Dimension of stationary measures with infinite entropy

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Based on a preprint "Dimension of Gibbs measures with infinite entropy" by Felipe Pérez.

Introduction

Consider a contractive IFS $f_1, ..., f_k : [0,1] \to [0,1]$ and the corresponding **coding map**

$$\pi : \{1, ..., k\}^{\mathbb{N}} \to [0,1], \ \pi(a_1, a_2, ...) = \bigcap_{n=1}^{\infty} f_{a_1} \circ \ldots \circ f_{a_n}([0,1]).$$

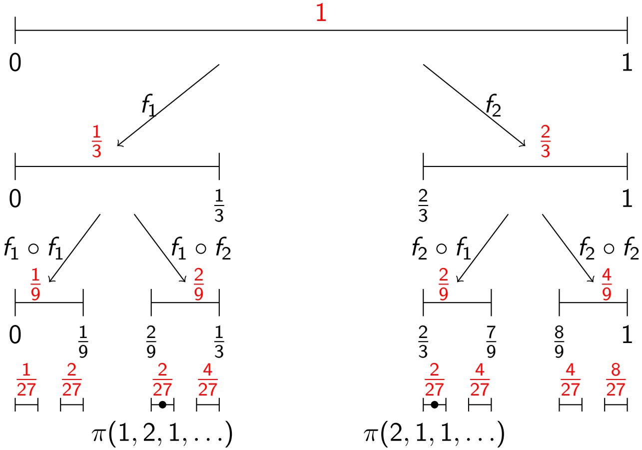
Let ν be a shift-invariant and ergodic measure on $\{1, \ldots, k\}^{\mathbb{N}}$.

We are interested in geometric properties of the measure $\mu = \pi_* \nu$ on $[0,1]$.

If $\nu = (p_1, ..., p_k)^{\otimes \mathbb{N}}$, then measure μ is the **stationary measure** for the random system $(\{f_1, ..., f_k\}, (p_1, ..., p_k))$, i.e. it satisfies

$$\mu = \sum_{j=1}^{k} p_j (f_j)_* \mu.$$

$$f_1(x) = \tfrac{1}{3}x,\ \ f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3},\ \ \nu = (\tfrac{1}{3}, \tfrac{2}{3})^{\otimes \mathbb{N}}$$



Local dimensions

Definition

Let μ be a Borel probability measure on $\mathbb{R}^n$. Define **lower** and **upper local dimension** of μ at point $x \in \mathrm{supp}(\mu)$ as

$$\underline{d}(\mu, x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ and } \overline{d}(\mu, x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If the limit exists, then $\mu(B(x,r)) \sim r^{d(\mu,x)}$.

μ is called **exact dimensional** if $\underline{d}(\mu, x) = \overline{d}(\mu, x) = const$ almost surely.

Definition

Lower and upper Hausdorff dimensions of μ:

$$\underline{\dim}_H(\mu) = \operatorname*{ess\,inf}_{x\sim\mu} \underline{d}(\mu, x), \ \overline{\dim}_H(\mu) = \operatorname*{ess\,sup}_{x\sim\mu} \underline{d}(\mu, x)$$

Lower and upper packing dimensions of μ:

$$\underline{\dim}_P(\mu) = \operatorname*{ess\,inf}_{x\sim\mu} \overline{d}(\mu, x), \ \overline{\dim}_P(\mu) = \operatorname*{ess\,sup}_{x\sim\mu} \overline{d}(\mu, x)$$

For exact dimensional measures, all of the above coincide.

Proposition

Let $\mathcal{F} = \{f_1, ..., f_k\}$ be a contractive IFS on $[0,1]$ consisting of similiarities, i.e.

$$f_i(x) = r_i x + t_i, \; r_i \in (0,1).$$

Assume that sets $f_i([0,1]), \; i = 1, ..., k$ have disjoint interiors. Let $\mathbf{p} = (p_1, ..., p_k)$ be a probability vector and let μ be the stationary measure $\mu = \pi_*(\mathbf{p}^{\otimes \mathbb{N}})$. Then μ is exact dimensional with

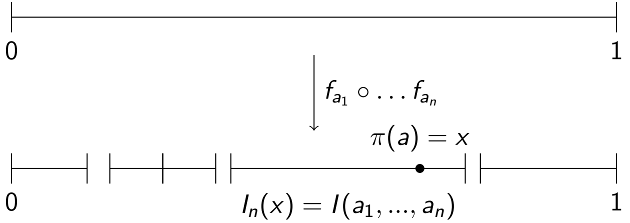
$$d(\mu, x) = \frac{\text{entropy}}{\text{Lyapunov exponent}} = \frac{h(\mu)}{\lambda(\mu)} := \frac{\sum\limits_{i=1}^{k} p_i \log \frac{1}{p_i}}{-\sum\limits_{i=1}^{k} p_i \log r_i}$$

almost surely.

