$ is defined as the state assumed by $\mathcal{A}$ in the root of t starting from state $\sigma(v)$ in leaves $v \in V$ and from q_0 in the remaining vertices.

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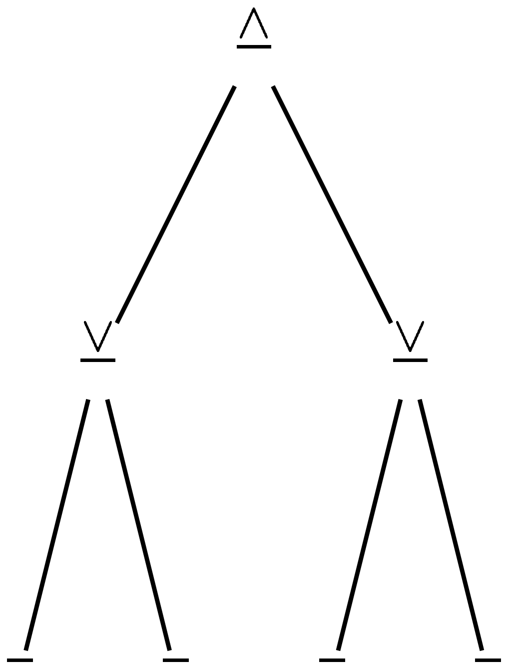
Df: Let $R \subseteq Q$. We say $\mathcal{A}$ contains R-*confusion* if there is a tree t with a designated set of leaves V such that for every $v \in V$ and every $q, q' \in R$, there is some assignment $\sigma : V \to R$ such that $t[\sigma[v := q]] = q'$.

Example of confusion



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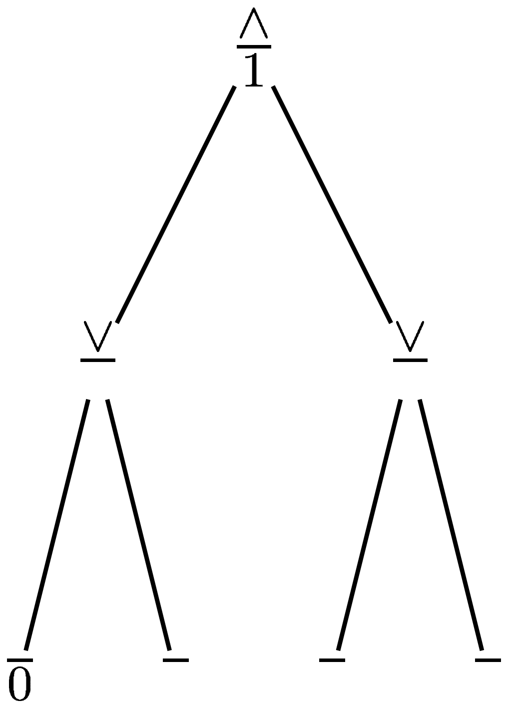

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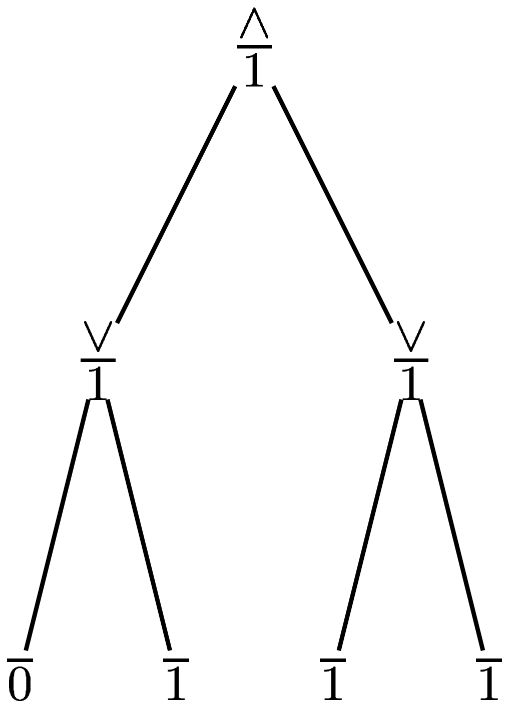

