Towards the solution of some fundamental questions concerning group actions on the circle and codimension-one foliations

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May 29, 2020

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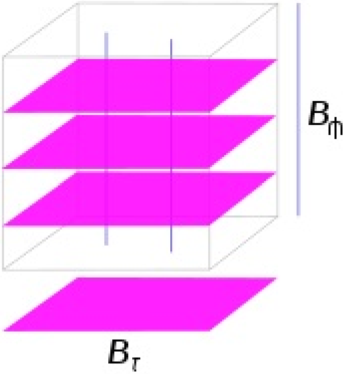
Examples

1. $M = \mathbb{R}^3$.Family of planes of the form $z = const$ form a 2-dimensional foliation.



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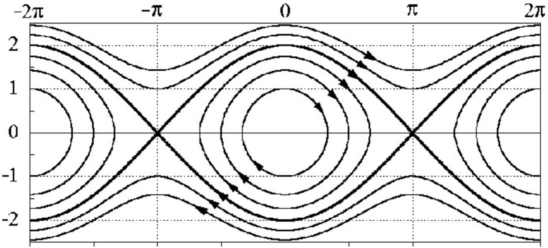

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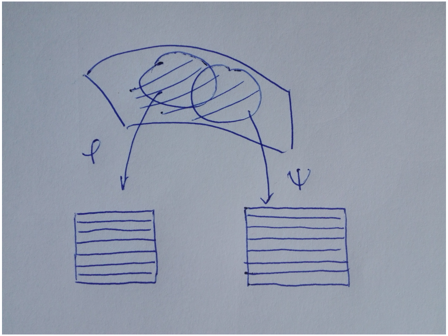

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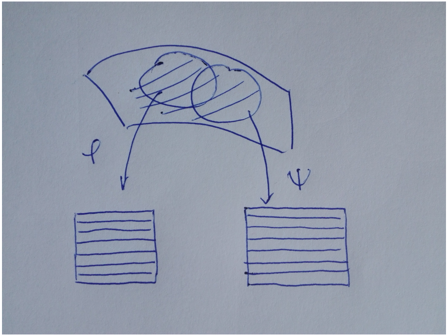

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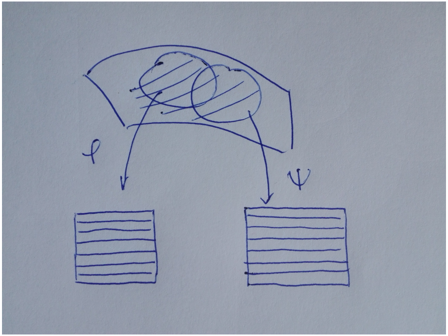

$\mathcal{F}$ is of class C^k if maps may be chosen such that $\psi \circ \varphi^{-1}$ is C^k.

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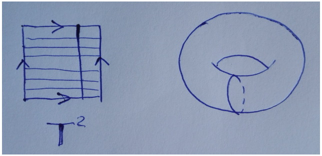

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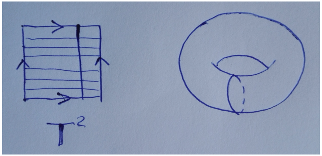

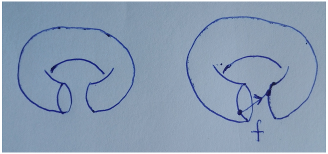

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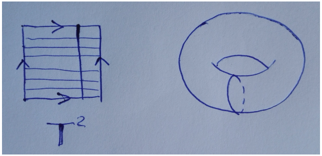

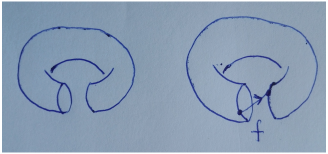

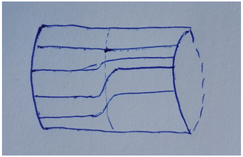

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Problem
If G is a finitely generated group of C^2 circle diffeomorphisms acting minimally, then is it ergodic with respect to the Lebesgue measure? In other words, if $A \subseteq \mathbb{S}^1$ is a G-invariant set, then it must be $Leb(A) \in \{0, 1\}$?

Minimal diffeomorphisms with irrational rotation number

Theorem

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It does not follow from Denjoy theorem. The reason is Denjoy theorem implies existence of conjugacy, but does not tell anything about its absolute continuity.

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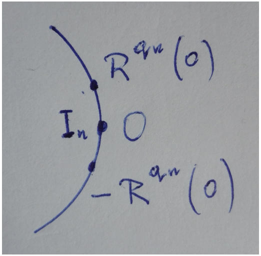

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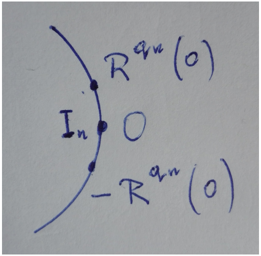

The interval I_n has the property that the sets $I_n, R(I_n), \ldots, R^{q_{n+1}-1}(I_n)$

- cover the circle,
- every point of the circle belongs to at most two of these sets.