This formula holds also for (well-behaved) infinite IFS and general ergodic measures, as long as $h(\mu)$ and $\lambda(\mu)$ are finite.

Main question: what if $h(\mu)$ and $\lambda(\mu)$ are infinite?

Proof: For $a = (a_1, a_2, \ldots) \in \{1, \ldots, k\}^{\mathbb{N}}$ and $n \in \mathbb{N}$ define the n-th level cylinder $I(a_1, \ldots, a_n) = f_{a_1} \circ \ldots \circ f_{a_n}([0,1])$.
For $x = \pi(a)$, let $I_n(x)$ be the n-th level cylinder containing x (it is unique for μ-almost every x), hence if $x = \pi(a_1, a_2, \ldots)$ then $I_n(x) = I(a_1, \ldots, a_n)$.



We want to calculate $\lim\limits_{r \to 0} \frac{\log \mu(B(\pi(a), r))}{\log r}$ for almost every a. First we will calculate the **symbolic dimension**

$$\delta(x) := \lim_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|}$$

For $x = \pi(a_1, a_2, \dots)$

$$\delta(x) = \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \lim_{n\to\infty} \frac{\log \mu(I(a_1, \dots, , a_n))}{\log |I(a_1, \dots, a_n)|} = \lim_{n\to\infty} \frac{\log p_{a_1} \cdots p_{a_n}}{\log r_{a_1} \cdots r_{a_n}} =$$

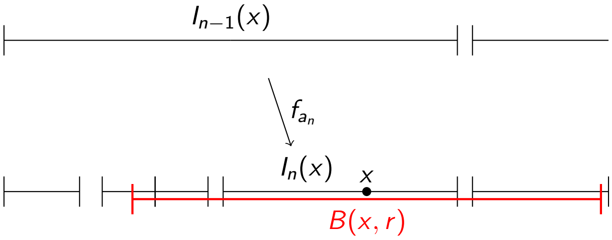
$$= \lim_{n\to\infty} \frac{\frac{1}{n} \sum_{j=1}^{n} \log p_{a_j}}{\frac{1}{n} \sum_{j=1}^{n} \log r_{a_j}} = \frac{\sum_{i=1}^{k} p_i \log p_i}{\sum_{i=1}^{k} p_i \log r_i} = \frac{h(\mu)}{\lambda(\mu)} \quad \nu\text{-a.s.}$$

How to relate $\delta(x)$ with $\underline{d}(x)$ and $\overline{d}(x)$?

Fix $x = \pi(a)$ and $r > 0$. There exists unique $n = n(r) \in \mathbb{N}$ such that

$$|I_n(x)| < r \leq |I_{n-1}(x)|.$$

Note that $n(r) \to \infty$ as $r \to 0$



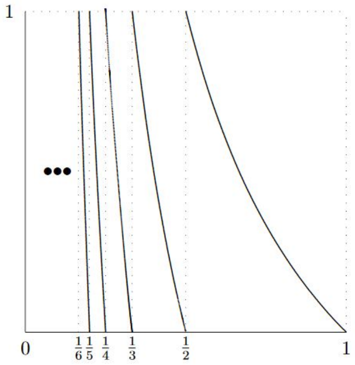
$$\log \mu(B(x,r)) \geq \log \mu(I_n(x)) \text{ and } \log r \leq \log |I_{n-1}(x)|, \text{ hence}$$

$$\frac{\log \mu(B(x,r))}{\log r} \leq \frac{\log \mu(I_n(x))}{\log |I_{n-1}(x)|} = \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \cdot \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} \to \delta(x) \text{ as}$$

$$\min\{r_i\}|I_{n-1}(x)| \leq |I_n(x)| \leq \max\{r_i\}|I_{n-1}(x)|.$$

We have proven $\overline{d}(x) \leq \delta(x)$ almost surely. One can similarly prove $\underline{d}(x) \geq \delta(x)$.

