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Möbius transformations and Furstenberg's theorem

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Theorem 1 - discrete version

Theorem

Let $F = \{f_1, \dots, f_N; p_1, \dots, p_N\}$, $p_j > 0, \sum_1^N p_j = 1$ *be an IFS (iterated function system) with probabilities of Möbius transformations* f_i *mapping the upper half-plane* $\mathbb{H}$ *onto itself, and having no common invariant (hyperbolic) line in* $\mathbb{H}$. *Then, for any initial point* $Z_0 \in \mathbb{H}$, *the orbit* $Z_n = Z_n(Z_0) = f_n \circ f_{n-1} \circ \cdots \circ f_1(Z_0)$, $n \geq 1$, *tends to* $\overline{\mathbb{R}}$ *almost surely, as* $n \to \infty$.

Theorem 1

Theorem

Let μ be a probability measure on the set of Möbius transformations mapping the upper half-plane $\mathbb{H} = \{Im\, z > 0\}$ onto itself. Assume that the transformations in the support of μ have no common fixed point in $\overline{\mathbb{H}}$ and no common invariant (hyperbolic) line in $\mathbb{H}$. Let $\{F_n\}$ be iid random variables with distribution μ. Then, for any initial point $Z_0 \in \mathbb{H}$, the orbit $\{Z_n\}_0^\infty$ tends to $\overline{\mathbb{R}}$ almost surely. That is, for arbitrarily fixed compact subset $K \subset \mathbb{H}$ and for almost all orbits $\{Z_n\}_{n \geq 1}$, only a finite number of points of the orbit belong to K.

Furstenberg's theorem

Theorem

Let μ be a probability measure on $SL(2, \mathbb{R})$, such that the following holds (if G_μ is the smallest closed subgroup of $SL(2, \mathbb{R})$ which contains the support of μ)

1. *G_μ is not compact;*
2. *there does not exist a subset L of $\mathbb{R}^2$ which is a finite union of one-dimensional subspaces, such that $M(L) = L$ for any M in G_μ.*

Then, with probability 1, the norms $\parallel Y_n \dots Y_1 x \parallel$ grow exponentially as $n \to \infty$, for all $x \in \mathbb{R}^2 \setminus \{0\}$, where $\{Y_n\}$ are iid random variables with values in $SL(2, \mathbb{R})$ and distribution μ.

Theorem 2

Theorem

Let μ be a probability measure on $SL(2, \mathbb{R})$. Assume that the matrices in suppμ have no common invariant elipse, and no common invariant set of type $l_1 \cup l_2$, with lines l_1, l_2 (not necessarily different) passing through the origin. Then, with probability 1, the norms $\| Y_n \dots Y_1 x \|$ grow exponentially, as $n \to \infty$, for all $x \in \mathbb{R}^2 \setminus \{0\}$, where $\{Y_n\}$ are iid random variables with values in $SL(2, \mathbb{R})$ and distribution μ.

Lemma 1

Lemma

Let $\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\}_A$ *be a subset of* $SL(2, \mathbb{R})$ *and let* $\{f_A(z) = \frac{az+b}{cz+d}\}_A$ *be the associated Möbius transformations. Then, for any fixed* $z \in \mathbb{H}$, $\{f_A(z)\}_A$ *is an unbounded set in* $\mathbb{H}$ *(in the hyperbolic sense) if and only if the norms* $\{\| A \|\}_A$ *are unbounded (in* $\mathbb{R}$ *with the ordinary Euclidean metric).*

Proof of Lemma 1 - part 1/3

Möbius transformations preserve hyperbolic distances. Hence, instead of considering general point $z \in \mathbb{H}$, we can consider the point $z = i$. Then

$$f_A(i) = \frac{ai + b}{ci + d} = \frac{(bd + ac) + i(ad - bc)}{c^2 + d^2}$$

. Since $1 = detA = ad - bc$, we get

$$f_A(i) = \frac{bd + ac}{c^2 + d^2} + i\frac{1}{c^2 + d^2}$$

.