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Distortion of diffeomorphism g on the interval $I \subseteq \mathbb{S}^1$:

$$\chi(g, I) := \sup_{x,y \in I} \log \frac{g'(x)}{g'(y)}.$$

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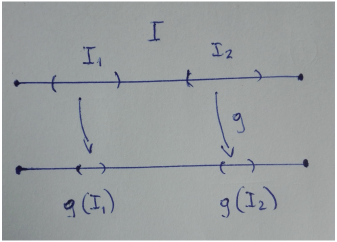

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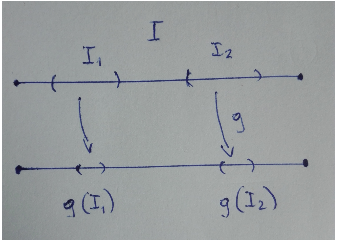

It follows that if $I_1, I_2 \subseteq I$ are two intervals, then:

$$\frac{|g(I_1)|}{|g(I)|} \leq \frac{|I_1|}{|I|} e^{\chi(g,I)}.$$

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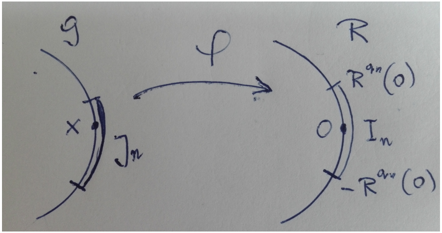

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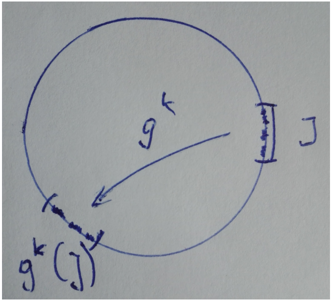

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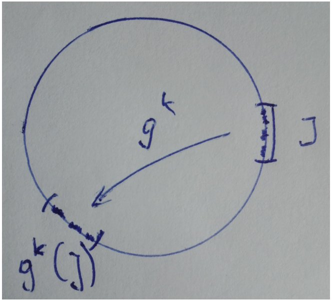

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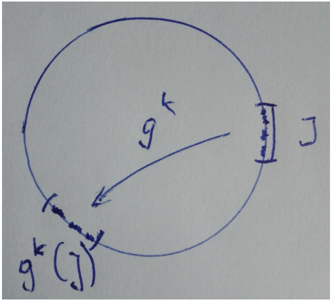

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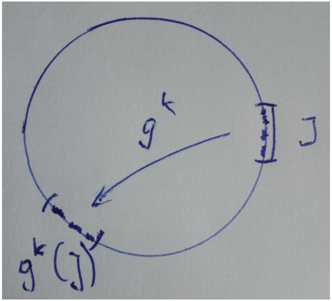

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$$\leq 2e^V \frac{Leb(J \setminus A)}{Leb(J)}.$$

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$$\chi\left(F_n, (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2})\right) \leq \log(2).$$

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Then $S = F_1'(y_n) + \ldots + F_n'(y_n) = \frac{1}{\lambda} + \ldots + \frac{1}{\lambda^n} < \sum_{j=1}^{\infty} \frac{1}{\lambda^j} :=$ D.

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Using the second estimate $\chi(F_n, V_n) \leq \log(2)$ for $V_n = (y_n - \varepsilon, y_n + \varepsilon)$, $\varepsilon := \log(2)/(2C_{\mathcal{G}}D)$

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Let $y_n := g_n \circ \ldots \circ g_1(x_0)$.

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Then $S = F_1'(y_n) + \ldots + F_n'(y_n) = \frac{1}{\lambda} + \ldots + \frac{1}{\lambda^n} < \sum_{j=1}^{\infty} \frac{1}{\lambda^j} :=$ D.(independent of n!).

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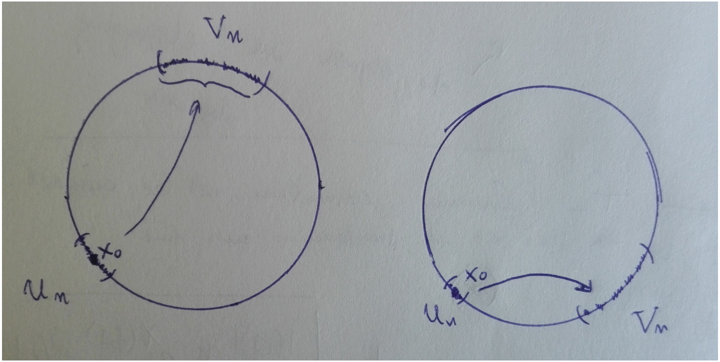

Compactness yield existence of an interval V of length 2ε with $Leb(V \setminus A) = 0$. Using minimality of the group action $Leb(A) = 1$.

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Problem: in all known examples of groups G where the constant is positive there are no points with $g'(x)\leq 1$ for all $g\in G$.