Gauss system - basic example of an infinite IFS



Gauss map

$$T : (0,1] \to (0,1], \ T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

Gauss system

$$f_i : [0,1] \to [0,1], \ f_i(x) = \frac{1}{x+i},$$
$$\mathcal{F} = \{f_i(x) = \frac{1}{x+i}\}_{i=1}^{\infty}$$

$$\pi : \mathbb{N}^{\mathbb{N}} \to [0,1], \ \pi(a) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

Setting

Assumptions

Let $T : (0,1] \to (0,1]$ be such that there exists a decomposition $(0,1] = \bigcup\limits_{n=1}^{\infty} I(n)$ into closed intervals with disjoint interiors with lengths $r_n = |I(n)|$ such that

(1) T is C^2 on $\bigcup\limits_{n=1}^{\infty} \mathrm{Int}(I(n))$

(2) there exists $k \geq 1$ such that $\inf \left\{ |(T^k)'(x)| : x \in \bigcup\limits_{n=1}^{\infty} \mathrm{Int}(I(n)) \right\} > 1$

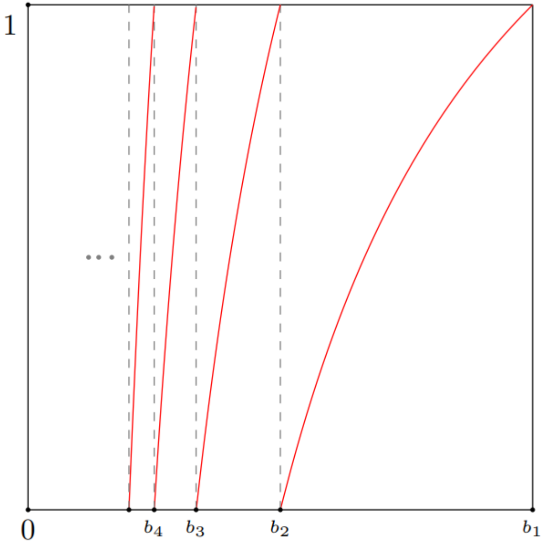
(3) $\sup\limits_{n\in\mathbb{N}} \sup\limits_{x,y,z\in I(n)} \frac{|T''(x)|}{|T'(y)||T'(z)|} < \infty$ (Rényi's condition)

(4) $T(I(n)) = (0,1]$, $I(n+1) < I(n)$ and $r_{n+1} < r_n$

(5) $0 < K \leq r_{n+1}/r_n \leq K' < \infty$ for some constants K, K'

(6) r_n decays polynomially, i.e. $\alpha = \sup\{t \geq 0 : \lim\limits_{n\to\infty} n^t r_n < \infty\}$ satisfies $1 < \alpha < \infty$

(7) T is orientation preserving on each $I(n)$


1
0
b_4
b_3
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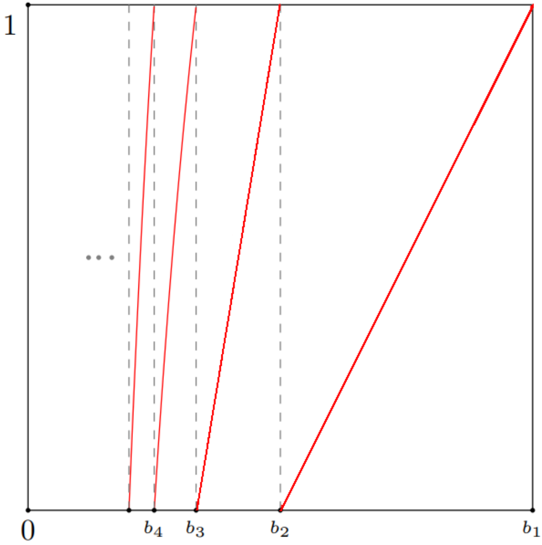
b_1


1
0
b_4
b_3
b_2
b_1

For $a_1, \ldots, a_n \in \mathbb{N}$ define the n-th level cylinder

$$I(a_1, ..., a_n) = I(a_1) \cap T^{-1}(I(a_2)) \cap \ldots \cap T^{-(n-1)}(I(a_n))$$

Let $\Sigma = \mathbb{N}^{\mathbb{N}}$ and define the natural projection $\pi : \Sigma \to (0, 1]$ by

$$\pi(a_1, a_2, \ldots) = \bigcap_{n=1}^{\infty} I(a_1, \ldots, a_n).$$

π is a continuous bijection satisfying $\pi \circ \sigma = T \circ \pi$, where σ is the left shift on Σ.