Proof of Lemma 1 - part 2/3

Now let's assume that $\{f_A(i)\}_A$ is an unbounded set in $\mathbb{H}$ in hyperbolic sense. This gives us three following alternatives:

1. $\{c^2 + d^2\}_A$ is unbounded (in $\mathbb{R}$),
2. $\{\frac{bd+ac}{c^2+d^2}\}_A$ is unbounded,
3. $\{c^2 + d^2\}_A$ contains element arbitrarily close to 0.

All these alternatives together with $detA = ad - bc = 1$ assure us that $\{|a| + |b| + |c| + |d|\}_A = \{\| A \|\}_A$ is unbounded in $\mathbb{R}$.

Proof of Lemma 1 - part 3/3

Now, conversely, $\{\| A \|\}_A$ is unbounded.
If, in addition, $\{|c| + |d|\}_A$ is unbounded, $\{f_A(i)\}_A$ has element arbitrarily close to real line.
If $\{|c| + |d|\}_A$ is bounded, $\{|a| + |b|\}_A$ is unbounded. In this case

$$|f_A(i)| = |\frac{ai+b}{ci+d}| = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$

and $|f_A(i)|_A$ is unbounded in $\mathbb{R}$. In either case the set $\{f_A(i)\}_A$ is an unbounded subset of $\mathbb{H}$.
Lemma 1 is proved.

Lemma 2

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Lemma

In the notation of Lemma 1, the matrices $\{A\}_A$ *have a common invariant ellipse if and only if the associated Möbius transformations* $\{f_A\}_A$ *have a common fixed point in* $\mathbb{H}$.

Proof of Lemma 2 - part 1/2

Let the set of Möbius transformations $\{f_A\}_A$ have a common fixed point $w \in \mathbb{H}$. Choose $B \in SL(2, \mathbb{R})$, such that for the associated Möbius transformation f_B we have $f_B(i) = w$. Then i is a common fixed point of the conjugate system $\{f_B^{-1} \circ f_A \circ f_B\}_A$.

Proof of Lemma 2 - part 2/2

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We should also note that $f_B^{-1} \circ f_A \circ f_B = f_{B^{-1}AB}$ and i is a fixed point of $f(z) = \frac{az+b}{cz+d}$ iff $a = d$ and $b = -c$.
If, in addition, $det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a rotation around the origin.
This shows that $\{B^{-1}AB\}_A$ have a common invariant circle centered at the origin in $\mathbb{R}^2$. Then, $\{A\}_A$ has a common invariant ellipse which is the image of the unit circle under B.
The converse is proved in the same manner.
Hence, Lemma 2 is proved.

Lemma 3

Lemma

Let μ be a probability measure on the set of Möbius transformations mapping the upper half-plane $\mathbb{H}$ onto itself. Assume that the transformations in the support of μ have no common fixed point in $\mathbb{H}$. Then for any initial fixed point $z \in \mathbb{H}$ the set $\{f(z)\}, f \in G_\mu$, is unbounded (in the hyperbolic sense), where G_μ denotes the smallest subgroup of Möbius transformations containing the support of μ.

Proof: too long.

Proposition 1

Proposition

Let μ be a probability measure on $SL(2, \mathbb{R})$ and let G_μ be the smallest closed subgroup of $SL(2, \mathbb{R})$ which contains the support of μ. Then G_μ is compact iff all the matrices in G_μ (or equivalently in $\mathrm{supp}\mu$) have a common invariant ellipse.

Proof of Proposition 1 - part 1/2

In one direction the assertion is evident: if all matrices in G_μ have a common invariant ellipse, then the group G_μ is compact. Now assume that the matrices in G_μ have no common invariant ellipse. Due to Lemma 2, this is equivalent to the fact that the associated Möbius transformations f_A, where $A \in G_\mu$, have no common fixed point in $\mathbb{H}$.

