Volume preserving homeomorphisms of the cube

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Introduction

We will deal with such objects:

- λ the standard Lebesgue measure (volume)
- $|\cdot|$ the Euclidean distance
- $I^n = [0,1]^n$ the unit cube
- $\mathcal{M} = \mathcal{M}[I^n, \lambda]$ the space of all volume preserving homeomorphisms of the cube
- $\mathcal{G} = \mathcal{G}[I^n, \lambda]$ the space of all volume preserving bimeasurable bijections of the cube (i.e. automorphisms)
- ergodic autmorphisms – having only trivial invariant sets (the empty set, the whole cube)

The talk is based on the book by Alpern and Prasad, *Typical dynamics of measure preserving homeomorphisms*. The book generalizes these ideas for homeo and automorphisms defined on manifolds and preserving any sufficiently nice measure.

A very important reminder

Definition

We call a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ volume preserving if for any measurable set E we have

$$\lambda(f^{-1}(E)) = \lambda(E).$$

Now when we assume a bijection f to be volume preserving, it means that both f and f^{-1} are volume preserving and we can either check that above condition or $\lambda(A) = \lambda(f(A))$, which we get if we take $A = f^{-1}(E)$.

Possible topologies

We can endow our spaces with (as usually) either strong (which here will be called uniform) or weak topology.

Definition (Uniform and weak topology)

The **uniform** topology on $\mathcal{G}$ is given by the metric $d(f,g) = \operatorname{ess\,sup}_{x \in I^n} |f(x) - g(x)| + |f^{-1}(x) - g^{-1}(x)|$.

The **weak** topology on $\mathcal{G}$ is given by the metric $\rho(f,g) = \inf_{\delta \geq 0} \left\{ \lambda\{x : |f(x) - g(x)| \geq \delta\} < \delta \right\}$. The convergence of a sequence of autmorphisms g_i to g in metric ρ is equivalent to saying that for all measurable sets $A \subset I^n$ we have $\lambda(g_i(A) \triangle g(A)) \to 0$, where $\triangle$ stands for symmetric difference between sets.

The space $\mathcal{G}$ of automorphisms is complete with respect to any of these topologies.

Dyadic permutations

Generally we divide the unit cube into small ones and swap them (in a, naturally, discontinuous manner).

A **cube of order** m is a product of intervals of the form: $[\frac{k}{2^m}, \frac{k+1}{2^m}]$. So, there are 2^{nm} such cubes and each has side length equal to 2^{-m}. Let us denote with $\mathcal{D}_m = \{\alpha_i : i = 1, 2, ..., 2^{mn}\}$ the set of all cubes of order m.

Now let us define a map $P : \mathcal{D}_m \to \mathcal{D}_m$ and think:

- ? When P is a permuation,
- ? when we could call it ergodic?

Dyadic permutations

- The map P is a permutation iff it is a bijection.

  That should be clear.

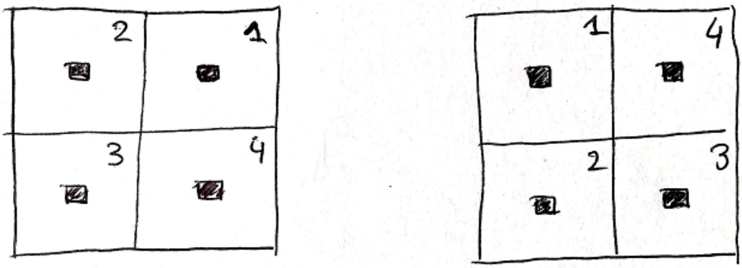
- It is ergodic iff it is a cyclic permutation. Why?

  Every permutation can be decomposed into a product of cycles and every cycle corresponds to an invariant set for P. Therefore only when the permutation is a single cycle, the only invariant sets are the empty set and the whole space $\mathcal{D}_m$.

Now look at P as an automorphism of the cube - the **good news** is that such mappings are great for approximating measure preserving homeomorphisms!

Watch out for this *ergodic*

The fact that we call the permutation ergodic does not mean that it is such viewed as a function on the unit cube.