For a symbolic cylinder $C(a_1, ..., a_n) \subset \Sigma$ we have

$$\pi(C(a_1, ..., a_n)) = I(a_1, ..., a_n).$$

Let $\mathcal{O} = \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} T^{-n}(\partial I(k))$. For every $x \in (0, 1] \setminus \mathcal{O}$ there is a unique n-th level cylinder $I_n(x)$ containing x.

Proposition (consequence of the Rényi condition)

There exists $D \geq 1$ such that

$$0 < D^{-1} \leq |(T^n)'(x)| \cdot |I(a_1, \ldots, a_n)| \leq D$$

holds for every sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and every $x \in \mathrm{Int}(I(a_1, \ldots, a_n))$

Proposition (consequence of the previous one)

There exist $D_1, D_2 > 0$ such that for every $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$ we have

(1) $\left| \log |I(a_1, \ldots, a_n)| - \sum_{k=1}^{n} \log r_{a_k} \right| \leq nD_1 + D_2$

(2) $\left| \log \left| \bigcup_{j=0}^{m} I(a_1, \ldots, a_n + j) \right| - \sum_{k=1}^{n-1} \log r_{a_k} - \log\left(\sum_{j=0}^{m} r_{a_n+j}\right) \right| \leq nD_1 + D_2$

(3) $\left| \log \left| \bigcup_{j=0}^{\infty} I(a_1, \ldots, a_n + j) \right| - \sum_{k=1}^{n-1} \log r_{a_k} - \log\left(\sum_{j=0}^{\infty} r_{a_n+j}\right) \right| \leq nD_1 + D_2$

Proof of $\left|\log|I(a_1,\ldots,a_n)| - \sum_{k=1}^{n} \log r_{a_k}\right| \leq nD_1 + D_2$

By Rényi's condition

$$-\log D \leq \log|I(a_1,\ldots,a_n)| + \log(T^n)'(x) \leq \log D \text{ for } x \in I(a_1,\ldots,a_n).$$

We have

$$\log(T^n)'(x) = \sum_{k=0}^{n-1} \log T'(T^k x)$$

and, as $T^k x \in I(a_{k+1})$,

$$-\log D \leq \log|I(a_{k+1})| + \log T'(T^k x) = \log r_{a_{k+1}} + \log T'(T^k) \leq \log D,$$

hence summing over $k = 0, \ldots, n-1$

$$-n\log D \leq \sum_{k=0}^{n-1} \log T'(T^k x) + \sum_{k=1}^{n} \log r_{a_k} \leq n\log D.$$

□

Gibbs measures

Definition

An ergodic shift invariant measure ν on Σ is called a **Gibbs measure** associated to the potential $\varphi : \Sigma \to \mathbb{R}$ if there exist constants $P \in \mathbb{R}$ and $A, B > 0$ such that for every point $x \in C(a_1, \ldots a_n)$

$$A \leq \frac{\nu(C(a_1, \ldots, a_n))}{\exp\Big(- nP + S_n\varphi(x)\Big)} \leq B,$$

where $S_n\varphi = \sum_{k=0}^{n-1} \varphi(\sigma^k x)$ is the Birkhoff sum of φ at x. We will assume $P = 0$ (otherwise take $\varphi - P$ as the potential).

Examples

(1) Bernoulli measures

(2) Markov measures

(3) $(\pi^{-1})_*\mathrm{Leb}$, where π is the natural projection for the Gauss map

Let $\mu = \pi_* \nu$.

Note: If $\nu = \mathrm{p}^{\otimes \mathbb{N}}$, then μ is stationary.

Set $p_n = \mu(I(n)) = \nu(C(n))$.

Assumptions

ν is a Gibbs measure for the potential $\varphi : \Sigma \to \mathbb{R}$ such that

$$\mathrm{var}_1(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in C(n), n \in \mathbb{N}\} < \infty$$

and

$$0 < K \leq p_{n+1}/p_n \leq K' < \infty \text{ for some constants } K, K'.$$

For ν-almost every $a \in \Sigma$

$$h(\nu) = \lim_{n\to\infty} -\frac{1}{n}\log \nu(C(a_1,\ldots,a_n)) = -\lim_{n\to\infty}\frac{1}{n}S_n\varphi(a) = -\int \varphi d\nu$$

Consequently $h(\nu)=\infty$ if and only if $\sum\limits_{n=1}^{\infty} p_n \log p_n = -\infty$, as for a choice $x_n \in I(n)$

$$\int \varphi d\nu \approx \sum_{n=1}^{\infty} p_n \varphi(x_n) \approx \sum_{n=1}^{\infty} p_n \log p_n$$

