Random quadratic Julia sets and quasicircles

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March 27, 2020

Introduction and notation

- We will consider compositions of functions $f_n(z) = z^2 + c_n$.
- Unlike in normal complex dynamics, c_n changes along iteration.
- Let us denote $F_n(z) = f_n(z) \circ f_{n-1}(z) \circ ... \circ f_1(z)$.
- We can ask questions about normality of the family $\{F_n\}$.

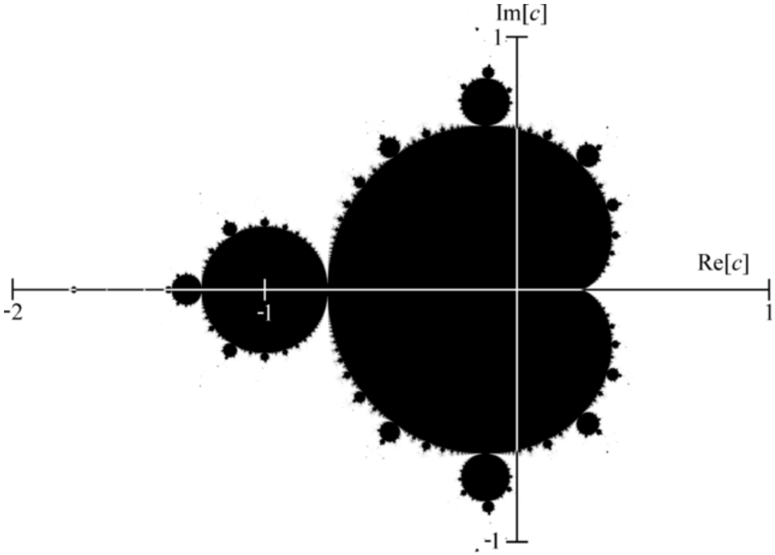
Non-autonomous definitions

- The Fatou set is defined by
  $\mathbb{F}_{(c_n)} = \{z \in \widehat{\mathbb{C}} : \{F_n\}$ *is normal on a neighbourhood of z* $\}$
- The Julia set $\mathbb{J}_{(c_n)}$ is the complement of the Fatou set.
- In the autonomous case these sets depend on a parameter c, since we investigate the normality of iterations of $z^2 + c$. In our case these sets depend on a sequence $\{c_n\}$.

Autonomous quadratic iteration

- The autonomous case where $\forall_n c_n = c$ has been studied extensively.
- The Julia set $\mathbb{J}_c$ is in this case either connected, or totally disconnected, i.e. every connected component is a single point.
- The set of points c for which the Julia set is connected is the famous Mandelbrot set.
- If c is in the interior of the main cardioid of the Mandelbrot set then the Julia set is a quasicircle.

Autonomous iteration: Mandelbrot set



Quasicircles

Definition

A *quasicircle* is the image of the unit circle under a quasiconformal homeomorphism of $\mathbb{C}$ onto itself.

Theorem (Ahlfors)

A Jordan curve $\gamma \subset \mathbb{C}$ is a quasicircle if and only if there exists a constant $M < \infty$ such that $|x - y| < M|x - z|$ holds for any y on the smaller diameter arc between $x, z \in \gamma$.

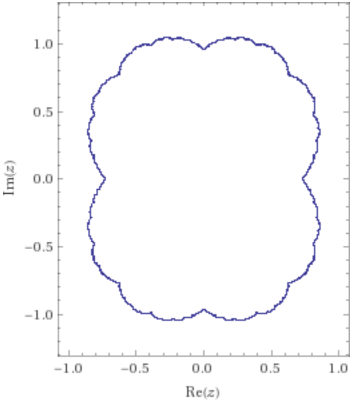


Figure: Julia set for $z^2 + \frac{1}{5}$

Some more pictures and geometric intuition

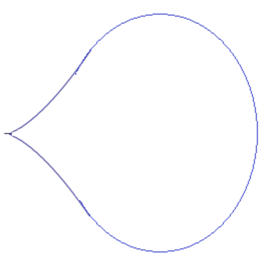


Figure: Not a quasicircle

Non autonomous quadratic dynamics

Theorem (Rainer Brück)

If $\forall_n |c_n| < \delta < \frac{1}{4}$ then the Julia set $\mathbb{J}_{c_n}$ is a quasicircle.

Theorem (Anna Zdunik, L.)

