Continuity of Hausdorff measure

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- Continuity of the Hausdorff Measure of Continued Fractions and Countable Alphabet Iterated Function Systems
  by M. Urbański and A. Zdunik

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

- We can think of an x as $x = (a1, a2, ...)$, $a_i \in \mathbb{N}$ - its continued fraction representation.
- We will be interested in $x \in [0, 1]$ such that $a_i \leq n$ for each i, where $n \in \mathbb{N}$ fixed.
- Set of such x we will denote as J_n.

Setup
Gauss map

$$G(x) = \frac{1}{x} - n \textit{ if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

$$g_n(x) = \frac{1}{n+x}$$

- The collection $\mathcal{G} := \{g_n\}_{n=1}^{\infty}$ forms conformal IFS and is called Gauss system.
- Let $g_\omega = g_{\omega_1} \circ g_{\omega_2} \circ \cdots \circ g_{\omega_n}$ for every $\omega \in \bigcup_{n \geq 1} \mathbb{N}^n$.

$$G^{|\omega|} \circ g_\omega|_{[0,1)} = Id|_{[0,1)}$$

- Our J_n can be described as $J_n = \bigcap_{k=1}^{\infty} \bigcup_{\omega \in \{1,\dots,n\}^k} g_\omega([0,1))$
- Let us denote by h_n Hausdorff measure of J_n

Hausdorff dimension and measure

t-dimensional (outer) Hausdorff measure

$$H_t(A) := \lim_{\delta \to 0} inf \left\{ \sum_{n=1}^{\infty} diam^t(U_n) : \bigcup_{n=1}^{\infty} U_n \supset A, diam(U_n) \leq \delta, \forall n \geq 1 \right\}$$

Hausdorff dimension

$$h(A) = inf\{t > 0 : H_t(A) = t\}$$

Theorem [D. Hensley]

$$\lim_{n\to\infty} n(1-h_n) = \frac{6}{\pi^2}$$

Theorem [M. Urbański, A. Zdunik]

$$\lim_{n\to\infty} H_{h_n}(J_n) = 1 = H_1(J)$$

where J is the set of all irrational numbers in the interval $[0,1]$

Hausdorff density theorems

Let X be a metric space, with $h(X) = h$, such that $H_h(X) \leq +\infty$. Then for H_h -a.e. $x \in X$

$$\lim_{r \to \infty} sup \left\{ \frac{H_h(F)}{diam^h(F)} : x \in F, \overline{F} = F, diam(F) \leq r \right\}$$

Theorem

Let X be a metric space if $0 < H_h(X) < +\infty$, then for H_h^1-a.e. $x \in X$

$$H_h(X) = \lim_{r \to 0} inf \left\{ \frac{diam^h(F)}{H_h^1(F)} : x \in F, \overline{F} = F, diam(F) \leq r \right\}$$

where H_h^1 is normed h-dimensional Hausdorff measure.

Corollary

If X is a subset of an interval $\Delta \subset \mathbb{R}$ and $0 < H_h(X) < +\infty$, then for H_h^1-a.e. $x \in X$ we have that

$$H_h(X) = \lim_{r \to 0} inf \left\{ \frac{diam^h(F)}{H_h^1(F)} : x \in F, F \subset \mathbb{R} \right.$$
$$\left. -closed\ interval, diam(F) \leq r \right\}$$

where H_h^1 is normed h-dimensional Hausdorff measure.

Main lemma of the proof

Lemma 3.9

Let m_n be normalised h_n-dimensional measure: $m_n := H^1_{h_n}$.

$$\lim_{n\to\infty,\ r\to 0} inf \left\{ \frac{r^{h_n}}{m_n([0,1])} \right\} \geq 1$$

- Proof: Fix $N \geq 2$ so large that $h_N \geq 3/4$ and let $n \geq N$. For every $r \in (0, 1/2)$ let $s_r \geq 1$ be unique integer s.t.

$$\frac{1}{s_r + 1} < r \leq \frac{1}{s_r}$$

$$m_n([0,r]) \leq \sum_{j=s_r}^{\infty} m_n(g_j([0,1])) \leq \sum_{j=s_r}^{\infty} ||g_j'||_{\infty}^{h_n} m_n([0,1])$$

$$\leq \sum_{j=s_r}^{\infty} j^{-2h_n} \leq \int_{s_r-1}^{\infty} x^{-2h_n} dx = (2h_n - 1)^{-1}(s_r - 1)^{1-2h_n}$$

Therefore:

$$\frac{m_n([0,1])}{r_n^h} \leq (2h_n - 1)^{-1}(s_r - 1)^{1-2h_n}(s_r + 1)^{h_n}$$

$$= (2h_n - 1)^{-1}(s_r - 1)^{1-h_n}(\frac{s_r + 1}{s_r - 1})^{h_n} = (2h_n - 1)^{-1}(s_r - 1)^{1-h_n}(1 + \frac{2}{s_r - 1})^{h_n}$$

$$\leq (2h_n - 1)^{-1}(s_r - 1)^{1-h_n}(1 + 4r)^{h_n}$$

If $0 < r \leq \frac{1}{n+1}$ then $m_n([0, r]) = 0$, hence we assume that $r > \frac{1}{n+1}$. Then $s_r < n + 1$ and for n large enough

$$\frac{m_n([0, r])}{r^{h_n}} \leq (2h_n - 1)^{-1} n^{1-h_n}(1 + 4r)$$

$$\leq (2h_n - 1)^{-1}(1 + 4r) n^{\frac{7}{\pi^2 n}} \leq (2h_n - 1)^{-1}(1 + 4r)(n^{\frac{1}{n}})^{\frac{7}{\pi^2}}$$

which implies

$$\lim_{n \to \infty,\ r \to 0} sup \left\{ \frac{r^{h_n}}{m_n([0, 1])} \right\} \leq 1$$