Proposition

There exist $G_1, G_2 > 0$ such that for every $(a_1,\ldots,a_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$ we have

(1) $\left|\log \mu(I(a_1,\ldots,a_n)) - \sum\limits_{k=1}^{n} \log p_{a_k}\right| \leq nG_1 + G_2$

(2) $\left|\log \mu(\bigcup\limits_{j=0}^{m} I(a_1,\ldots,a_n+j)) - \sum\limits_{k=1}^{n-1} \log p_{a_k} - \log(\sum\limits_{j=0}^{m} p_{a_n+j})\right| \leq nG_1 + G_2$

(3) $\left|\log \mu(\bigcup\limits_{j=0}^{\infty} I(a_1,\ldots,a_n+j)) - \sum\limits_{k=1}^{n-1} \log p_{a_k} - \log(\sum\limits_{j=0}^{\infty} p_{a_n+j})\right| \leq nG_1 + G_2$

Define the **Lyapunov exponent** of μ as

$$\lambda(\mu) = \int_{[0,1]} \log |T'(x)| d\mu(x).$$

Similarly as before $\lambda(\mu) = \infty$ if and only if $-\sum_{n=1}^{\infty} p_n \log r_n = \infty$.

Fact

If $h(\mu) = \infty$, then $\lambda(\mu) = \infty$.

Proof:

$$h(\mu) = h(\nu) = -\sum_{n=1}^{\infty} p_n \log p_n \leq -\sum_{n=1}^{\infty} p_n \log r_n = \lambda(\mu),$$

as both $(p_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ are probability vectors. $\square$

Theorem (Volume Lemma)

If $h(\mu) < \infty$ or $\lambda(\mu) < \infty$, then μ is exact dimensional with

$$\dim(\mu) = \frac{h(\mu)}{\lambda(\mu)}.$$

Main result

Assumptions

Assume that $h(\mu) = \infty$ and **the decay ratio** $s = \lim_{n \to \infty} \frac{\log p_n}{\log r_n}$ exists.

By the Stolz-Cesàro theorem (a.k.a. L'Hôpital's rule for sequences)

$$s = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} p_k \log p_k}{\sum_{k=1}^{n} p_k \log r_k}.$$

Theorem (F. Pérez)

Under all the Assumptions (including $h(\mu) = \infty$)

(1) the symbolic dimension $\delta(x)$ exists and equals s for μ-a.e. $x \in (0, 1]$

(2) $\overline{d}(x) = s$ for μ-a.e. $x \in (0, 1]$

(3) $\underline{d}(x) = 0$ for μ-a.e. $x \in (0, 1]$

More on s

Recall:

$$\alpha = \sup\{t \geq 0 : \lim_{n\to\infty} n^t r_n < \infty\} \text{ satisfies } 1 < \alpha < \infty$$

and we assume

$$-\sum_{n=1}^{\infty} p_n \log p_n = -\sum_{n=1}^{\infty} p_n \log r_n = \infty.$$

Proposition

$$s = \frac{1}{\alpha} = s_\infty,$$

where

$$s_\infty = \inf\left\{t \geq 0 : \sum_{n=1}^{\infty} r_n^t < \infty\right\}.$$

Proof of $\delta(x) = s$ almost surely

Let $x = \pi(a) = \pi(a_1, a_2, \ldots)$. We have

$$\overline{\delta}(x) = \limsup_{n\to\infty} \frac{\log \mu(I(a_1,\ldots,a_n))}{\log |I(a_1,\ldots,a_n)|} = \limsup_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \log p_{a_k} + O(n)}{\sum\limits_{k=1}^{n} \log r_{a_k} + O(n)} = \limsup_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \log p_{a_k}}{\sum\limits_{k=1}^{n} \log r_{a_k}},$$

as $\frac{1}{n}\sum\limits_{k=1}^{n} \log r_{a_k} \to -\infty$ and $\frac{1}{n}\sum\limits_{k=1}^{n} \log p_{a_k} \to -\infty$ almost surely. Similarly

$$\underline{\delta}(x) = \liminf_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \log p_{a_k}}{\sum\limits_{k=1}^{n} \log r_{a_k}} \text{ almost surely.}$$

Define

$$f_{n,k}(x) = \#\{i \in \{1, \ldots, n\} : a_i = k\}.$$

By the ergodic theorem

$$\frac{1}{n} f_{n,k}(x) \to p_k \text{ almost surely.}$$

Proof of $\delta(x) = s$ almost surely

Denote $m(n) = \max\{a_k : 1 \le k \le n\}$. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that

$$\left| \frac{\log p_k}{\log r_k} - s \right| < \varepsilon \text{ for } k \ge N.$$

