DYNAMICAL SYSTEMS WORKING SEMINAR
2016-2017

Poincaré functions with spiders' webs (14/10/16)

Krzysztof Lech

At the seminar we discussed the paper "Poincaré functions with spiders' webs" by Helena Mihaljević-Brandt and Jörn Peter. We started with a long discussion of the basic definitions such as the escaping set and the fast escaping set. We talked about some basic examples and the geometry involved. Then we moved on to discuss the appendix of the paper. The propositions 4.1 and 4.2 were presented and proved in detail. Thus we have shown the equality of the sets of omitted of the linearizer and the set of exceptional values of f with the exclusion of the fixed repelling point at which linearization is performed.

The Hausdorff dimension of graphs of Weierstrass functions (21/10/16)

Reza Mohammadpour Bejargafsheh

In the seminar we discussed about "The Hausdorff dimension of graphs of Weierstrass functions" by Brian R.Hunt. As we know , the graph of the Weierstrass function, that is $\omega(x) = \sum_{k=0}^{\infty} a^k cos(2\pi b^k x)$ is continuous but nowhere differentiable but in this paper we discussed about $\omega_\Theta(x) = \sum_{k=0}^{\infty} a^k cos(2\pi b^k x + \Theta_n)$. At the beginning of the seminar we presented some definitions like Hausdorff and Capacity dimension and the relation between them and t-energy. Then we showed the process that follows this article. This article has two theorems; theorem 1 says that if each n is chosen independently with respect to the uniform probability measure in [0,1], then with probability one the Hausdorff dimension of the graph ω_Θ is $2 + \frac{\log a}{\log b}$. Theorem 2 is similar to Theorem 1 so we only proved Theorem 1. The idea of the proof was to first use the t-energy to find a lower bound for the Hausdorff dimension and then by using the s-dimensional Hausdorff measure we found an upper bound for Hausdorff dimension by using t-energy.

Introduction to infinite ergodic theory (4 & 18/11/16)

Łukasz Treszczotko

In the talk I presented some basic notions studied in the field of infinite ergodic theory such as conservativity, pointwise dual ergodicity, the inducing method and quite a few more. I mentioned the negativer results given by Aaronson's Ergodic Theorem and proved Ratio Ergodic Theorem. I briefly mentioned a distributional approach to IET.
http://mat.univie.ac.at/ zweimueller/MyPub/SurreyNotes.pdf

Hausdorff dimension of uniformly wiggly sets (2/12/16)

Adam Śpiewak

For $E \subset \mathbb{R}^2$ and a square $Q \subset \mathbb{R}^2$ with side length $s(Q)$ define

$$\beta_E(Q) = \inf_{L \in \mathcal{L}} \sup_{z \in E \cap 3Q} \frac{\operatorname{dist}(z, L)}{s(Q)},$$

where $3Q$ is a square with the same center as Q and 3 times larger side length and $\mathcal{L}$ is the set of all lines intersecting Q. A connected set $E \subset \mathbb{R}^2$ is called uniformly wiggly with constant $\beta_0 > 0$ if $\beta_E(Q) \geq \beta_0$ for every Q with $\frac{1}{3}Q \cap E \neq \emptyset$ and $\operatorname{diam}(Q) \leq \operatorname{diam}(E)$. The talk was devoted to proving the following theorem: if $E \subset \mathbb{R}^2$ is closed, connected and uniformly wiggly with constant β_0, then $\dim_H(E) \geq 1 + C\beta_0^2$, where C is an absolute constant. This is based on the paper *Wiggly sets and limit sets*, C. J. Bishop, P. W. Jones, Ark. Mat. 35 (1997), no. 2, 201–224.

Inner functions (9 & 16/12/16)

Krzysztof Lech

At the seminar we discussed the paper "The Dynamics of Inner Functions" by Claus Doering and Ricardo Mane. The basic definitions were presented including inner functions themselves, recurrence and the Denjoy-Wolff point. Then we discussed the dichotomy dictated by theorems 4.1 and 4.2. That is, if f is an inner function and there exists a point $z \in D$ such that the series $1 - |f^n(z)|$ converges then the Denjoy-Wolff point p of f belongs to the boundary of D and the iterates of almost every point of the boundary converge to this point p. On the other hand, if there exists a point $z \in D$ such that the series diverges, then the boundary map of f is recurrent. Both theorems were proven in full, exactly as in the paper.