Let V be an open and bounded set such that $\mathbb{D}(0, \frac{1}{4}) \subset V$ and $V \neq \mathbb{D}(0, \frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ equipped with the product $\mathbb{P}$ of uniform distributions on V. Then for $\mathbb{P}$–almost every sequence $\omega \in \Omega$ the Julia set J_ω is totally disconnected.

Comparison of non-autonomous and autonomous dynamics

- In the autonomous quadratic dynamics, if the Julia set is disconnected, then it must be totally disconnected. This is not true for non-autonomous iteration.
- Indeed, it is easy to produce sequences for which the Julia set is disconnected but not totally disconnected. Take c_1 to be some large number, and $c_n = 0$ for $n > 1$.
- In the above example, the Julia set has 2 connected components, the preimages of the unit circle under $z^2 + c_1$.

Comparison of non-autonomous and autonomous dynamics

- In the autonomous case the Julia set is a quasicircle if c is from the interior of the main cardioid of the Mandelbrot set.
- In the non-autonomous case if $|c_n| < \delta < \frac{1}{4}$ then the Julia set is a quasicircle. But if c_n are chosen from the interior of the main cardioid of the Mandelbrot set, then the sequences for which the Julia set is totally disconnected are of full measure.

John domains

Before we look at Brück's proof we need one more auxilliary result.

Definition

A domain $G \subset \widehat{\mathbb{C}}$ with $\partial G \subset \mathbb{C}$ is called a *John domain*, if there exists a constant $b > 0$ and a point $w_0 \in G$ such that for any $z_0 \in G$, there is an arc $\gamma = \gamma(z_0) \subset G$ joining z_0 and w_0 and satisfying
$dist(z, \partial G) \geqslant b|z - z_0|$ for any $z \in \gamma$.

Theorem (Raimo Näkki, Jussi Väisälä)

If the two complementary components of a Jordan curve are John domains, then that curve is a quasicircle.

The Fatou set has two components

- From now on let $\forall_n |c_n| < \delta$ for some $\delta < \frac{1}{4}$.
- For any r let $D_r = \{z : |z| < r\}$ and $\Delta_r = \{z : |z| > r\}$
- Let $R \geqslant R_\delta := \frac{1}{2}(1 + \sqrt{1 + 4\delta})$ and $r_\delta := \frac{1}{2}(1 + \sqrt{1 - 4\delta}) > r > \frac{1}{2}$
- Then $\forall_{c_n \in D(0,\delta)}\ f_{c_n}(\Delta_R) \subset \Delta_R$ and $f_{c_n}(D_r) \subset D_r$
- The Fatou set has two connected components, one which contains infinity, and one which contains 0, let us denote them by $A_{(c_n)}(\infty)$ and $A_{(c_n)}(0)$ respectively.
- Finally, we have: $\partial A_{(c_n)}(0) = \mathbb{J}_{(c_n)} = \partial A_{c_n}(\infty)$.

Brück's proof of the quasicircle theorem

Let us recall again:

Theorem (Brück)

If $\delta < \frac{1}{4}$ *and* $(c_n) \in D_\delta^{\mathbb{N}}$ *then* $\mathbb{J}_{(c_n)}$ *is a quasicircle.*

Brück proves this by showing that both $A_{(c_n)}(0)$ and $A_{(c_n)}(\infty)$ are John domains. The remainder of the slides are a presentation of his proof, exactly as in his paper.

Brück's proof of the quasicircle theorem

Lemma

Let $\delta < \frac{1}{4}$*,* $(c_n) \in D_\delta^{\mathbb{N}}$ *and* $\frac{1}{2} < r < r_\delta$*. Let* $\gamma : [0,1] \to V$ *be a rectifiable curve in* $V := \Delta_r$*. Let* $z := \gamma(0)$*,* $w := \gamma(1)$ *and let* F_n^{-1} *be an analytic branch of the inverse function of* F_n *on some disk* $D \subset V$ *with center at* z*. Finally, we denote the analytic continuation of* F_n^{-1} *along* γ *also by* F_n^{-1}*. Then we have*

$$\left|\frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)}\right| \leqslant 1 + \alpha\ell(\gamma)e^{\alpha\ell(\gamma)}$$

for some constant α*.*

Proof of the lemma

Proof.

