Book of Abstracts

Summer School on Calculus of Variations: variational and geometrical methods

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Thematic Research Programme

Variational and geometrical methods in partial differential equations

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- Bruno Premoselli (*Université Libre de Bruxelles*)
- Susanna Terracini (*Universitá di Torino*)
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Abstracts

On a pseduorelativistic fractional magnetic Schrödinger operator

Federico Bernini

Universitá degli Studi di Milano

In this talk, we present some results on Schrödinger fractional operators in the presence of an external magnetic field, obtained in a joint work with Pietro d'Avenia (Politecnico di Bari). After formally introducing the operators, we will define the suitable functional setting. We then study the limit behavior of these operators as the fractional order s goes to 1^-. We will conclude providing an example of a subcritical nonlinear problem driven by this operator: in this context, we prove the existence of both radial and non-radial solutions.

Amann-Zehnder type results for resonance quasilinear elliptic systems

Natalino Borgia

Universitá degli Studi di Bari Aldo Moro

In my talk I will present an Amann-Zehnder type result for resonance systems of quasilinear elliptic equations with homogeneous Dirichlet boundary conditions, involving nonlinearities growing asymptotically (p, q)-linear at infinity. The proof relies on a cohomological linking in a product Banach space where the properties of cones of the sublevels are missing, differently from the single quasilinear equation. Critical groups of the energy functional have been computed at the origin, in spite of the lack of its C^2 regularity, to exclude that the found mini-max solution is trivial. Finally a local condition is given to guarantee that the found solution is not semi-trivial. This is a joint paper with S. Cingolani and G. Vannella.

Graph Based Semi-supervised Learning Using Spatial Segregation Theory

Farid Bozorgnia

Técnico Lisboa

In this talk, we briefly explain various models of Reaction-Diffusion Systems characterized by high competition rates. We investigate the existence and uniqueness of solutions for each model, and numerical approximation of their singular limit. Next, I address graph-based semi-supervised learning leverage the theory of these competitive-type systems of PDEs to classify data when only a few labels are available.

We define a discrete counterpart over connected graphs by using a direct analogue of the corresponding competitive system. Then we consider a model motivated by the recent numerical results on the spatial segregation of reaction-diffusion systems. Finally, we present some numerical experiments showing the efficiency of the method.

Existence and uniqueness of large solutions of a degenerate and/or singular semilinear elliptic problem

Raj Narayan Dhara

Palacky University Olomouc

We show an existence and a uniqueness result for large solution to a degenerate and/or singular semilinear elliptic problem with its blow-up rate which is influenced by the degeneracy or singularity. It has been proved by showing, however, of an independent interest, an existence and a uniqueness of a weak solution of a degenerate and/or singular semilinear elliptic boundary value (nonhomogeneous) problem by penalty method for given weak sub- and supersolution.

Melius abundare: multiple paths for a solution of a critical mass problem

Marco Gallo

Università Cattolica del Sacro Cuore (Brescia)

Searching for a solution of the nonlinear Schrödinger equation ($\omega > 0$)

$$-\Delta u + \omega u = g(u) \quad \text{in } \mathbb{R}^N,$$

is a classical problem. When the *mass* $\int_{\mathbb{R}^N} u^2 = m$ is prescribed in advance, it is also known that how $g(t)$ relates with the power $\bar{g}(t) := |t|^{\frac{4}{N}} t$ is of key importance. The homogeneous case $g \equiv \bar{g}$ can be easy handled by scaling: on the other hand, when only $g(t) \sim \bar{g}(t)$ as $t \to \pm\infty$ and $t \to 0$, the problem becomes much more tricky. In this talk I will present some tools exploited to achieve the existence of a solution.

These results are in collaboration with Silvia Cingolani, Hirohisha Ikoma and Kazunaga Tanaka.

