Denjoy examples and their dimension

Rudimentary slides

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Bibliography

This talk is partially based on:

- *Diophantine classes, dimension and Denjoy maps* by B. Kra and J. Schmelling, Acta Arithmetica 105.4 (2002);
- Work in progress with M. Urbański;
- Some results from my PhD, *someday perhaps published...*

For those who have forgotten:

1) In the 1890's Poincaré proved that any orientation preserving circle homeomorphism $f\colon S^1 \to S^1$ defines a unique parameter $\alpha \in (0, 1]$ called *the rotation number*, and should this number be irrational, then the map f is semi-conjugate to a rotation by α.

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2) In the 1930's Denjoy proved that f is in fact 'fully' conjugate to the rotation, provided that f' is of bounded variation.
3) Moreover, he gave examples of $\mathcal{C}^1$ diffeomorphisms that are not conjugate to the rotation. Herman (in 1979) gave such examples for any $\mathcal{C}^{1+\delta}$, where $\delta \in (0, 1)$.

Definition

We will call any orientation preserving circle homeomorphism with an irrational rotation number that is not conjugate to the rotation a *Denjoy map*.

Definition

A set Ω is called *minimal* (for the homeomorphism f) if it is non-empty, compact, invariant and has no proper subset with these properties.
In other words, Ω is non-empty, $f(\Omega) = \Omega$ and each forward orbit of a point $x \in \Omega$ is dense in Ω.

Poincaré's result gives that if S^1 is minimal, then f is conjugate to the rotation.

Enter the sequence

Our intervals J_n will have lengths satisfying:

i) $\sum_{n\in\mathbb{Z}} \ell_n \leq 1$, *but in fact we will assume equal to 1.*

ii) $\lim_{n\to\pm\infty} \frac{\ln|\ell_n - \ell_{n+1}|}{\ln \ell_n} = 1+\delta$,

for some $\delta \in (0,1)$.

Model sequence

A model example of such a sequence is

$$\ell_n = c_\delta(|n|+1)^{-1/\delta}, \qquad \text{where } c_\delta^{-1} = \sum_{n\in\mathbb{Z}}(|n|+1)^{-1/\delta}$$

The metric

Set $\Omega_\alpha^\delta = S^1 \setminus \sum_{n\in\mathbb{Z}} J_n$.

Let $h\colon \Omega_\alpha^\delta \to S^1$ be the semi-conjugacy and remember that we assume $h(J_0) = 0$.

For x, y not in the orbit (by rotation) of 0 h^{-1} is well-defined and we have:

$$d(h^{-1}(x), h^{-1}(y)) = \sum_{n:n\alpha\in(x,y)} |J_n| = \sum_{n:n\alpha\in(x,y)} \ell_n.$$

And if x (and/or y) are in the orbit of zero, then $h^{-1}(x)$ consists of two points and we have to take the correct preimage (in such a way that the arc between $h^{-1}(x)$ and $h^{-1}(y)$ is the shortest possible.

Diophantine class

Definition

An irrational α has a Diophantine class $\nu > 0$, if

$$\inf_{p \in \mathbb{Z}} |q\alpha - p| \leq \frac{1}{q^{\mu}}$$

has infinitely many solution for $\mu < \nu$ and at most finitely many for $\mu > \nu$.

Remark

The golden ratio ϕ has class 1. (And the class cannot be lower).
The Liouville numbers have class $+\infty$.
The set of points of class ν has Hausdorff dimension $\frac{1}{\nu}$.

Denote by $[a_1, a_2, \ldots]$ the standard continued fraction expansion of α; and by q_n the denominators of the convergents (finite fractions). Recall that

$$q_{n+1} = a_n q_n + q_{n-1}$$

and

$$\frac{1}{q_n(a_n+2)} \leq \inf_{p \in \mathbb{Z}} |q_n \alpha - p| \leq \frac{1}{a_n q_n}.$$

So we may think of a number with Diophantine class ν as one satisfying a sequence

$$q_{n+1} \approx q_n^{\nu}.$$

The only theorem

Theorem (Kra–Schmelling)

Assume $\delta \in (0,1)$ *and* α *has Diophantine class* ν. *Then an orientation preserving* $\mathcal{C}^{1+\delta}$ *diffeomorphism of the circle with rotation number* α *and the minimal set* Ω_α^δ *satisfies*

$$\dim_B \Omega_\alpha^\delta \geq \delta \qquad \text{and} \qquad \dim_H \Omega_\alpha^\delta \geq \frac{\delta}{\nu},$$

and taking the model sequence gives equalities.

Box dimension - upper bound for model case

We want to prove $\limsup_{r \to +\infty} \frac{\log N(r)}{-\log(r)} \leq \delta$ for a reasonably dense sequence of r's.

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Fix $n \in \mathbb{N}$ and consider $\mathcal{J}_n = S^1 \setminus \Big(\bigcup_{-n\leq k\leq n} J_k \Big)$. This is a sum of $2n+1$ disjoint intervals of total length $\Big(1 - \sum_{-n\leq k\leq n} \ell_k \Big) \approx n^{1-1/\delta}$.

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This set may be covered *almost trivially* by intervals of average length, i.e. $n^{1-1/\delta}/(2n+1)$. How many such intervals do we need?

Box dimension - upper bound, cont.

We need 1 to cover every set of length shorter than the average (at most $2n+1$ sets). And at most twice the total length for the longer sets. This yields

$$N(n^{1-1/\delta}/(2n+1)) \leq 6n+3.$$

Box dimension - upper bound, cont.

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Plugging this into the dimension formula gives

$$\frac{\log N(r)}{-\log(r)} \leq \frac{\log(6n+3)}{-\log\left(n^{1-1/\delta}/(2n+1)\right)} \approx \delta$$

Box dimension - lower bound for model case

Fix $m, n \in \mathbb{N}$ and observe the set $Z = S^1 \setminus \left(\bigcup_{-m \le k \le n} J_k \right)$. The set

$$\{l\alpha : -m - n - 1 \le l \le m + n + 1\}$$

contains at least one point in each of the $m + n + 1$ intervals of Z. This means that the length of any of those intervals is bounded from below by $c_\delta(2n + 2m + 3)^{-1/\delta}$ and the distance of any two intervals is at least $c_\delta(\max(n, m))^{-1/\delta}$.

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So to cover Ω by intervals of length $c_\delta (2n + 2m + 3)^{-1/\delta}$ we must use at least $m + n + 1$ intervals. This proves the lower bound for model case.

For the general case we need to additionally prove that the assumptions on ℓ_n give that $\ell_n > n^{-1/\theta}$ for all $0 < \theta < \delta$.

Here we will assume that (X, d) is a metric space and $T\colon X \to X$ a Borel measurable map; μ is a T–invariant, ergodic, probability, Borel measure on X.

Theorem

With the assumptions on the dynamical system as above, for any $\beta > 0$ and for μ – almost every $x \in X$ we have

$$\liminf_{n\to\infty} n^{1/\beta} d(T^n(x), x) \le g(x)^{1/\beta}, \text{ where } g(x) = \limsup_{r\to 0} \frac{H_\beta(B_x(r))}{\mu(B_x(r))}.$$

Remark

Note that $g(x)$ may be equal to 0 or $+\infty$. The statement still holds.