We have almost surely

$$\frac{\sum\limits_{k=1}^{n} \log p_{a_k}}{\sum\limits_{k=1}^{n} \log r_{a_k}} = \frac{\sum\limits_{k=1}^{N} f_{n,k} \log p_k + \sum\limits_{k=N+1}^{m(n)} f_{n,k} \log p_k}{\sum\limits_{k=1}^{N} f_{n,k} \log r_k + \sum\limits_{k=N+1}^{m(n)} f_{n,k} \log r_k} = \frac{O(n) + \sum\limits_{k=N+1}^{m(n)} f_{n,k} \log p_k}{O(n) + \sum\limits_{k=N+1}^{m(n)} f_{n,k} \log r_k} =$$

$$= \text{(in the limit)} = \frac{\sum\limits_{k=N+1}^{m(n)} f_{n,k} \log p_k}{\sum\limits_{k=N+1}^{m(n)} f_{n,k} \log r_k} \le \frac{(s+\varepsilon) \sum\limits_{k=N+1}^{m(n)} f_{n,k} \log r_k}{\sum\limits_{k=N+1}^{m(n)} f_{n,k} \log r_k} = s + \varepsilon$$

and similarly from below by $s - \varepsilon$. □

Local dimensions

Recall: for $r > 0$ there exists unique $n = n(r) \in \mathbb{N}$ such that $|I_n(x)| < r \leq |I_{n-1}(x)|$. It satisfies

$$\frac{\log \mu(B(x,r))}{\log r} \leq \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \cdot \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|}.$$

Proposition

For μ-almost every $x \in (0,1]$

$$\liminf_{n\to\infty} \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} = 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} = \infty.$$

This gives upper bound

$$\underline{d}(x) \leq s,$$

but no bound on $\overline{d}(x)$.

Better upper bound on $\underline{d}(x)$

Define the **tail decay ratio** $\hat{s} = \lim\limits_{n\to\infty} \frac{\log \sum\limits_{m=n}^{\infty} p_m}{\log \sum\limits_{m=n}^{\infty} r_m}$.

Theorem

$\underline{d}(x) \leq \hat{s}$ almost surely.

Lemma

If $s = \lim\limits_{n\to\infty} \frac{\log p_n}{\log r_n}$ exists, then $\hat{s} = 0$.

Lemma

If $s = \lim_{n \to \infty} \frac{\log p_n}{\log r_n}$ exists, then $\hat{s} = 0$.

Proof: Fix small $\varepsilon > 0$. For large $m \in \mathbb{N}$ we have

$$\frac{C}{m^{\alpha+\varepsilon}} \leq r_m \leq \frac{C'}{m^{\alpha-\varepsilon}} \quad \text{and} \quad p_m \geq r_m^{s+\varepsilon} \geq \frac{C}{m^{(\alpha+\varepsilon)(s+\varepsilon)}} = \frac{C}{m^{1+\varepsilon'}},$$

hence

$$\sum_{m=n}^{\infty} r_m \leq \sum_{m=n}^{\infty} \frac{C'}{m^{\alpha-\varepsilon}} \leq \frac{C'}{(\alpha-1-\varepsilon)(n-1)^{\alpha-1-\varepsilon}} \leq \frac{C''}{n^{\alpha-1-\varepsilon}}$$

and

$$\sum_{m=n}^{\infty} p_m \geq \sum_{m=n}^{\infty} \frac{C}{m^{1+\varepsilon'}} \geq \frac{C}{\varepsilon' n^{\varepsilon'}}.$$

Taking logarithms

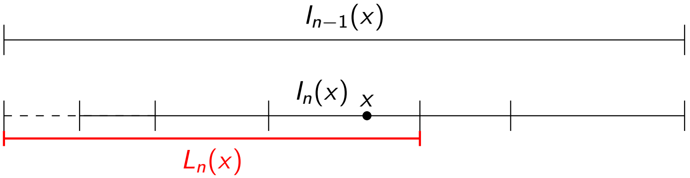
$$\frac{\log \sum\limits_{m=n}^{\infty} p_m}{\log \sum\limits_{m=n}^{\infty} r_m} \leq \frac{\log C - \log \varepsilon' - \varepsilon' \log n}{\log C'' - (\alpha - \varepsilon - 1)\log(n-1)} \to \frac{\varepsilon'}{\alpha - 1 - \varepsilon}.$$

