Solenoid

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Dynamical Systems student/PhD seminar, May 2020

Attractors

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- Function $f|_\Lambda$ is called chain transitive if $\forall_{\delta>0} \forall_{x,y \in \Lambda} \exists_{x=x_0,x_1,x_2,\ldots,x_n=y}$ $d(f(x_i), x_{i+1}) \leq \delta$ for all $i = 0, 1, \ldots, n$

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- An attrator with a hyperbolic structure is called a hyperbolic attractor.

Attractors

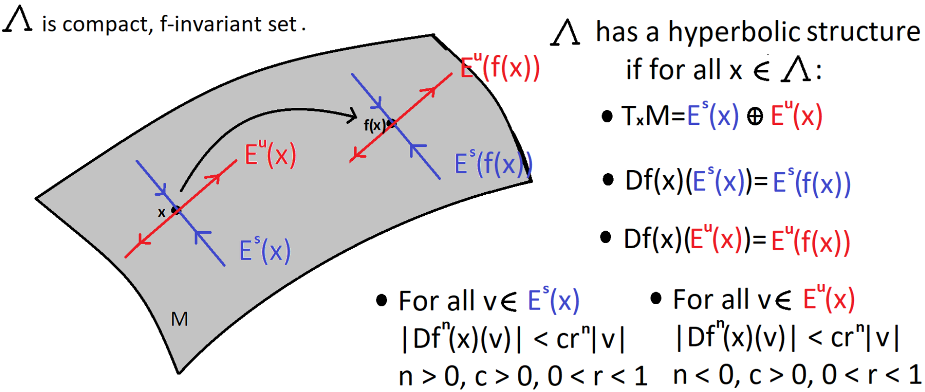

Figure: Hyperbolic structure

Attractors

Proposition: Let Λ be a compact invariant set in a finite-dimensional manifold. Then Λ is an attracting set if and only if there a exists an arbitrarily small neighbourhood V such that $V \subset \Lambda$, V is positively invariant and for all $p \in V$ $\omega(p) \subset \Lambda$.

Theorem: Let Λ be an attracting set for f. Assume either that $p \in \Lambda$ is a hyperbolic periodic point or Λ has a hyperbolic structure and $p \in \Lambda$. Then $W^u(p) \subset \Lambda$.

Proof:

Recall $W^u(p) = \{x \in N : |f^n(x) - f^n(p)| \longrightarrow 0 \text{ as } n \to -\infty\}$ and $W^u_\epsilon(p) = \{x \in N : \forall_{n<0} \ |f^n(x) - f^n(p)| < \epsilon\}$. $\Lambda \subset intN$, where N is a trapping region. So there exists $\epsilon > 0$ such that $W^u_\epsilon(f^k(p)) \subset N$ for all $k \in \mathbb{Z}$. Therefore for all $k \geq 0$ we have $W^u(f^{-k}(p)) = \bigcup_{j\geq 0} f^j(W^u_\epsilon(f^{-j-k}(p))) \subset N$ and $W^u(p) = f^k(W^u(f^{-k}(p))) \subset f^k(N)$. Thus $W^u(p) \subset \bigcap_{k\geq 0} f^k(N) = \Lambda$. $\square$

Attractors

Definitions:

- The definition of topological dimension is given inductively. A set Λ has topological dimension 0 provided for each point $p \in \Lambda$ there exists arbitrarily small neighbourhood U of p such that $\partial U \cap \Lambda = \emptyset$. Then, inductively, a set Λ is said to have topological dimension n provided for each point $p \in \Lambda$ there exists arbitrarily small neighbourhood U of p such that $\partial U \cap$ has dimension $n - 1$.

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- Hyperbolic attractor Λ is an expanding attractor provided the topological dimension of Λ is equal to the dimension of the unstable splitting.

The Solenoid Attractor

Let

$$D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$$
$$S^1 = \{z \in \mathbb{R} \mod 1\}$$

And consider solid torus $N = S^1 \times D^2$.
Let $g : S^1 \longrightarrow S^1$ be a doubling map, given by $g(t) = 2t \mod 1$.

Definition: The solenoid map is the embedding $f : N \longrightarrow N$ of the form $f(t,z) = (g(t), \frac{1}{4}z + \frac{1}{2}e^{2\pi it})$

The Solenoid Attractor

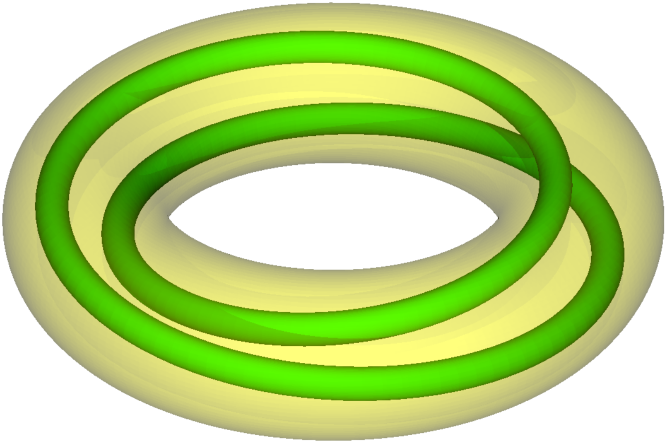

Figure: Smale-Williams Solenoid.