For $k = 0, 1, ..., n-1$ we set $F_{n,k} := f_{c_n} \circ ... \circ f_{c_{k+1}}$. Since:

$$(F_n^{-1})'(z) = \frac{1}{F_n'(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_j(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_{n,j}^{-1}(z)}$$

and V is backward invariant we have

$|(F_n^{-1})'(z)| \leqslant q^n$ for $z \in V$

and even

$|(F_{n,k}^{-1})'(z)| \leqslant q^{n-k}$

where $q := \frac{1}{2r} < 1$. This yields

$$|F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)| \leqslant |\int_\gamma |(F_{n,k}^{-1})'(\zeta)||d\zeta|| \leqslant q^{n-k}\ell(\gamma)$$

Finally:

$$\frac{|(F_n^{-1})'(z)|}{|(F_n^{-1})'(w)|} = |\prod_{k=0}^{n-1} \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)}| = \prod_{k=0}^{n-1} |1 + \frac{F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)}{F_{n,k}^{-1}(z)}| \leqslant \prod_{k=0}^{n-1} 1 + 2q^{n-k+1}\ell(\gamma)$$

Proof of the lemma

Proof.

We finish the proof by:

$$\prod_{k=0}^{n-1}(1+2q^{n-k+1}\ell(\gamma)) = \prod_{k=2}^{n+1}(1+2q^{k}\ell(\gamma)) \leqslant \prod_{k=0}^{\infty}(1+2q^{k}\ell(\gamma)) =$$

$$\exp(\sum_{k=0}^{\infty}\log(1+2q^{k}\ell(\gamma))) \leqslant \exp(\sum_{k=0}^{\infty}2q^{k}\ell(\gamma)) = e^{\alpha\ell(\gamma)} \leqslant 1+\alpha\ell(\gamma)e^{\alpha\ell(\gamma)}$$

□

The basin of infinity is a John domain

Theorem

Let $\delta < \frac{1}{4}$ and $c_n \in D_\delta^{\mathbb{N}}$. Then $A_{c_n}(\infty)$ is a John domain.

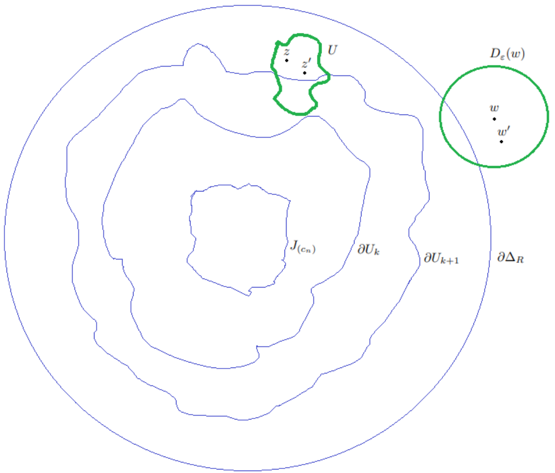
Proof of the basin of infinity being a John domain

Proof.

Let $R > R_\delta$ such that $R^2 + \delta - R \leqslant \frac{1}{2}$, $\varepsilon := R - R_\delta < 1$, and let $U_k := F_k^{-1}(\Delta_R)$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset A_{(c_n)}(\infty)$ and $A_{(c_n)}(\infty) = \bigcup_{k=1}^{\infty} U_k$. For $z \in A_{(c_n)}(\infty)$ let $d(z) := dist(z, \mathbb{J}_{(c_n)})$.
Finally let $z \in U_k$ and set $w := F_k(z)$. If U is the component of $F_k^{-1}(D_\varepsilon(w))$ containing z, then $U \subset A_{(c_n)}(\infty)$. Now let $\rho > 0$ be such that $D_\rho(z) \subset U$. Let $z' \in D_\rho(z)$ and $w' := F_k(z')$.
We begin the proof by finding a lower bound for $d(z)$.

Proof of the basin of infinity being a John domain

Proof.



Proof of the basin of infinity being a John domain

Proof.