Proof of Proposition 1 - part 2/2

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Next we use Lemma 3, which says that in this case the set $\{f_A(z)\}_A$ is unbounded (in hyperbolic sense) in $\mathbb{H}$, for any fixed $z \in \mathbb{H}$. Finally, Lemma 1 shows that $\{\| A \|\}_A$ is unbounded and, hence, that G_μ is non-compact, which finishes the proof. Proposition 1 is proved.

Proposition 2

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Proposition

Assume that the group G_μ in Proposition 1 is non-compact and that there exists a subset $L = \cup l_i$ of $\mathbb{R}^2$ of a finite union of n different one-dimensional subspaces l_i, $i = 1, \ldots, n$, such that $A(L) = L$ for all $A \in G_\mu$. Then $n \leq 2$.

Proof of Proposition 2 - part 1/2

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Let Δ be a unit disc in $\mathbb{R}^2$. The lines l_i, where $i = 1, \ldots, n$, intersect at the origin and divide Δ into $2n$ parts. Let's denote them by D_i, where $i = 1, \ldots, i = 2n$.
Let $A \in G_\mu$. $A(\Delta)$ is an ellipse centered at the origin. We also know that $A(L) = L$. The set L divides $A(\Delta)$ into $2n$ parts, which we'll denote as S_i, for $i = 1, \ldots, 2n$.
Now, Proposition 2 will be proved by showing that if $n > 2$, at least one S_i will have an arbitrarily small area, provided that the norm of A is sufficiently large.

Proof of Proposition 2 - part 2/2

First we assume that $n = 3$ ($L = l_1 \cup l_2 \cup l_3$). If $\parallel A \parallel$ is large, $A(\Delta)$ is a long and thin origin-centered ellipse with unit area. Let's denote by P_1, P_2 points of the ellipse having maximal distance from each other, and denote by l a straight line passing through these points. Then l passes through the origin as well. Let l_1 form the minimal angle with l among all l_i.
Then the parts S_i of $A(\Delta)$ lying between the lines l_2 and l_3 and not containing points P_i will have arbitrarily small area if $A(\Delta)$ is sufficiently long and thin.
The proof proceeds in similar fashion for $n > 3$. Proposition 2 is proved.

Theorem 2

Theorem

Let μ be a probability measure on $SL(2, \mathbb{R})$. Assume that the matrices in suppμ have no common invariant elipse, and no common invariant set of type $l_1 \cup l_2$, with lines l_1, l_2 (not necessarily different) passing through the origin. Then, with probability 1, the norms $\| Y_n \dots Y_1 x \|$ grow exponentially, as $n \to \infty$, for all $x \in \mathbb{R}^2 \setminus \{0\}$, where $\{Y_n\}$ are iid random variables with values in $SL(2, \mathbb{R})$ and distribution μ.

Proof of Theorem 2

Furstenberg's theorem and Propositions 1,2 are sufficient to prove Theorem 2.
Let's assume that the matrices in suppμ have no common invariant elipse (1) and no common invariant set of type $l_1 \cup l_2$, with lines l_1, l_2 (not necessarily different) passing through the origin (2).
Due to Proposition 1 and (1), G_μ is non-compact. (3)
Due to Proposition 2, (2) and (3), there does not exist a subset L of $\mathbb{R}^2$ which is a finite union of one-dimensional subspaces, such that $M(L) = L$ for any $M \in G_\mu$.
Now, due to Furstenberg's theorem, Theorem 2 is proved.

Theorem 1

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Theorem

Let μ be a probability measure on the set of Möbius transformations mapping the upper half-plane $\mathbb{H} = \{Im\, z > 0\}$ onto itself. Assume that the transformations in the support of μ have no common fixed point in $\overline{\mathbb{H}}$ and no common invariant (hyperbolic) line in $\mathbb{H}$. Let $\{F_n\}$ be iid random variables with distribution μ. Then, for any initial point $Z_0 \in \mathbb{H}$, the orbit $\{Z\}_0^\infty$ tends to $\overline{\mathbb{R}}$ almost surely. That is, for arbitrarily fixed compact subset $K \subset \mathbb{H}$ and for almost all orbits $\{Z_n\}_{n\geq 1}$, only a finite number of points of the orbit belong to K.