The Solenoid Attractor

Proposition: Let $D(t) = \{t\} \times D^2$. Then $f : D(t) \longrightarrow D(t)$ is a contraction by a factor of $\frac{1}{4}$.

Proof:

Let $p_1 = (t, z_1), p_2 = (t, z_2) \in D(t)$. Then
$|f(p_1) - f(p_2)| = |(g(t), \frac{1}{4}z_1 + \frac{1}{2}e^{2\pi it}) - (g(t), \frac{1}{4}z_2 + \frac{1}{2}e^{2\pi it})| =$
$|(0, \frac{1}{4}(z_1 - z_2)| = \frac{1}{4}|(t, z_1) - (t, z_2)| = \frac{1}{4}|p_1 - p_2|$. $\square$

Notation: $D([t_1, t_2]) = \bigcup\{D(t) : t \in [t_1, t_2]\}$.

The Solenoid Attractor

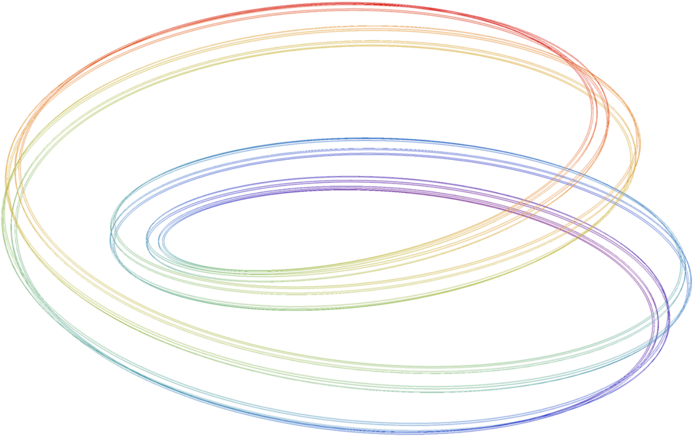

Figure: Smale-Williams Solenoid 2.

The Solenoid Attractor

Theorem: Let $\Lambda = \bigcap_{k\geq 0} f^k(N)$. Then Λ is a hyperbolic expanding attractor for f, of topological dimension 1, called the solenoid.
Proof: Conclusion of this lecture. $\square$

Proposition: For all t_0 the set $\Lambda \cap D(t_0)$ is a Cantor set.
Proof:
If $f(t,z) \in D(t_0)$, then $g(t) = t_0 \mod 1$, so $t = \frac{t_0}{2}$ or $t = \frac{t_0}{2} + \frac{1}{2}$. Notice that $f(D(\frac{t_0}{2})) = (t_0, \frac{1}{4}D^2 + \frac{1}{2}e^{\pi i t_0})$ and $f(D(\frac{t_0}{2} + \frac{1}{2})) = (t_0, \frac{1}{4}D^2 - \frac{1}{2}e^{\pi i t_0})$. Now, since $\frac{1}{2} - \frac{1}{4} > 0$, equality $f(D(\frac{t_0}{2})) \cap f(D(\frac{t_0}{2} + \frac{1}{2})) = \emptyset$ is true. Since $\frac{1}{2} + \frac{1}{4} < 1$, inclusion $f(D(\frac{t_0}{2})), f(D(\frac{t_0}{2} + \frac{1}{2})) \subset D(t_0)$ is true. As a consequence $f(N) \subset N$. Let

$$\mathcal{N}_k = \bigcap_{j=0}^{k} f^j(N) = f^k(N)$$

The Solenoid Attractor

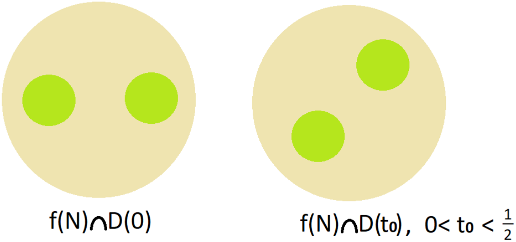

Figure: Cross section of $f(N)$

The Solenoid Attractor

Lemma: For all $t \in S^1$ the set $\mathcal{N}_k \cap D(t)$ is the union of 2^k discs of radius $(\frac{1}{4})^k$.