Approximation

Theorem (P. Lax)

Let h be a a volume preserving homeomorphism of I^n and $\varepsilon > 0$. Then there exists a dyadic permutation P such that $d(P, h) < \varepsilon$.

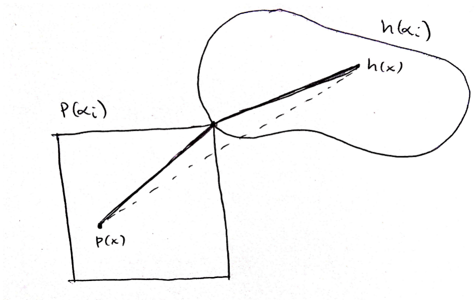
That means that dyadic permutations are dense in $\mathcal{M}$ in the uniform topology!

Proof (It is really nice!)
Let us recall the notation: $\mathcal{D}_m = \{\alpha_i : i = 1, 2, ..., N\}$ is the set of all cubes of order m with $N = 2^{mn}$. We will choose m later.
Firstly, we show that it suffices to find a dyadic permutation P such that for all $i = 1, 2, ..., N$ we have

$$P(\alpha_i) \cap h(\alpha_i) \neq \emptyset. \quad (1)$$

If condition (1) is satisfied, then for all $x \in I^n$ we have that

$$|P(x) - h(x)| \leq \text{diam}(\alpha_1) + \max_{1 \leq i \leq N} \text{diam}(h(\alpha_i)).$$



Let us choose m_1 such that $\text{diam}(\alpha_1) = \frac{\sqrt{2}}{2^{m_1}} < \varepsilon/2$.

The second term can also be made arbitrarily small:
h continuous on a compact set $\rightarrow$ h uniformly continuous. Indeed,

$$\forall\, \varepsilon > 0 \quad \exists\, m_2 \quad \forall\, |x - y| < \frac{\sqrt{2}}{2^{m_2}} \quad |h(x) - h(y)| < \varepsilon/2.$$

Now let us take the final m to be the smaller one, i.e
$m = \min(m_1, m_2)$. Hence we get

$$|P(x) - h(x)| \leq \text{diam}(\alpha_1) + \max_{1 \leq i \leq N} \text{diam}(h(\alpha_i)) < \varepsilon$$

So now we have to prove that we can find P of a chosen order m such that condition (1) is satisfied, that is for any cube α_i we have $P(\alpha_i) \cap h(\alpha_i) \neq \emptyset$.

We need some help

Lemma (Hall's Marriage Theorem)

There are N girls and N boys. We assume that if a girls ***likes*** *a boy, he would not turn her down. If any $k \leq N$ girls like, in total, k boys, then it is possible to pair everyone up.*

Here we say that a cube α_i **likes** α_j if $\alpha_j \cap h(\alpha_i) \neq \emptyset$.
Take any k cubes $\rightarrow$ their image has the volume of k cubes $\rightarrow$ their image must intersect at least k cubes.

Therefore, any k cubes like, in total, at least k cubes $\rightarrow$ the condition from the Marriage Theorem is satisfied!

We can pair up the cubes – for any cube α_i we can find a cube α_j that the former likes and set $P(\alpha_i) = \alpha_j$ so that $P(\alpha_i) \cap h(\alpha_i) \neq \emptyset$. □

Cyclic dyadic permutations

Theorem

Let h be a a volume preserving homeomorphism of I^n and $\varepsilon > 0$. Then there exists a **cyclic** dyadic permutation P such that $d(P, h) < \varepsilon$.

The proof goes exactly like the previous one, requires just one additional combinatorial fact:

Lemma

Given any permutation ρ of $J = \{1, 2, ..., N\}$ there is a cyclic permuation σ of J with $|\rho(j) - \sigma(j)| \leq 2$ for all $j \in J$.

Some remarks

We can even strengthen these result to the following version, which we will later use.

Theorem (Cyclic approximation)

Let h be a a volume preserving homeomorphism of I^n and $\varepsilon > 0$. Then there exists a **cyclic** dyadic permutation P of order m such that $d(P, h) + \frac{\sqrt{2}}{2^m} < \varepsilon$.

