Wiman–Valiron discs and the Hausdorff dimension of Julia sets of meromorphic functions

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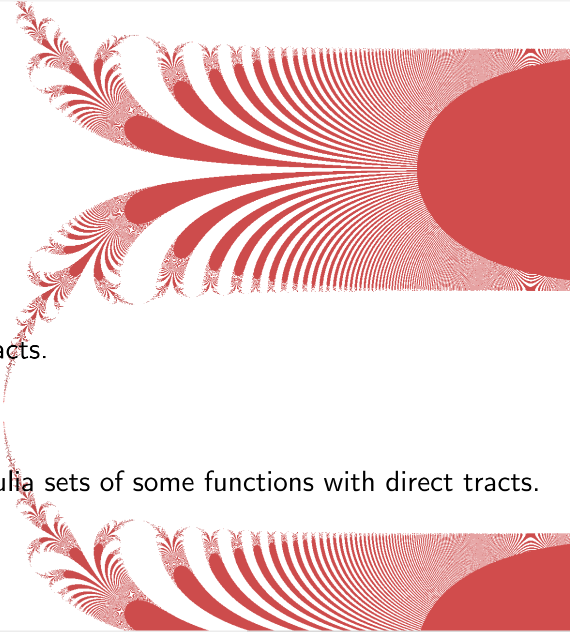
Outline



- Basic definitions.
- Direct and logarithmic tracts.
- Hausdorff dimension.
- Wiman–Valiron theory.
- Hausdorff dimension of Julia sets of some functions with direct tracts.

Basic definitions

- Let $f : \mathbb{C} \to \mathbb{C}$ be analytic.
- Denote by f^n the nth iterate of f.

Definition

The **Fatou set** is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighborhood of } z\}.$$

Definition

The **Julia set** is

$$J(f) = \mathbb{C} \setminus F(f).$$

The escaping set of a polynomial

Definition

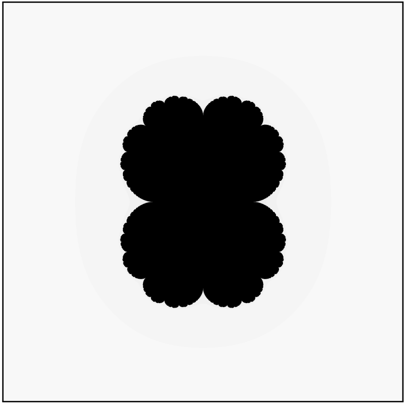
The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

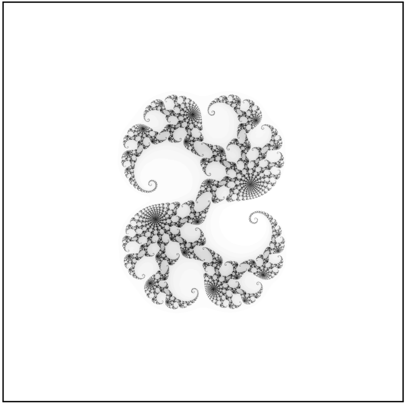
- $I(f)$ is a neighborhood of ∞.
- $\partial I(f) = J(f)$.
- $I(f) \subset F(f)$.
- Points in $I(f)$ all have the same rate of escape.

Denote by $K(f)$ the set of points with bounded orbit.

Examples of the escaping set of some polynomials (in white)

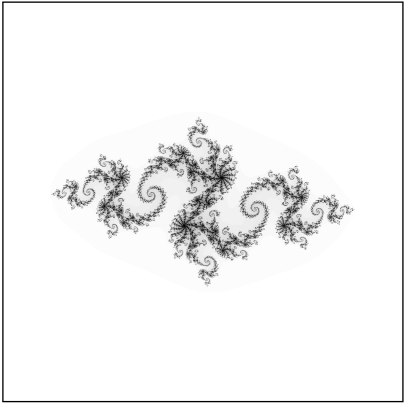


$z^2 + 0.25$

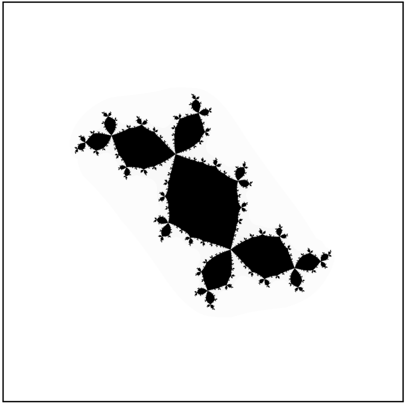


$z^2 + .28 + .008i$

More examples of the escaping set of some polynomials (in white)