□

Proof of $\underline{d}(x) \leq \hat{s}$ almost surely

For $x = \pi(a_1, a_2, \ldots)$ and $n \in \mathbb{N}$ define

$$L_n(x) = \bigcup_{m=0}^{\infty} I(a_1, \ldots, a_n + m) \quad \text{and} \quad \rho_n = |L_n(x)|.$$



$$\log \mu(B(x, \rho_n)) \geq \log \mu(L_n(x)) \geq \sum_{k=1}^{n-1} \log p_{a_k} + \log(\sum_{m=0}^{\infty} p_{a_n+m}) - nG_1 - G_2$$

$$\log \rho_n = \log |L_n(x)| \leq \sum_{k=1}^{n-1} \log r_{a_k} + \log(\sum_{m=0}^{\infty} r_{a_n+m}) + nD_1 + D_2.$$

Proof of $\underline{d}(x) \leq \hat{s}$ almost surely

$$\frac{\log \mu(B(x,\rho_n))}{\log \rho_n} \leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + \log(\sum_{m=0}^{\infty} p_{a_n+m}) - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log(\sum_{m=0}^{\infty} r_{a_n+m}) + nD_1 + D_2} \leq$$

if n and a_n are large enough, then almost surely

$$\leq \frac{(s+\varepsilon)\sum_{k=1}^{n-1} \log r_{a_k} + (\hat{s}+\varepsilon)\log(\sum_{m=0}^{\infty} r_{a_n+m}) - nG_1 - G_2}{\sum_{k=1}^{n-1} \log r_{a_k} + \log(\sum_{m=0}^{\infty} r_{a_n+m}) + nD_1 + D_2}$$

This has limit $\hat{s}+\varepsilon$ along subsequences such that

$$a_n \to \infty \quad \text{and} \quad \frac{\log(\sum_{m=0}^{\infty} r_{a_n+m})}{\sum_{k=1}^{n-1} \log r_{a_k}} \to \infty$$

Proof of $\underline{d}(x) \leq \hat{s}$ almost surely

Along a subsequence we have

$$\infty \leftarrow \frac{\log |I_n(x)|}{\log |I_{n-1}(x)|} = \frac{\sum_{k=1}^{n} \log r_{a_k}}{\sum_{k=1}^{n-1} \log r_{a_k}} = 1 + \frac{\log r_{a_n}}{\sum_{k=1}^{n-1} \log r_{a_k}}, \text{ so } \frac{\log r_{a_n}}{\sum_{k=1}^{n-1} \log r_{a_k}} \to \infty.$$

On the other hand, for small δ and a_n large enough

$$\sum_{m=0}^{\infty} r_{a_n+m} \leq \frac{C_1}{a_n^{\alpha-\delta-1}} = C_2 \left(\frac{C}{a_n^{\alpha+\delta}} \right)^{\frac{\alpha-\delta-1}{\alpha+\delta}} \leq C_2 r_{a_n}^{\frac{\alpha-\delta-1}{\alpha+\delta}}, \text{ hence}$$

$$\log \left(\sum_{m=0}^{\infty} r_{a_n+m} \right) \leq \frac{\alpha-\delta-1}{\alpha+\delta} \log r_{a_n} + \log(C_2), \text{ so}$$

$$\frac{\log(\sum_{m=0}^{\infty} r_{a_n+m})}{\sum_{k=1}^{n-1} \log r_{a_k}} \geq \frac{\frac{\alpha-\delta-1}{\alpha+\delta} \log r_{a_n} + \log(C_2)}{\sum_{k=1}^{n-1} \log r_{a_k}} \to \infty$$

□

Proof of $\overline{d}(x) \leq s$

Lemma

For every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $n \in \mathbb{N}$

$$\frac{\log \sum_{m=k}^{k+n} p_m}{\log \sum_{m=k-1}^{k+n} r_m} \leq s + \varepsilon.$$

Instead of the proof: for $n = 0$

$$\frac{\log p_k}{\log(r_{k-1} + r_k)} \leq \frac{\log p_k}{\log Cr_k} = \frac{\log p_k}{\log r_k + \log C} \approx s.$$

For large $n \in \mathbb{N}$

$$\frac{\log \sum_{m=k}^{k+n} p_m}{\log \sum_{m=k-1}^{k+n} r_m} \approx \frac{\log \sum_{m=k}^{\infty} p_m}{\log \sum_{m=k}^{\infty} r_m + \log C} \approx \hat{s} = 0.$$

Proof of $\overline{d}(x) \leq s$

Fix $x = \pi(a)$ and small $\varepsilon > 0$.