Proof of Theorem 1 - part 1/4

By A_f we will denote matrices corresponding to Möbius transformations $f \in supp\mu$ in Theorem 1. We assume that $A_f \in SL(2, \mathbb{R})$. Our first goal is to show that these matrices satisfy all assumptions of Theorem 2:

1. no common invariant ellipse,
2. no common invariant line l passing through the origin,
3. no common invariant set $l_1 \cup l_2$, where $l_1, l_2 (l_1 \neq l_2)$ are lines in $\mathbb{R}^2$ passing through the origin.

Proof of Theorem 1 - part 2/4

The existence of common invariant ellipse is excluded by Lemma 2, since otherwise the functions $f \in supp\mu$ would have common invariant point in $\mathbb{H}$.
Now, if the matrices A_f have a common invariant line l passing through the origin, then we can assume that l is the x-axis (otherwise we could consider a conjugate system). In this case any matrix A in $\{A_f\}$ has a form $A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. Then, the corresponding Möbius transformations are linear functions, and, consequently, have fixed point $z = \infty$ on $\overline{\mathbb{H}}$. But this contradicts the assumption in Theorem 1.

Proof of Theorem 1 - part 3/4

Now, let's assume that the matrices A_f have a common invariant $l_1 \cup l_2 (l_1 \neq l_2)$. We can make additional assumption that these lines are x- and y-axes (due to possibility of conjugation, as before). In such case, any matrix A i $\{A_f\}$ has one of following forms

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix}.$$

The first matrix corresponds to the situation when each line l_i is separately invariant under A. The second matrix corresponds to the situation when l_i change places under transformation A. In either case the corresponding Möbius transformations $(f(z) = a^2 z, f(z) = -b^2 z)$ have a common invariant hyperbolic line in $\mathbb{H}$ - $\{Rez = 0, Imz > 0\}$. This contradicts the assumptions of Theorem 1.
Hence, A_f satisfy all the conditions of Theorem 2.

Proof of Theorem 1 - part 4/4

Since all conditions of Theorem 2 are satisfied, we can use it.
Now we know that, with probability 1, $\| Y_n \dots Y_1 \| \to \infty$, as $n \to \infty$. Reminder: Y_i are iid random variables in $SL(2, \mathbb{R})$ and distribution μ.
Then Lemma 1 gives that $Z_n(Z_0) = F_n \circ \cdots \circ F_1(Z_0)$ tends to $\overline{\mathbb{R}}$, as $n \to \infty$, almost surely.
Therefore, Theorem 1 is proved.

Corollary 1

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Corollary

If the Möbius transformations in Theorem 1*, mapping* $\mathbb{H}$ *onto* $\mathbb{H}$*, have no common fixed point in* $\mathbb{H}$*, and in addition, no common* 2*-periodic point on* $\overline{\mathbb{R}}$*, then for any initial point* $Z_0 \in \mathbb{H}$*, the orbit* $\{Z_n\}_0^\infty$ *tends to* $\overline{\mathbb{R}}$ *almost surely.*

Corollary 2

Corollary

If the system in Theorem 1*, of Möbius transformations of* $\mathbb{H}$ *onto* $\mathbb{H}$*, having no common fixed point in* $\mathbb{H}$*, contains at least one Möbius transformation* $\frac{az+b}{cz+d}$*, whose matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has an eigenvalue* $\lambda = \alpha + i\beta$*, with* $\alpha\beta \neq 0$*, then the random orbit* $\{Z_n\}_0^\infty$ *converges to* $\overline{\mathbb{R}}$ *almost surely.*

The end

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Thank you!