Proof(Lemma): Induction. For $k = 0$ thesis is trivially true. Suppose lemma is true for some k. Then $\mathcal{N}_k \cap D(t) = f(\mathcal{N}_{k-1} \cap D(\frac{t}{2})) \cup f(\mathcal{N}_{k-1} \cap D(\frac{t+1}{2}))$. By induction $\mathcal{N}_{k-1} \cap D(\frac{t}{2}))$ and $\mathcal{N}_{k-1} \cap D(\frac{t+1}{2})$ are union of 2^{k-1} discs of radius $(\frac{1}{4})^{k-1}$. Since f is $\frac{1}{4}$-contraction, the sets $\mathcal{N}_k \cap D(t) = f(\mathcal{N}_{k-1} \cap D(\frac{t}{2})), f(\mathcal{N}_{k-1} \cap D(\frac{t+1}{2}))$ are the union of 2^{k-1} discs of radius $(\frac{1}{4})^k$. Together they the union of 2^k discs of the stated radius. $\square$

Now, $\Lambda = \bigcap_{j \geq 0} f^j(N) = \bigcap_{j \geq 0} \mathcal{N}_j$, so $\Lambda \cap D(t_0)$ is a Cantor set. $\square$

The Solenoid Attractor

Proposition: The set Λ has the following properties:

- Λ is connected.

Proof:

- $\mathcal{N}_j$ are compact, connected and nested. Hence $\Lambda = \bigcap_{j \geq 0} \mathcal{N}_j$ is connected.

The Solenoid Attractor

Proposition: The set Λ has the following properties:

- Λ is connected.
- Λ is not locally connected.

Proof:

- $\mathcal{N}_j$ are compact, connected and nested. Hence $\Lambda = \bigcap_{j \geq 0} \mathcal{N}_j$ is connected.
- If $t_2 - t_1 \in (0, 1)$, then $D[t_1, t_2] \cap \mathcal{N}_k$ is the union of 2^k tubes. For all U there exists t_2, t_1, k such that U contains two of these tubes. Since each one contains some point of Λ, Λ is not locally connected.

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- Fix $p \in \Lambda$ and create sequence $q_k \in \Lambda \cap D(t_0)$ such that q_k and q_{k-1} are in the same component of $\mathcal{N}_{k-1} \cap D(t_0)$ and any path from p to q_k in $\mathcal{N}_k$ must go around S^1 at least 2^{k-1} times. By construction q_k is Cauchy sequence and let q be its limit.

The Solenoid Attractor

Proposition: The set Λ has the following properties:

- Λ is connected.
- Λ is not locally connected.
- Λ is not path connected.
- The topological dimension of Λ is one.

Proof:

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Since Λ is closed, $q \in \Lambda$. This limit point q is in the same component of $\mathcal{N}_k \cap D(t_0)$ as q_k and any path from p to q in $\mathcal{N}_k$ must go around S^1 at least 2^{k-1} times. Thus any continuous path form p to q have to go around S^1 infinitely many times. Contradiction.

- $\Lambda \cap D(t_0)$ is totally disconnected, hence have topological dimension 0. Since $\Lambda \cap D([t_1, t_2])$ is homeomorphic to $(\Lambda \cap D(t_1)) \times [t_1, t_2]$ and so has topological dimension 1.

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Proposition: The map $f|_\Lambda$ has the following properties:

- Periodic points of $f|_\Lambda$ are dense in Λ.

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Proposition: The map $f|_\Lambda$ has the following properties:

- Periodic points of $f|_\Lambda$ are dense in Λ.
- $f|_\Lambda$ is topologically transitive.
- $f|_\Lambda$ has a hyperbolic structure on Λ.

The Solenoid Attractor

Proof:

Lemma: The periodic points of g are dense in S^1.

- If $g^k(t_0) = t_0$, then $f^k(D(t_0)) \subset D(t_0)$. f^k takes $D(t_0)$ into itself with a contraction factor of 4^{-k}, so f^k has a fixed point in $D(t_0)$. By lemma, fibers with a periodic point for f are dense in the set of all fibers. Take $p \in \Lambda$ and a neighbourhood U of p. There exists k, t_1, t_2 such that $f^k(D[t_1, t_2]) \subset U$. We showed above that f has periodic point in $D[t_1, t_2]$ and so in U.

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- Let U and V be two open subsets of Λ. Thus there exists sets U', V' such that $U' \cap \Lambda = U$ and $V' \cap \Lambda = V$ and constants k, t_1, t_1', t_2, t_2' such that $f^k(D[t_1, t_2]) \subset U'$ and $f^k(D[t_1', t_2']) \subset V'$. There is $j > 0$ such that $f^j(D[t_1, t_2] \cap \Lambda) \cap D[t_1', t_2')] \cap \Lambda \neq \emptyset$.
Thus $f^j(f^k(D[t_1, t_2]) \cap \Lambda) \cap f^k(D[t_1', t_2')]) \cap \Lambda \neq \emptyset$ and
$f^j(U) \cap V = f^j(U' \cap \Lambda) \cap (V' \cap \Lambda) \neq \emptyset$.