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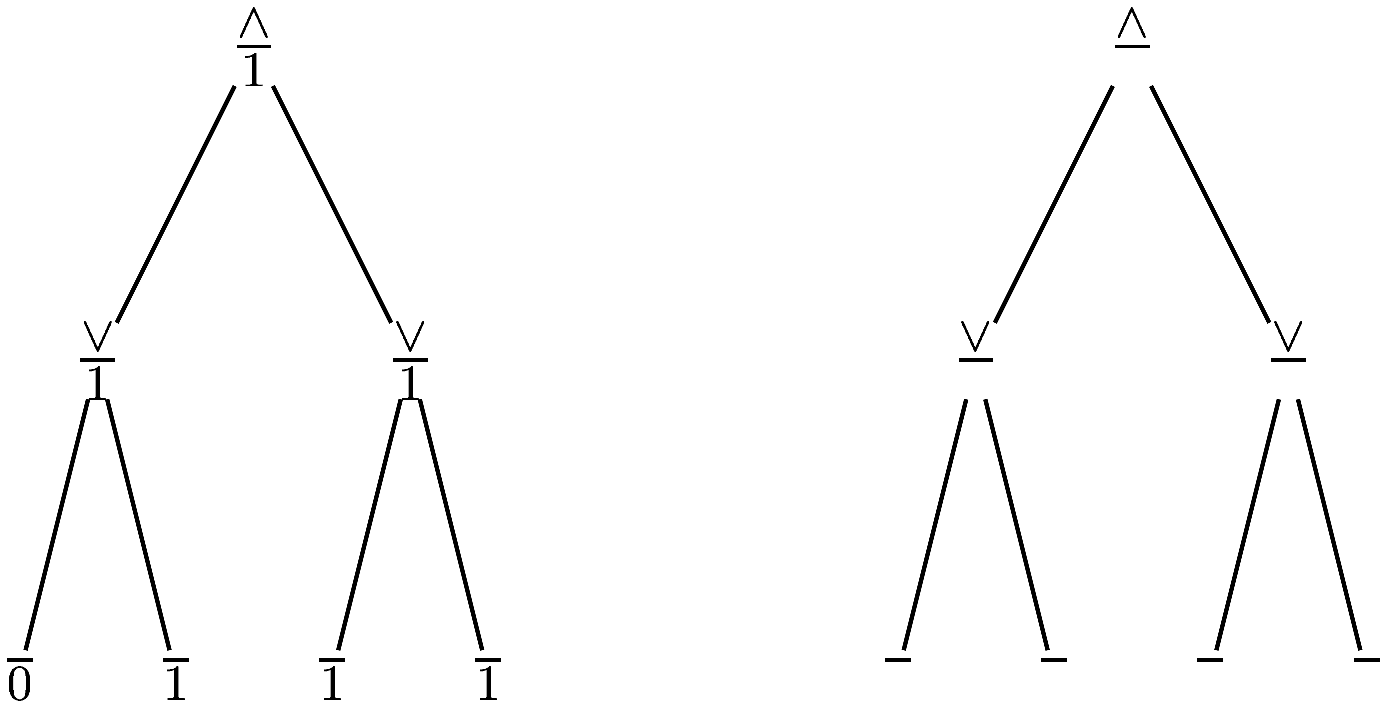

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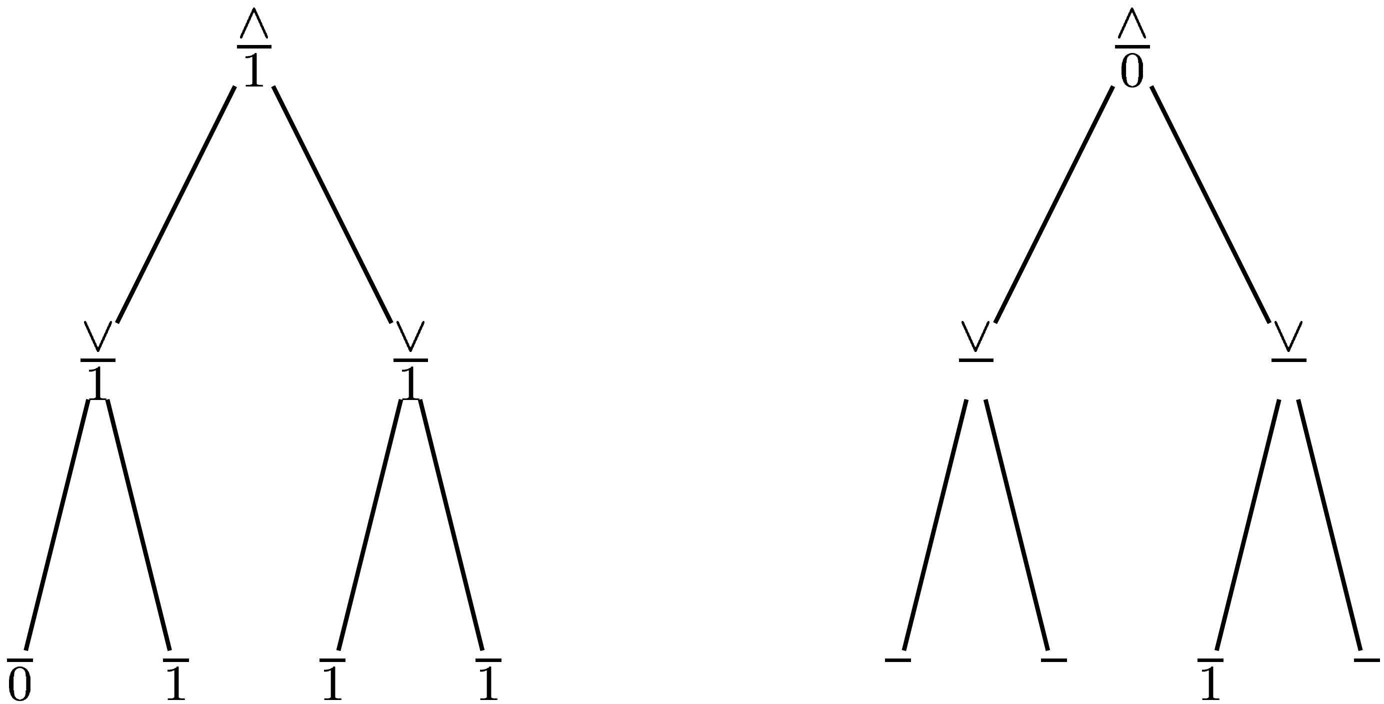