Let k_0 be as in the Lemma.

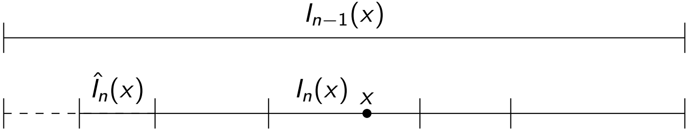
For $r > 0$ there exists unique $n = n(r) \in \mathbb{N}$ such that

$$|I_n(x)| < r \leq |I_{n-1}(x)|.$$

Let $\hat{I}_n(x) = I(a_1, \ldots, a_{n-1}, k_0)$.

Proof of $\overline{d}(x) \leq s$

Case 1: $a_n \leq k_0$



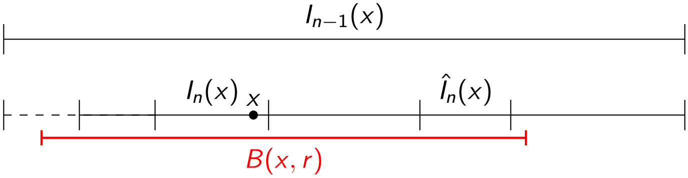
Then $p_{a_n} \geq C(k_0) = p_1 A^{k_0}$, as $A \leq \frac{p_{n+1}}{p_n} \leq B$.

$$\frac{\log \mu(B(x,r))}{\log r} \leq \frac{\log \mu(I_n(x))}{\log |I_{n-1}(x)|} \leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + \log p_{a_n} + O(n)}{\sum_{k=1}^{n-1} \log r_{a_k} + O(n)} \leq$$

$$\leq \frac{\sum_{k=1}^{n-1} \log p_{a_k} + O(n)}{\sum_{k=1}^{n-1} \log r_{a_k} + O(n)} \approx s$$

Proof of $\overline{d}(x) \leq s$

Case 2: $a_n > k_0$ but $I(a_1, \ldots, a_{n-1}, k_0) \subset B(x, r)$

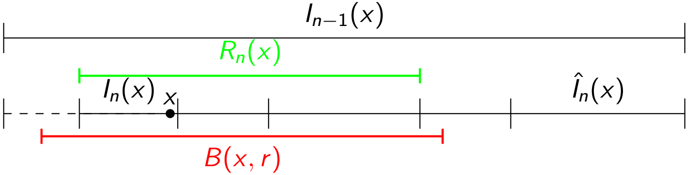


Same as before but with $\mu(B(x, r)) \geq \mu(\hat{I}_n(x)) = \mu(I(a_1, \ldots, a_{n-1}, k_0))$.

Proof of $\overline{d}(x) \leq s$

Case 3: $a_n > k_0$ and $\hat{I}_n(x) \not\subset B(x, r)$

Let $R_n(x) = \bigcup_{m=j}^{a_n} I(a_1, \ldots, a_{n-1}, m)$ be such that $j > k_0,\ R_n(x) \subset B(x, r)$, but $|R_n(x) \cup I(a_1, \ldots, a_{n-1}, j-1)| > r$



$$\log \mu(B(x, r)) \geq \log \mu(R_n(x)) \geq \sum_{k=1}^{n-1} \log p_{a_k} + \log \sum_{m=j}^{a_n} p_m + O(n)$$

$$\log r \leq \log |R_n(x) \cup I(a_1, \ldots, a_{n-1}, j-1)| \leq \sum_{k=1}^{n-1} \log r_{a_k} + \log \sum_{m=j-1}^{a_n} r_m + O(n)$$

Proof of $\overline{d}(x) \leq s$

$$\frac{\log \mu(B(x,r))}{\log r} \leq \frac{\sum\limits_{k=1}^{n-1} \log p_{a_k} + \log \sum\limits_{m=j}^{a_n} p_m + O(n)}{\sum\limits_{k=1}^{n-1} \log r_{a_k} + \log \sum\limits_{m=j-1}^{a_n} r_m + O(n)} \leq$$

$$\leq \frac{(s+\varepsilon) \sum\limits_{k=1}^{n-1} \log r_{a_k} + (s+\varepsilon) \log \sum\limits_{m=j}^{a_n} r_m + O(n)}{\sum\limits_{k=1}^{n-1} \log r_{a_k} + \log \sum\limits_{m=j-1}^{a_n} r_m + O(n)} \leq s + \varepsilon.$$

□

Thank you for your attention!