We have:

$w' - w = F_k(z') - F_k(z) = \int_{[z,z']} F_k'(\zeta) d\zeta =$

$F_k'(F_k^{-1}(w)) \int_{[z,z']} \frac{F_k'(\zeta)}{F_k'(F_k^{-1}(w))} d\zeta = F_k'(z) \int_{[z,z']} \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} d\zeta$

From the lemma we obtain:

$\frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} \leqslant 1 + \alpha e^{\alpha\varepsilon} |w - F_k(\zeta)| \leqslant 1 + \alpha\varepsilon e^{\alpha\varepsilon} \leqslant 1 + \alpha e^{\alpha}$

Which yields:

$|w - w'| \leqslant |F_k'(z)||z' - z|(1 + \alpha e^{\alpha}) \leqslant |F_k'(z)|\rho(1 + \alpha e^{\alpha})$

We have shown that we can now set $\rho = \frac{\varepsilon}{|F_k'(z)|(1+\alpha e^{\alpha})}$ and have $D_\rho(z) \subset U$. This finally implies that $d(z) \geqslant \rho = \frac{\varepsilon}{|F_k'(z)|(1+\alpha e^{\alpha})} = \frac{\alpha_1}{|F_k'(z)|}$ for any arbitrary $z \in U_k$.

Proof of the basin of infinity being a John domain

Proof.

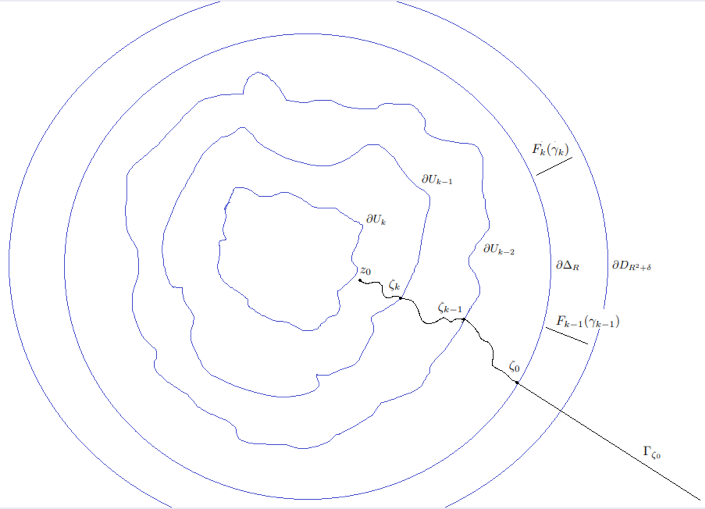
In order to prove the John property, let $w_0 = \infty$ and $z_0 \in A_{c_n}(\infty)$. We may assume that $z_0 \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$. Then $R < |F_k(z_0)| \leqslant R^2 + \delta$. Finally let us denote by Γ_ζ the ray from ζ to ∞ for which the extension to a line passes through 0. We construct an arc in U_k joining z_0 and w_0 as follows:

- Join z_0 with ∂U_{k-1} by an arc $\gamma_k \subset U_k \setminus U_{k-1}$ such that $F_k(\gamma_k) \subset \Gamma_{F_k(z_0)}$ and denote the endpoint of γ_k on ∂U_{k-1} by ζ_{k-1}
- Now join ζ_{k-1} with ∂U_{k-2} by a curve $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$ such that $F_{k-1}(\gamma_{k-1}) \subset \Gamma_{F_{k-1}(\zeta_{k-1})}$ and denote the endpoint of γ_{k-1} on ∂U_{k-2} by ζ_{k-2}
- Proceed inductively, and set $\gamma := \gamma_k \cup ... \cup \gamma_1 \cup \Gamma_{\zeta_0}$

We claim that γ is the curve for z_0 that has the John property. It is important to note that all line segments $F_j(\gamma_j)$ lie in $\Delta_R \cap D_{R^2+\delta}$, thus have length at most $\frac{1}{2}$.

Proof of the basin of infinity being a John domain

Proof.



Proof of the basin of infinity being a John domain

Proof.

Let us now assume $z \in \gamma$. For the purpose of showing the John property, we may assume without loss of generality that $z \in D_R$. First, let $z \in U_k \setminus U_{k-1}$. We deduce an upper estimate for $|z - z_0|$. We have:
$z - z_0 = F_k^{-1}(F_l(z)) - F_k^{-1}(F_k(z_0)) = \int_{[F_k(z_0),F_k(z)]}(F_k^{-1})'(\zeta)d\zeta =$
$(F_k^{-1})'(F_k(z)) \int_{[F_k(z_0),F_k(z)]} \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))}d\zeta$. Using the lemma in the same way as before yields:
$|\frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))}| \leqslant 1 + \alpha e^{\alpha}|F_k(z) - \zeta| \leqslant 1 + \alpha e^{\alpha}|F_k(z) - F_k(z_0)| \leqslant 1 + \alpha e^{\alpha}$
and thus:
$|z - z_0| \leqslant |(F_k^{-1})'(F_k(z))|(1 + \alpha e^{\alpha})|F_k(z) - F_k(z_0)| \leqslant \frac{1+\alpha e^{\alpha}}{|F_k'(z)|} = \frac{\alpha_2}{|F_k'(z)|}$ for all $z \in \gamma \setminus U_{k-1}$.