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If the group G of C^2 diffeomorphisms satisfies $(*)$, then the action is ergodic with respect to the Lebesgue measure.

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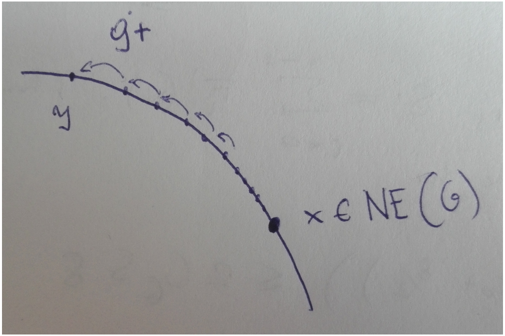

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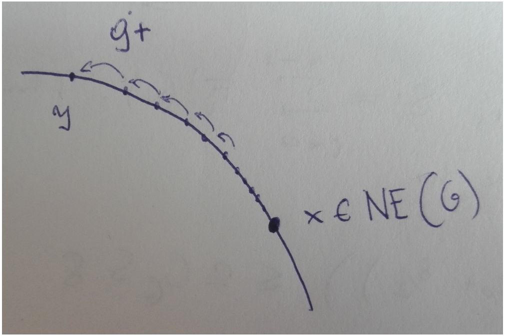

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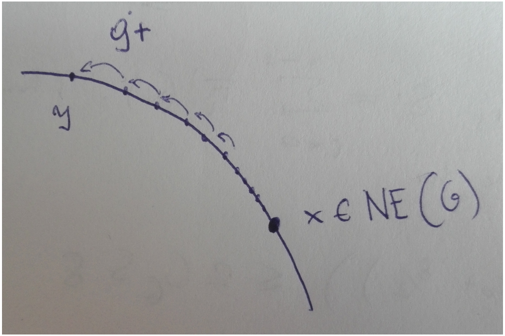

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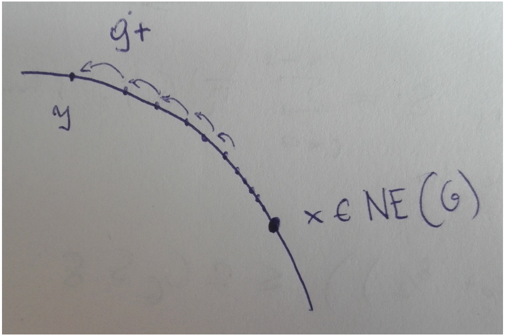

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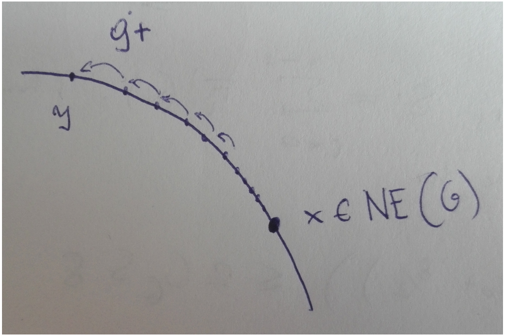

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For all y sufficiently close to x from the right there exist n such that $(g_+^n)'(x) > 2$. The same may be proved from the left side.

What is known about the set $NE(G)$?

- it may be nonempty (smooth representation of Thompson's group, $\mathrm{PSL}(2, \mathbb{Z})$) (DKN "On the question of ergodicity...")

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- it may be nonempty (smooth representation of Thompson's group, $\mathrm{PSL}(2,\mathbb{Z})$) (DKN "On the question of ergodicity...")
- it is not invariant under change of coordinates (but property $(*)$ is!)
- in the case of free, minimal, analytic groups property $(*)$ always holds (DKN "On the ergodic theory of free group actions by real-analytic circle diffeomorphisms")

Conjecture (Y. Guivarc'h, V. Kaimanovich, F. Ledrappier)

For any finitely supported measure m on a lattice $\Gamma < PSL(2, \mathbb{R})$ *whose support generates* Γ*, the corresponding stationary measure on the circle is singular w. r. t. Lebesgue.*

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Problem

Given group G *of* C^2 *diffeomorphisms, does there exists a distribution on* G *such that its support generates* G, *and the corresponding stationary measure on the circle is absolutely continuous to the Lebesgue measure?*

Let $\mathbb{P}$ be a distribution on a group G acting minimally such that support of the measure generates G.

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From the Birkhoff ergodic theorem for μ almost every point $x \in \mathbb{S}^1$

$$\frac{\log(g_{\omega_n} \circ \ldots \circ g_{\omega_1})'(x)}{n} \to \lambda \qquad \text{a.s}$$

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is positive μ a.e.

Corollary

If the Lyapunov expansion exponent is zero Lebesgue a.e., then for every probability distribution $\mathbb{P}$ on the group such that its support generates G, the corresponding stationary measure is singular w.r.t the Lebesgue.

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Proof in "Global rigidity of conjugations for locally non-discrete subgroups of $Diff^{\omega}(\mathbb{S}^1)$", Eskif, Rebelo.

Thank you!