Similarities

Let us consider case where $F_k : [b_k, b_{k-1}] \to [0,1]$ is a linear function such that $F_k(b_k) = 1$ and $F_k(b_{k-1}) = 1$, where $(b_k)_{k=0}^{\infty} \subset (0,1]^{\mathbb{N}}$ is a sequence such that $b_0 = 1$, $b_n \searrow 0$ as n goes to infinity.

- We define $f_k := F_k^{-1}$, so $f_k : [0,1] \to [b_k, b_{k-1}]$ and $f_k(0) = b_{k-1}$ and $f_k(1) = b_k$.
- We analogously define J_n as Julia set created by iterating first n functions f_n.

Similarities

Because this IFS fulfills OSC, the h_n - Hausdorff dimension is the solution to following implicit equation:

$$\sum_{k=1}^{n}(b_k - b_{k-1})^{h_n} = 1$$

It is easy to see that $h_n \to 1$ as n goes to infinity.

We are interested if

$$H_{h_n}(J_n) \to 1 = 1 = H_1(J)$$

for a given sequence $(b_k)_{k=0}^{\infty}$

Similarity dimension estimate

Let $b_k = \frac{1}{n^\alpha}$, $\alpha > 0$. Then there exists $0 < h_n^* < 1$, $h_n^* \to \infty$, as n goes to infinity and

$$1 - h_n \leq 1 - h_n^*$$

Moreover

$$\lim_{n\to\infty} n^\alpha(1 - h_n^*) = \frac{\alpha}{\alpha + 1}$$

Similarity measure continuity

$$\lim_{n\to\infty} H_{h_n}(J_n) = 1$$

for $b_k = \frac{1}{k^\alpha}$, $\alpha > 0$.

Main lemma of the proof

Main lemma

$$\liminf_{\substack{n\to\infty \\ r\to 0}} \{\frac{r^{h_n}}{m_n([0,r])}\} \geq 1$$

Proof. Fix $N \geq 2$, s.t. $h_N > \frac{1}{\alpha+1}$ and $n \geq N$. For every $r \in (0, 1/2)$ let $k \in \mathbb{N}$ be such that

$$b_{k+1} < r \leq b_k$$

Then:

$$m_n([0,r]) \leq \sum_{j=k}^{\infty} m_n(g_j([0,1]))$$

$$\leq \sum_{j=k}^{\infty} ||g_j'||_{\infty}^{h_n} m_n([0,1])$$

$$\leq \sum_{j=k}^{\infty} (b_j - b_{j+1})^{h_n} = \sum_{j=k}^{\infty} \left(\frac{1}{j^{\alpha}} - \frac{1}{(j+1)^{\alpha}} \right)^{h_n} \leq \sum_{j=k}^{\infty} \left(\frac{\alpha^{h_n}}{j^{(\alpha+1)h_n}} \right)$$

$$\leq \int_{k-1}^{\infty} \alpha^{h_n} \frac{1}{x^{(\alpha+1)h_N}} dx \leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1} (k-1)^{1-(\alpha+1)h_n}$$

Thus:

$$\frac{m_n([0,r])}{r^{h_n}} \leq \frac{\frac{\alpha^{h_n}}{(\alpha+1)h_n-1}(k-1)^{1-(\alpha+1)h_n}}{(b_{k+1})^{h_n}} \leq$$

$$\frac{\alpha^{h_n}}{(\alpha+1)h_n - 1} \left(\frac{b_{k-1}}{b_{k+1}}\right)^{h_n} (k-1)^{1-h_n} \leq$$

$$\leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(\frac{k+1}{k-1})^{\alpha h_n}(k-1)^{1-h_n}$$

$$= \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n}(k-1)^{1-h_n}$$

$$\leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n}(k-1)^{1-h_n}$$

$$\frac{m_n([0,r])}{r_n^h} \leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n} k^{1-h_n}$$

$$\leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n} n^{1-h_n} \leq (*)$$

Now, using estimate for $1 - h_n$, we get:

$$(*) \leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n} n^{1-h_n^*}$$

$$\leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n} (n^{\alpha})^{\frac{2\alpha}{(\alpha+1)n^{\alpha}}}$$

$$\leq \frac{\alpha^{h_n}}{(\alpha+1)h_n - 1}(1 + \frac{2}{k-1})^{\alpha h_n} ((n^{\alpha})^{\frac{1}{n^{\alpha}}})^{\frac{2\alpha}{\alpha+1}}$$

for sufficiently large n we have $n^{\alpha}(1 - h_n^*) < \frac{2\alpha}{\alpha+1}$, which ends the proof.

Following problems

- $b_k = e^{-k}$
- $b_k = \frac{1}{ln(k+e)}$

The End

Thank you for your attention!