Proof of the basin of infinity being a John domain

Proof.

So for $z \in \gamma \setminus U_{k-1}$ we have $d(z) \geqslant \alpha_3 |z - z_0|$. Now let $z \in U_{k-m} \setminus U_{k-m-1}$ for some $m = 1, 2, ..., k-1$. Let us recall that we have $d(z) \geqslant \frac{\alpha_1}{|F'_{k-m}(z)|}$. Now we have:
$|z - z_0| \leqslant |z_0 - \zeta_{k-1}| + |\zeta_{k-1} - \zeta_{k-2}| + ... + |\zeta_{k-m} - z| \leqslant \alpha_2 (\frac{1}{|F'_k(\zeta_{k-1})|} + \frac{1}{|F'_{k-1}(\zeta_{k-2})|} + ... + \frac{1}{|F'_{k-m}(z)|})$. Thus we have $\frac{d(z)}{|z-z_0|} \geqslant \frac{\alpha_3}{1+\sum_{j=1}^{m} |\frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})}|}$. We now need to estimate the denominator for the right hand side. Consider the term
$\frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})} = \frac{1}{2^j F_{k-m+j-1}(\zeta_{k-m+j-1}) ... F_{k-m}(\zeta_{k-m+j-1})} \cdot \frac{F'_{k-m}(z)}{F'_{k-m}(\zeta_{k-m+j-1})}$. Now note that since $|F_{k-m+j-1(\zeta_{k-m+j-1})}| = R$, and D_r is invariant, and also $R > r$, we can estimate
$|\frac{F'_{k-m}(z)}{F'_{k-m+j}(\zeta_{k-m+j-1})}| \leqslant q^j |\frac{F'_{k-m}(z)}{F'_{k-m}(\zeta_{k-m+j-1})}|$ where $q = \frac{1}{2r} < 1$.

Proof of the basin of infinity being a John domain

Proof.

For the last estimates let us denote $p = k - m$ for brevity. We write: $\left|\frac{F_p'(z)}{F_p'(\zeta_{p+j-1})}\right| = \left|\frac{(F_p^{-1})'(F_p(\zeta_{p+j-1}))}{(F_p^{-1})'(F_p(z))}\right| \leqslant 1 + \alpha\ell(\sigma)e^{\alpha\ell(\sigma)}$, the last inequality being our lemma. Here σ is the curve $F_p(\gamma_p' \cup \gamma_{p+1} \cup ... \cup \gamma_{p+j-1})$, and where γ_p' is the part of γ_p joining ζ_p with z. Hence we have $\ell(\sigma) \leqslant \ell(F_p(\gamma_p)) + ... + \ell(F_p(\gamma_{p+j-1}))$. Recall that $\ell(F_p(\gamma_p)) \leqslant \frac{1}{2}$ and $F_p(\gamma_{p+\nu}) = F_{p+\nu,p}^{-1}(s_{p,\nu})$ where $s_{p,\nu} := F_{p+\nu}(\gamma_{p+\nu})$ is again a curve of length at most $\frac{1}{2}$. Furthermore we know that $F_p(\gamma_{p+\nu}) \subset \Delta_r$. Therefore we get

$\ell(F_p(\gamma_{p+1})) = \int_{s_{p,1}} \frac{|dw|}{2\sqrt{|w-c_{p+1}|}} \leqslant \frac{\ell(s_{p,1})}{2r} \leqslant \frac{1}{4r}$. By induction we get $\ell(F_p(\gamma_{p+\nu})) \leqslant \frac{1}{2}q^{\nu}$ and thus $\ell(\sigma) \leqslant \frac{1}{2}(1 + q + ... + q^{j-1}) = \alpha_4$.

Finally we finish with $\frac{d(z)}{|z-z_0|} \geqslant \frac{\alpha_3}{1+\alpha_5 \sum_{j=1}^{m} q^j} \geqslant \frac{\alpha_3(1-q)}{\alpha_5}$.

□

Bibliography I

Rainer Brück
Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$.
Pacific Journal of Mathematics, 2001.

Raimo Näkki, Jussi Väisälä
John disks
Expositiones Mathematicae, 1991