- We can naturally take n-fold products of $[\frac{i}{k^m}, \frac{i+1}{k^m}]$ for any k.
- Once we find a threshold M of the order of the cubes, we can find an approximating permutation for any $m \geq M$.
- Intriguing - why should we approximate something continuous with something that is highly not?
- In particular, these theorems show that $\mathcal{M}$ is not open in $\mathcal{G}$ in uniform topology.

Measure preserving Lusin Theorem

We equip our automorphisms with the norm
$||g|| = \operatorname{ess\,sup} |g(x) - x| = d(g, id)$, where *id* is the identity map.

Theorem (Measure preserving Lusin Theorem)

Let g be an automorphism of I^n with the norm $||g|| < \varepsilon$. Then for any $\delta > 0$ there exists h, a volume preserving homeomorphism of I^n, satisfying

1. $||h|| < \varepsilon$
2. h is identity on the boundary of I^n
3. $\lambda \{x : |g(x) - h(x)| \geq \delta\} < \delta$.

Property 1 (norm preservation) is a key problem here but is crucial to applications. An even stronger result is true that $\lambda \{x : g(x) \neq h(x)\} < \delta$ but is less useful.

What the theorem actually says

That $\mathcal{M}$ is dense in $\mathcal{G}$ with respect to the weak topology. But it preserves the uniform norm!

The **weak metric**:
$\rho(f,g) = \inf_{\delta \geq 0} \{\lambda\{x : |f(x) - g(x)| \geq \delta\} < \delta\}$.

Theorem (Measure preserving Lusin Theorem)

Let g be an automorphism of I^n with the norm $||g|| < \varepsilon$. Then for any $\delta > 0$ there exists h, a volume preserving homeomorphism of I^n, satisfying

1. $||h|| < \varepsilon$
2. h is identity on the boundary of I^n
3. $\lambda\{x : |g(x) - h(x)| \geq \delta\} < \delta$
   $\rightarrow \rho(f,g) \leq \varepsilon$, i.e any ball centered at g has a nonempty intersection with $\mathcal{M}$.

An important corollary

Theorem

Let $\mathcal{V}$ be a G_δ subset of $\mathcal{G}$ in the weak topology. Assume that $\mathcal{M} \subset \bar{\mathcal{V}}$, where the closure is taken wrt the **uniform** topoology. Then $\mathcal{V} \cap \mathcal{M}$ is a dense G_δ subset of $\mathcal{M}$ in the **uniform** topology.

For a dense G_δ set there even is a special name – **generic**.

Lemma

Both metrics are right-invariant, that is for any $f \in \mathcal{G}$ we have $d(f, g) = d(id, gf^{-1})$ and $\rho(f, g) = \rho(id, gf^{-1})$.

Quick proof. For the uniform metric it is merely the fact that f is bijective. For the weak metric one observes that since f is an automorphism, then

$$\lambda \{x : |f(x) - g(x)| \geq \delta\} = \lambda \{f(x) : |f(x) - g(x)| \geq \delta\},$$

which transalates into $\lambda \{y : |y - g(f^{-1}(y))| \geq \delta\}$ and proves what we need. □

Proof of the corollary

We prove the crucial part where G_δ is replaced with **open**.

Theorem

Let $\mathcal{V}$ be an open subset of $\mathcal{G}$ in the weak topology. Assume that $\mathcal{M} \subset \bar{\mathcal{V}}$, where the closure is taken wrt the **uniform** topoology. Then $\mathcal{V} \cap \mathcal{M}$ is a dense open subset of $\mathcal{M}$ in the **uniform** topology.

Proof. Uniform topology is finer, so $\mathcal{V}$ is also open in $\mathcal{G}$ in uniform topology and hence $\mathcal{V} \cap \mathcal{M}$ is open in $\mathcal{M}$ with the induced uniform topology.