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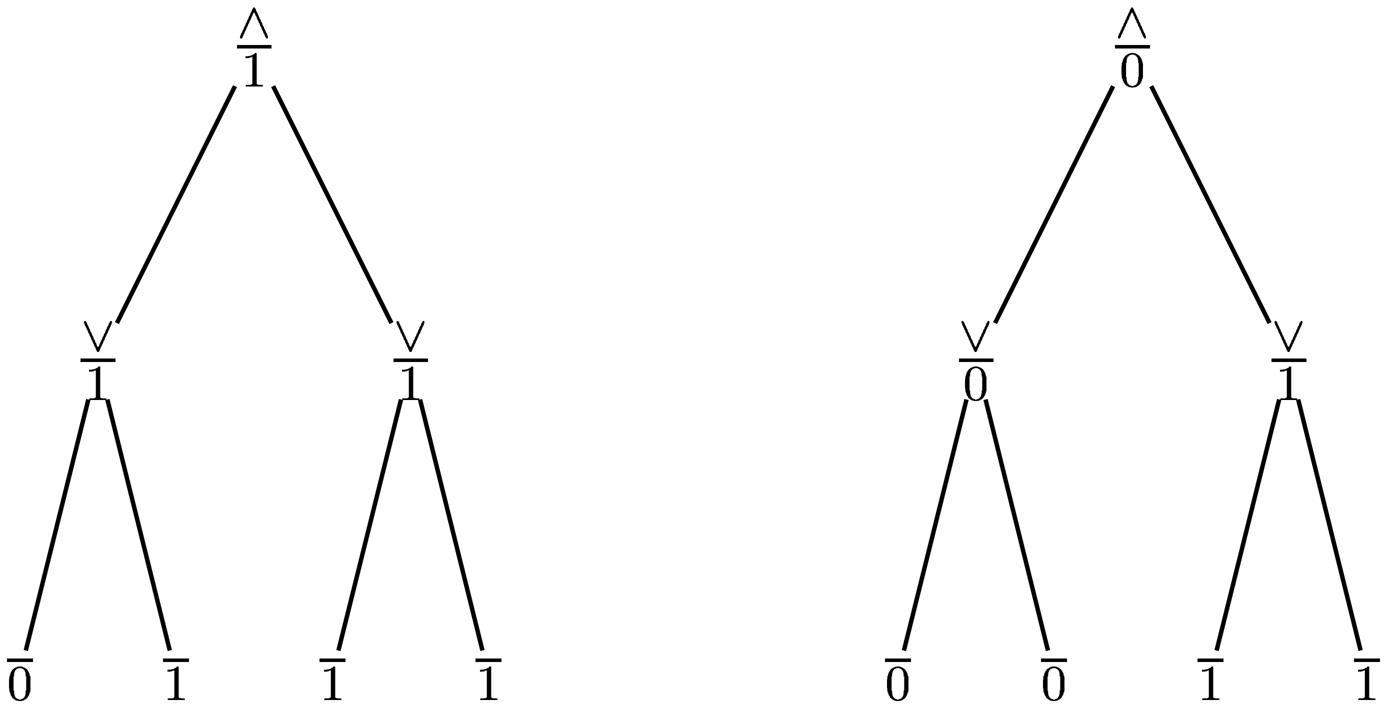

Confusion conjecture

Df: A language L *contains confusion* if the minimal deterministic bottom-up automaton recognizing L contains confusion. Otherwise L is *non-confusing*.

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Conjecture: A language is chain definable iff it is non-confusing

Arguments in favor of the conjecture

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- Works for languages with two types (i. e. whose minimal deterministic bottom-up automaton has two states)

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- Nonconfusion behaves like a logic.

Yield languages

Df: The *yield* $\mathrm{y}(t)$ of a tree t is the word consisting of the labels in the leaves of t, read from left to right.

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Thm: A yield language is in CL iff it is in FOL iff it is non-confusing.

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Thm: Nonconfusing languages are closed under homomorphic images, direct and wreath products.

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Cor: Nonconfusing languages are closed under boolean operations and chain quantification.

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The base case: There is no congruence in $\mathcal{A}$ ($\mathcal{A}$ is a *simple* algebra).

Separation

The L-type of a tree t is the state assummed in the root of t by the minimal deterministic bottom-up automaton recognizing L.
An automaton $\mathcal{A}$ separates two types τ, σ if $\mathcal{A}$ accepts all trees of type τ and rejects all trees of type σ.

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Fact: If no deterministic top-bottom automaton can separate any two types then boolean combination of such automata can do it.

Summary and future work

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- Try to characterise other logics such as CTL, MPL
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- Do something easier