$z^2 - 0.79 + .15i$



$z^2 - 0.122565 + 0.744864i$

The escaping set of a transcendental entire function

Definition

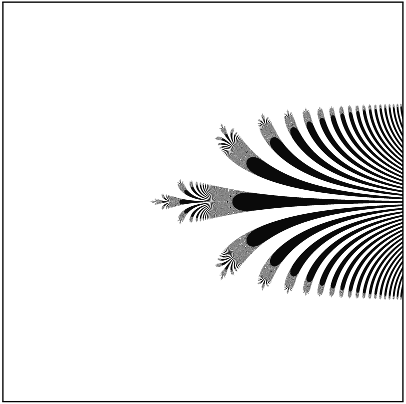
The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

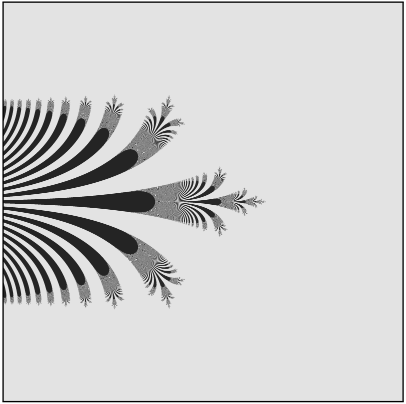
- $I(f)$ is not a neighborhood of ∞.
- $I(f)$ can meet $F(f)$ and $J(f)$.
- Points in $I(f)$ have different rates of escape.
- Eremenko (1989) showed $I(f)$ has the following properties:
  - $I(f) \cap J(f) \neq \emptyset$,
  - $\partial I(f) = J(f)$,
  - $\overline{I(f)}$ has no bounded components.
- Eremenko's conjecture: All components of $I(f)$ are unbounded.

Denote by $K(f)$ the set of points with bounded orbit.

Examples of the escaping set of some transcendental entire functions (in black and gray)



$\frac{1}{4}\exp(z)$



$z + 1 + \exp(-z)$

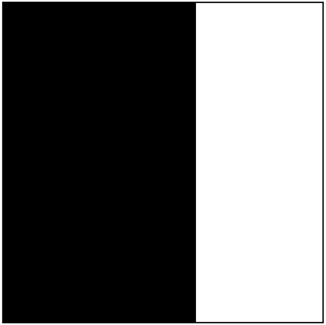
Tracts

Definition

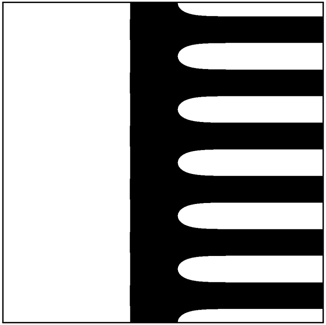
Let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves. Further suppose that the complement of D is unbounded and let f be a complex valued function whose domain of definition includes the closure $\bar{D}$ of D.
Then, D is a **direct tract** if f is analytic in D, continuous on $\bar{D}$, and if there exists $R > 0$ such that $|f(z)| = R$ for $z \in \partial D$ while $|f(z)| > R$ for $z \in D$. If in addition the restriction $f : D \to \{z \in \mathbb{C} : |z| > R\}$ is a universal covering, then D is a **logarithmic tract**.

- Every transcendental entire function has a direct tract.