The Solenoid Attractor

- In terms of the coordinates on $S^1 \times D^2$

$$Df(t,z) = \begin{pmatrix} 2 & 0 \\ \pi i e^{2\pi i} & \frac{1}{4} Id_{\mathbb{C}} \end{pmatrix}.$$

Let $E^s = \{0\} \times \mathbb{R}^2$. Then for $(0, v) \in E^s$

$$Df(t,z) \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} v \end{pmatrix}$$

and

$$Df^k(t,z) \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ (\frac{1}{4})^k v \end{pmatrix}$$

which goes to 0 as k tends to ∞. Therefore E^s is indeed the stable bundle at each $(t,z) \in \Lambda$.

To find E^u it is necessary to use cones.

The Solenoid Attractor

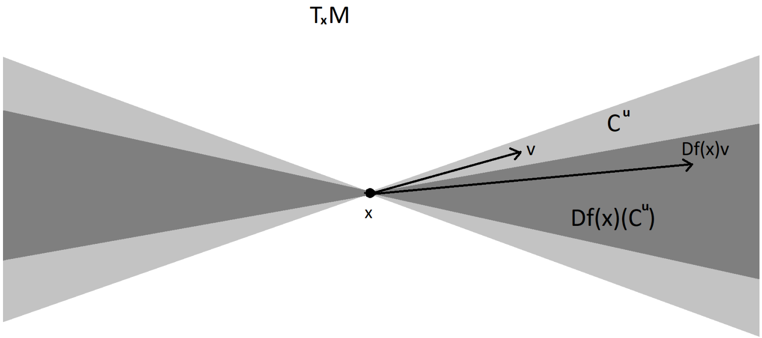

Figure: Cones

The Solenoid Attractor

Let $C_p^u = \{(v_1, v_2) : v_1 \in TS^1, v_2 \in \mathbb{R}^2 \text{ such that } |v_1| \geq \frac{1}{2}|v_2|\}$. We will prove our statement in three steps.

STEP 1: $Df(p)(C_p^u) \subset C_{f(p)}^u$.

$$Df(p)\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 2v_1 \\ \pi i e^{2\pi t i} v_1 + \frac{1}{4}v_2 \end{pmatrix} = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.$$

Then $|v_1'| = 2|v_1| = \frac{1}{2}|4v_1| > \frac{1}{2}(\pi|v_1| + \frac{1}{2}|v_1|) \geq \frac{1}{2}(\pi|v_1| + \frac{1}{4}|v_2|) \geq \frac{1}{2}|v_2'|$.

STEP 2: $\bigcap_{k\geq 0} Df^k(f^{-k}(p))(C_{f^{-k}(p)}^u) = E^u$ is a line in the tangent space. The sets

$$\bigcap_{j=0}^{k} Df^j(f^{-j}(p))(C_{f^{-j}(p)}^u) = Df^k(f^{-k}(p))(C_{f^{-k}(p)}^u)$$

are nested.

The Solenoid Attractor

We will prove that the angle between any two vectors in these finite intersection goes to 0 as $k \to \infty$.
Let

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in C^u_{f^{-k}(p)} : v_1, w_1 > 0.$$

$$Df^k(f^{-k}(p)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1^k \\ v_2^k \end{pmatrix}$$

and so for w_j. Then

$$\left| \frac{v_2^1}{v_1^1} - \frac{w_2^1}{w_1^1} \right| = \left| \frac{\pi i e^{2\pi i t} v_1 + \frac{1}{4} v_2}{2v_1} - \frac{\pi i e^{2\pi i t} w_1 + \frac{1}{4} w_2}{2w_1} \right| = \frac{1}{8} \left| \frac{v_2}{v_1} - \frac{w_2}{w_1} \right|.$$

So $Df^k(f^{-k}(p))$ is a contraction on the slopes. By induction

$$\left| \frac{v_2^k}{v_1^k} - \frac{w_2^k}{w_1^k} \right| = \left(\frac{1}{8} \right)^k \left| \frac{v_2}{v_1} - \frac{w_2}{w_1} \right|$$

Last expression goes to 0 as $k \to \infty$.

STEP 3: $Df(p)|_{E^u}$ is an expansion.

Let $\left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|_\star = |v_1|$ be a norm on the cone. Then

$$Df(p)\left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|_\star = \left|\begin{pmatrix} 2v_1 \\ \dots \end{pmatrix}\right|_\star = 2|v_1| = 2\left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|_\star .$$

□

THANK YOU FOR YOUR ATTENTION!

ANY QUESTIONS?