Partial Blow-up Phenomena in the $SU(3)$ Toda System on Riemann Surfaces

Zhengni Hu

Justus-Liebig-Universität Gießen

This talk discusses the partial blow-up phenomena for the $SU(3)$ Toda system on compact Riemann surfaces with boundary. We consider the following coupled Liouville system with Neumann boundary conditions:

$$\begin{cases} -\Delta_g u_1 = 2\rho_1 \left(\frac{V_1 e^{u_1}}{\int_\Sigma V_1 e^{u_1}\, dv_g} - \frac{1}{|\Sigma|_g} \right) - \rho_2 \left(\frac{V_2 e^{u_2}}{\int_\Sigma V_2 e^{u_2}\, dv_g} - \frac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma} \\ -\Delta_g u_2 = 2\rho_2 \left(\frac{V_2 e^{u_2}}{\int_\Sigma V_2 e^{u_2}\, dv_g} - \frac{1}{|\Sigma|_g} \right) - \rho_1 \left(\frac{V_1 e^{u_1}}{\int_\Sigma V_1 e^{u_1}\, dv_g} - \frac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u_1 = \partial_{\nu_g} u_2 = 0 & \text{on } \partial\Sigma \end{cases},$$

where (Σ, g) is a compact Riemann surface with the interior $\mathring{\Sigma}$ and smooth boundary $\partial\Sigma$, ρ_i is non-negative parameter and the positive potential function V_i is smooth for $i = 1, 2$.

We construct a family of blow-up solutions via the Lyapunov-Schmidt reduction and variational methods, where one component remains uniformly bounded from above, and the other exhibits partial blow-up at a prescribed number of points, both in the interior and on the boundary. This construction is based on a non-degeneracy hypothesis for singular mean field equations.

I will discuss constructing solutions for the Toda system with partial blow-ups and analyze the non-degeneracy hypothesis that is central to our approach. This is joint work with Prof. Dr. Thomas Bartsch and Prof. Dr. Mohameden Ahmedou.

Semiclassical states to curl-curl problem

Adam Konysz

Nicolaus Copernicus University in Toruń

We show the existence of the so-called semiclassical states $\mathbf{U} : \mathbb{R}^3 \to \mathbb{R}^3$ to the following curl-curl problem

$$\varepsilon^2 \nabla \times (\nabla \times \mathbf{U}) + V(x)\mathbf{U} = g(\mathbf{U}),$$

for sufficiently small $\varepsilon > 0$. We study the asymptotic behaviour of solutions as $\varepsilon \to 0^+$ and we investigate also a related nonlinear Schrödinger equation involving a singular potential. The problem models large permeability nonlinear materials satisfying the system of Maxwell equations. This talk is based on joint work with B. Bieganowski and J. Mederski.

Nondegeneracy and stability properties of some minimal hypersurfaces

Matteo Rizzi

Mathematisches Institut, Justus Liebig Universität

In this seminar we discuss some results concerned with existence and stability properties of embedded minimal manifolds asymptotic to the Lawson cone in high dimension. We also discuss generalizations of such results in the framework of more general minimal cones, including the lower dimensional case. Nondegeneracy and stability properties of such manifolds are discuss with the use of standard tools from theory of PDEs. These results can be applied to the construction of entire solutions to the Allen-Cahn equation.

On the Brezis-Nirenberg problem

Serena Rocci

Universitá degli Studi di Roma "La Sapienza"

I will provide an overview of the equation introduced by Brezis and Nirenberg (see [1]) with Dirichlet boundary condition:

$$\begin{cases} -\Delta u = |u|^{p-1}u + \varepsilon u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\varepsilon > 0$ and $p = 2^\star - 1$ where $2^\star$ is the critical Sobolev exponent.

It is well-known that the existence and multiplicity of positive and sign-changing solutions to (1) are strongly influenced by the geometry of the domain and the dimension of the Euclidean space in which the domain is situated. We are interested in *blow-up* solutions to (1) and this is a possible feature to this problem due to the critical non-linearity. I will discuss the existence of blowing-up solutions to this problem involving the critical Sobolev exponent, as well as some new results ([2] joint work with Monica Musso and Giusi Vaira, [3] joint work with Angela Pistoia).

[1] Brézis H., Nirenberg L. - Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. pure appl. math., 36(4): 437-477 (1983).

[2] Musso M., Rocci S., Vaira G. - Nodal cluster solutions for the Brezis-Nirenberg problem in dimensions $N \geq 7$. Calc. Var. 63, 119 (2024).