Examples (tracts in white)

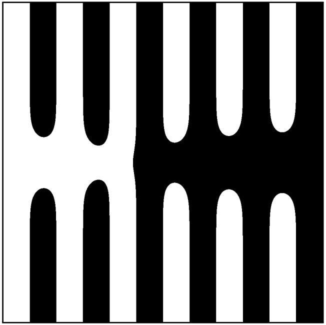


$\exp(z)$

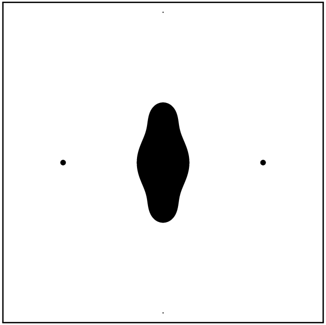


$\exp(\exp(z) - z)$

More examples (tracts in white)



$\exp(\sin(z) - z)$



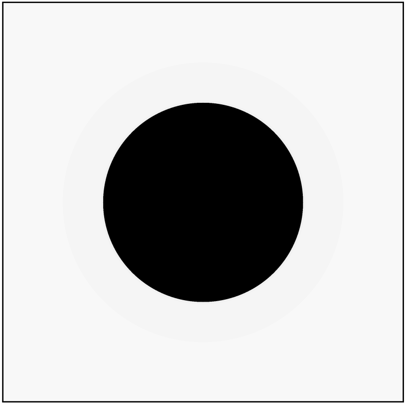
$\sin(z)\cosh(z)$

Hausdorff dimension

Denote by $\dim J(f)$ the Hausdorff dimension of the Julia set of f.

- If f is a quadratic map, then $0 < \dim J(f) \leq 2$.
- 'Difficult' to find functions, f, for which $\dim J(f) = 2$.

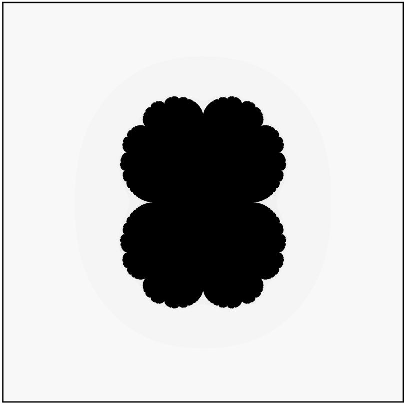
Hausdorff dimension for quadratic maps



z^2

- $I(f) = \{z : |z| > 1\}$ is in white
- $J(f) = \{z : |z| = 1\}$ is the boundary of the black region
- $\dim J(f) = 1$

Hausdorff dimension for quadratic maps



$z^2 + 0.25$

- $I(f)$ is in white
- $J(f)$ is the boundary of the black region
- $1 < \dim J(f) < 3/2$

Hausdorff dimension for quadratic maps

$z^2 - 3/2 + 2i/3$

- $I(f)$ is in white
- $J(f)$ is in black
- $J(f)$ is totally disconnected
- $\dim J(f) < 1$

Hausdorff dimension for transcendental entire functions

In general, for a transcendental entire function f:

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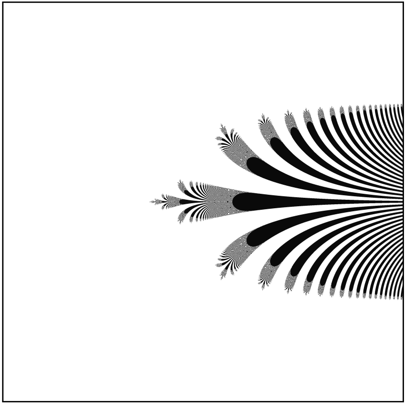
- 'Difficult' to find functions, f, for which $\dim J(f) = 1$.
- Bishop (2018) constructed a transcendental entire function f with $\dim J(f) = 1$.