Now we have to prove that for any $f \in \mathcal{M}$ and $\varepsilon > 0$ an open ball $\mathcal{B} = B_d(f, \varepsilon)$ has a nonempty intersection with $\mathcal{V} \cap \mathcal{M}$.
$\rightarrow$ $\mathcal{M} \subset \bar{\mathcal{V}}^d$, so $\mathcal{B}$ has a nonempty intersection with $\mathcal{V}$.
$\rightarrow$ There exists $g_0 \in \mathcal{V}$ with $d(f, g_0) < \varepsilon$.
$\rightarrow$ Since d is right-invariant, $d(id, g_0 f^{-1}) < \varepsilon$, so $||g_0 f^{-1}|| < \varepsilon$.

Since $\mathcal{V}$ is weak open, so is $\mathcal{V}f^{-1} = \{gf^{-1} : \quad g \in \mathcal{V}\}$.
$\rightarrow$ There exists a ball $B_\rho(g_0 f^{-1}, \eta) \subset \mathcal{V}f^{-1}$.

Measure preserving Lusin Theorem says that there exists $h \in \mathcal{M}$ with $||h|| < \varepsilon$ and $h \in B_\rho(g_0 f^{-1}, \eta)$
$\rightarrow$ $h \in \mathcal{V}f^{-1}$ and so $hf \in \mathcal{V}$ and $d(f, hf) < \varepsilon$.

Since both h and f were homeomorphisms, so is hf and so hf belongs to the intersection of M, $\mathcal{V}$ and $\mathcal{B}$. □

Was the uniform norm preservation important...?

Ergodic measure preserving homeomorphisms

Theorem

The ergodic homemorphisms form a dense G_δ subset of the volume preserving homeomorphisms of I^n in the uniform topology. That is: ergodicity is generic for volume preserving homeomorphisms.

Proof.

Basically we would like to apply the previous Corollary for $\mathcal{V}$ – ergodic automorphisms.

Lemma (Halmos)

The set $\mathcal{V}$ of ergodic automorphisms is a G_δ set in $\mathcal{G}$ in the weak topology.

Now we want to prove that $\mathcal{M} \subset \bar{\mathcal{V}}^d$, i.e. for any $h \in \mathcal{M}$ and $\varepsilon > 0$ there is an ergodic automorphism $f \in \mathcal{V}$ with $d(f, h) < \varepsilon$.

Construction of f

By our Cyclic Approximation Theorem we find a permutation P of order m such that $d(h, P) + \frac{\sqrt{2}}{2^m} < \varepsilon$.

Let us number all the cubes α_i of order m so that: $P(\alpha_i) = \alpha_{i+1}$, $i = 1, 2, ..., N-1$ and $P(\alpha_N) = P(\alpha_1)$. Naturally, $\text{diam}(\alpha_1) = \frac{\sqrt{2}}{2^m}$.

Now we take $\tilde{f}$ to be an ergodic automorphism from α_1 to α_1 and identity elsewhere (let's believe the authors that it is *easy*).

Let $f = \tilde{f}P$, then $d(f, P) < \text{diam}(\alpha_1)$. Hence

$$d(f, h) \leq d(f, P) + d(P, h) < \frac{\sqrt{2}}{2^m} + d(h, P) < \varepsilon$$

.

Is f ergodic?

1. Let us assume that there exists a nontrivial invariant set S.
2. We call $S_i = S \cap \alpha_i$ and claim that S must intersect all cubes. Indeed, WLOG let us assume it does not intersect α_1. Then it would not intersect α_N and then α_{N-1}... And would be empty.
3. For $x \in \alpha_1$, $f^N(x) = \tilde{f} \circ P \circ ... \circ \tilde{f} \circ P(x) = \tilde{f}(x)$.
4. We have $f^N(S_1) = f^N(S \cap \alpha_1) = f^N(S) \cap f^N(\alpha_1) = S \cap \alpha_1 = S_1$
5. On the other hand, $f^N(S_1) = \tilde{f}(S_1)$.
6. So we have $\tilde{f}(S_1) = S_1$...
7. Which contradicts the fact that $\tilde{f}$ is an ergodic automorphism of α_1.
8. Great, f is indeed ergodic.

The end

We found an ergodic automorphism f such that $d(f, h) < \varepsilon$, which is exactly what we needed.

$\square$

Constructions like the one above (permutation and something applied to a few cubes) seem to be popular in ergodic theory and are referred to as *skyscraper constructions*.

Any questions? Thank you for attention!