[3] Pistoia A., Rocci S. - The Brezis-Nirenberg problem in 4D. Discrete Contin. Dyn. Syst. - S. 17(4): 1562-1572 (2024).

The solution to Dirichlet's problem $\Delta_g u = -1$ in a convex Riemannian domain

Daniel Raom Santiago

Universitá degli Studi di Roma "La Sapienza"

Let Ω be a geodesically convex domain on a Riemannian manifold (M, g) and let u be a solution of the boundary value problem

$$\begin{cases} \Delta_g u = -1 \quad \text{in} \quad \Omega \subset M, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

called the *torsion function*.

We aim to discuss some results on how the torsion function u may inherit the convexity of the domain. We will go through the classical results of Makar-Limanov [1] when $M = \mathbb{R}^2$, the *continuation method* of Caffarelli & Friedman [2] and its generalisation by Korevaar [3] ($M = \mathbb{R}^n$), and a log-concavity result by Lee & Wang [5] for the first eigenfunction of the Dirichlet eigenvalue problem when $M = \mathbb{S}^n$. We will then show how to obtain a similar log-concavity result for the torsion function u when $M = \mathbb{S}^n$ by means of an adaptation of the *continuation method*. As a consequence, it also provides an alternative way to obtain the uniqueness of the critical point of the torsion function u for the cases where M has nonnegative constant curvature, as studied by Grossi & Provenzano [4].

[1] L. G. Makar-Limanov, *Solution of Dirichlet's problem for the Equation* $\Delta u = -1$ *in a convex region*, Matematicheskie Zametki; Volume 9, No. 1, January (1971), pp. 89-92.

[2] L. A. Caffarelli and A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J., 52(2), June (1985), pp. 431-456.

[3] N. J. Korevaar, *Convexity Properties of Solutions to Elliptic P.D.E.'S*, Variational Methods for Free Surface Interfaces; Springer-Verlag, (1987), pp. 115-121.

[4] M. Grossi and L. Provenzano, *On the critical points of semi-stable solutions on convex domains of Riemannian surfaces*, Mathematische Annalen; October (2023).

[5] Y. Lee and A. I. Wang, *Estimate of* $\lambda_2 - \lambda_1$ *on Spheres*, Chinese Journal of Mathematics; Volume 15, Number 2, June (1987).

Third order estimates and the regularity of the stress field for solutions to p-Laplace equations

Domenico Vuono

University of Calabria

We consider solutions to

$$-\Delta_p u = f(x) \quad \text{in } \Omega\,,$$

when p approaches the semilinear limiting case $p = 2$ and we get third order estimates. As a consequence we deduce improved regularity properties of the stress field. The results presented in this seminar are contained in a recent paper with D. Baratta and B. Sciunzi.

Outline of Lavrentiev Phenomena in One Dimension

Wiktor Wichrowski

University of Warsaw

I will introduce the Lavrentiev phenomena and present examples illustrating the existence and nonexistence of gaps within the context of absolute continuity. Specifically, I will focus on a recent example highlighted in the work by Cerf and Mariconda (2024), where non-occurrence of the phenomenon was shown. Additionally, I will examine theorems addressing the absence of gaps, including the original Lavrentiev theorem from 1926 and contemporary theorems based on convexity assumptions.

Stability of the Caffarelli-Kohn-Nirenberg inequality: the existence of minimizers

Yuanze Wu

China University of Mining and Technology

In this talk, I will report our recent results, based on the joint work with Professor Jucheng Wei, on the stability of the Caffarelli-Kohn-Nirenberg inequality. We find a new curve which is above the well-known Felli-Schneider curve such that the stability inequality of the Caffarelli-Kohn-Nirenberg inequality has a minimizer if the parameters are above or on this new curve. Our computation also make us to think that the stability inequality of the Caffarelli-Kohn-Nirenberg inequality may have no minimizers if the parameters are below this new curve.

Keywords: Caffarelli-Kohn-Nirenberg inequality, Stability, Minimizers, new Felli-Schneider curve.

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