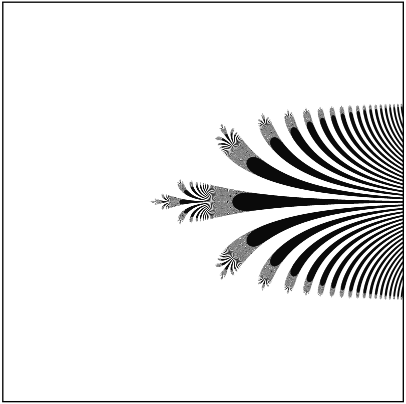
Hausdorff dimension for transcendental entire functions



$\frac{1}{4}\exp(z)$

- $I(f)$ is in black and is a Cantor bouquet of curves (without some endpoints)
- $J(f)$ is in black
- $J(f)$ is $I(f)$ along with all the endpoints
- $\dim J(f) = \dim I(f) = 2$ (McMullen, 1987)

Hausdorff dimension for transcendental entire functions



$\frac{1}{4}\exp(z)$

Karpińska's paradox

- The set of curves without the endpoints has dimension 1 (Karpińska, 1999).
- The set of endpoints has dimension 2.

Hausdorff dimension for transcendental entire functions

Theorem (Barański, Karpińska, and Zdunik, 2009)

The Hausdorff dimension of the set of points with bounded orbits in the Julia set of a meromorphic map with a logarithmic tract is greater than 1.

Wiman–Valiron theory (Power series)

- Let $f = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function.
- The main result of Wiman–Valiron theory gives how much of this power series is needed to obtain a good estimate on f near maximum modulus points.
- Used by Eremenko to show $I(f)$ is non-empty.

Wiman–Valiron theory (Power series)

- $M(r) = \max_{|z|=r} |f(z)|$ is the maximum modulus of f
- $\mu(r) = \max_{n \geq 0} |a_n| r^n$ is the maximum term
- $\nu(r) = \max_{n \geq 0} \{n : |a_n| r^n = \mu(r)\}$ is the central index
- A set $E \in [1, \infty)$ has finite logarithmic measure if $\int_E dt/t < \infty$.

Main result of Wiman–Valiron theory (Power series)

Theorem (Wiman, Valiron, Macintyre, and Hayman (1916-1974))

There exists a set E of finite logarithmic measure such that if $|z_r| = r \notin E$, if $|f(z_r)| = M(r)$, and if z is sufficiently close to z_r, then

$$f(z) \sim \left(\frac{z}{z_r}\right)^{\nu(r)} f(z_r)$$

as $r \to \infty$.

Wiman–Valiron theory (Without power series)

- $B(r) = \log M(r)$.
- $B(r)$ is a convex function of $\log r$, so

$$a(r) = \frac{dB(r)}{d \log r}$$

exists except, perhaps, for a countable set of values of r and is non-decreasing.

- Macintyre (1938) proved that

$$f(z) \sim \left(\frac{z}{z_r} \right)^{a(r)} f(z_r)$$

for $z \in D(z_r, r/(B(r)^{1/2+\varepsilon})$ if $\varepsilon > 0$.

Wiman–Valiron theory in direct tracts

- Let D be a direct tract of f.
- The subharmonic function $v(z) = \log|f(z)|/R$ if $z \in D$ and 0 if $z \notin D$.
- $B(r, v) = \max_{|z|=r} v(z)$, so $a(r, v) = \frac{dB(r,v)}{d\log r} = rB'(r, v)$.

Theorem (Bergweiler, Rippon, Stallard 2008)

Let D be a direct tract of f and let $\tau > \frac{1}{2}$. Let v be the associated subharmonic function and let $z_r \in D$ be a point satisfying $|z_r| = r$ and $v(z_r) = B(r, v)$. Then there exists a set $E \subset [1, \infty)$ of finite logarithmic measure such that if $r \in [1, \infty) \setminus E$, then $D(z_r, r/a(r, v)^\tau) \subset D$. Moreover,

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,v)} f(z_r), \quad \text{for } z \in D(z_r, r/a(r, v)^\tau),$$

as $r \to \infty$, $r \notin E$.

How large can the disc around z_r be chosen?

Wiman–Valiron discs

How large can the disc around z_r be chosen?

Theorem (Bergweiler, 2011)

- *Let* $\psi : [t_0, \infty) \to (0, \infty)$ *satisfy* $1 \leq \frac{t\psi'(t)}{\psi(t)} < 2$.
- *If*

$$\int_{t_0}^{\infty} \frac{dt}{\psi(t)} < \infty$$

and $r \notin E$ *is sufficiently large, then* $D(z_r, r/\sqrt{\psi(a(r,v))}) \subset D$.

- *However, if*

$$\int_{t_0}^{\infty} \frac{dt}{\psi(t)} = \infty$$

then there exists an entire function such that for r *sufficiently large and* $|z| = r$, $D(z, r/\sqrt{\psi(a(r,v))})$ *contains a zero of* f.

Wiman–Valiron discs

How large can the disc around z_r be chosen?

Theorem

Let f be a meromorphic function with a direct tract D with a simply connected direct tract, then for $1/2 > \tau > 0$ and for $r \in [1, \infty) \setminus E$, where E has finite logarithmic measure, there exists $D(z_r, r/a(r,v)^\tau) \in D$.

Wiman–Valiron discs

What is the estimate on these discs?

Theorem

There exists a set $E \in [1, \infty)$ *such that if, for* $\tau > 0$, *there exists a disc* $D(z_r, r/a(r,v)^\tau) \subset D$ *for* $r \notin E$ *sufficiently large, then there exists an analytic function* g *in* $D(z_r, r/a(r,v)^\tau)$ *such that*

$$\log f(z) = \log f(z_r) + a(r,v) \log \frac{z}{z_r} + g(z), \quad \text{for } z \in D(z_r, r/a(r,v)^\tau),$$

where

$$g(z) = \begin{cases} O(a(r,v)^{\xi(\tau)}) & \text{for } z \in D(z_r, r/a(r,v)^\tau) \text{ and } \tau < 1/2, \\ o(1) & \text{for } z \in D(z_r, r/a(r,v)^\tau) \text{ and } \tau > 1/2, \end{cases}$$

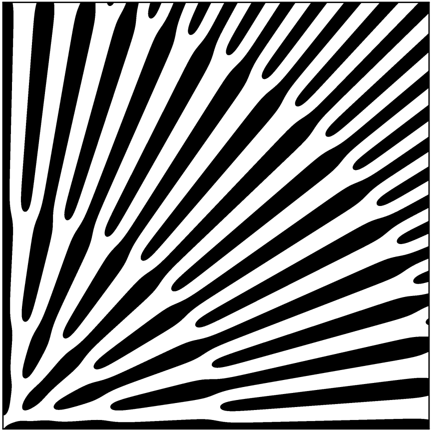
and $\xi(\tau) = \sqrt{1 - 2\tau}$ *as* $r \to \infty$, $r \notin E$.

Hausdorff dimension of Julia sets of meromorphic maps with simply connected direct tracts

Theorem

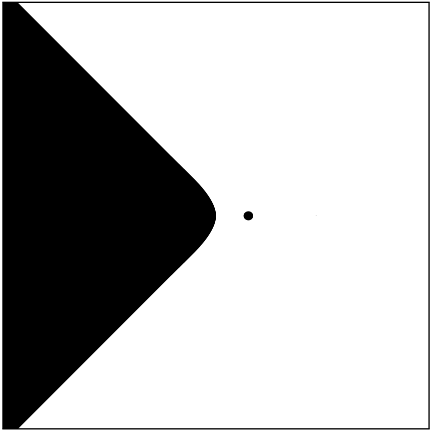
Let f be a transcendental meromorphic function with a simply connected direct tract D. Suppose that there exists $\lambda > 1$ such that for arbitrarily large r there exists an annulus $A(r/\lambda, \lambda r)$ containing no singular values of the restriction of f to D. Then $\dim J(f) \cap K(f) > 1$.

An example



$$\exp\left(-\sum_{k=1}^{\infty}\left(\frac{z}{2^k}\right)^{2^k}\right)$$

Another example



$\cos(z)\exp(z)$

Thank you for your attention!