Rectifiability of measures (23/12/16)

Jacek Gałęski

We were discussing the paper written by Xavier Tolsa titled: "Characterization of n-rectifiability in terms of Jones' square function: part I". This work concerns the n-dimensional finite radon measure μ in $\mathbb{R}^d$. The author proves that if the considered measure is n-rectifiable, which means that the support is contained in some rectifiable set and the measure is absolutely continuous with respect to the n-dimensional Hausdorff measure, then the integral expression of the so-called "beta number", or "Jones' square function" as the title suggests, is finite almost everywhere with respect to μ. Beta number is the mean distance between the measure μ and n-dimensional plane L, more precisely:

$$\beta^n_{\mu,p}(x, r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y, L)}{r} \right)^p \, d\mu(y) \right)^{1/p}.$$

The integral expression mentioned above is the Dini condition for beta numbers, so the main theorem is the following, if the measure μ is as above, then

$$\int_0^\infty \beta^n_{\mu,p}(x, r)^2 \, \frac{dr}{r} < \infty \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

The proof is based on the division of $\mathbb{R}^d$ into dyadic cubes then grouping them into families depending on the concentration of the measure μ and size of beta numbers. Then we introduce stopping conditions meaning that we do not add all dyadic cubes (arbitrarily small) but stop at certain size and apply the procedure of division into families again.

There exists a Part II of this paper. It is written by Xavier Tolsa and Jonas Azzam and they prove that the above Dini condition for beta numbers is in fact a characterisation of rectifiable measures absolutely continuous with respect to the n-dimensional Hausdorff measure.

Random Julia sets of exponential functions (27/1/16)

Krzysztof Lech

The talk was about normality of function series of the form $e^z \times \lambda_n$ where $\lambda_n > 0$. We discussed that the Julia set for a series where $\lambda_n > \lambda > 1/e$ is the whole plane, and a sketch of a proof given by Urbański-Zdunik was presented. After that we discussed various theorems where the assumptions on λ_n are weaker. In particular, one only needs $\lambda_n > \lambda > 1/e$ for an infinite amount of n, if we assume that for the rest of the series $\lambda_n \geq 1/e$ holds. Finally series of the form $\lambda_n = 1/e + 1/n^p$ were discussed, and we proved the fact that the Fatou set is non-empty for $p \geq 2$.

Iteration of transcendental entire functions, Part I: The escaping set and spiders' webs (10/3/17)

Vasiliki Evdoridou

In this talk we made an introduction to the iteration of transcendental entire functions and we gave some basic definitions and results. We focused on the escaping set, which consists of the points whose iterates tend to infinity. Because of its interesting properties and its connection to some of the most famous open problems in the area, the escaping set has become a very famous object of study. For several transcendental entire functions the escaping set has the structure of a 'spider's web'. We showed that this is the case for a function that was first studied by Fatou. We also described a whole class of functions which share the same property. Finally, we commented on functions whose escaping set cannot be a spider's web and saw that in some cases a superset of the escaping set is a spider's web.

Iteration of transcendental entire functions, Part II: Non-escaping endpoints (17/3/17)

Vasiliki Evdoridou

Let $f_a(z) = e^z + a, a < -1$. The Julia set of f_a consists of an uncountable union of disjoint curves going to infinity (a Cantor bouquet). Following several interesting results on the endpoints of these curves, we considered the set of non-escaping endpoints, that is, the endpoints whose iterates do not tend to infinity. We showed that the union of non-escaping endpoints with infinity is a totally separated set by finding continua that separate these endpoints from infinity. In fact we showed this result for a larger class of functions in the exponential family. This is a complementary result to the very recent result of Alhabib and Rempe-Gillen that for the same family of functions the set of escaping endpoints together with infinity is connected. Moreover, we showed that a function that was first studied by Fatou shares the same property of the non-escaping endpoints. Finally, we presented a well-studied class of functions outside the exponential family for which the result also holds.

Non-existence of absolutely continuous invariant probabilities for exponential maps, Part I (24/3/17)

Krzysztof Lech

We discussed a paper by Neil Dobbs and Bartlomiej Skorulski ("Non existence of absolutely continous invariant probabilities for exponential maps") in which they prove that for the map $\lambda \to \lambda e^z$ there can be no probabilistic invariant absolutely continous (wrt to Lebesque) measure, under appropriate assumptions. That is λ has to be such that the postsingular set is bounded, the Julia set is the whole plane, and there is a set of positive Lebesque measure for which the omega limit set is not contained in the postsingular set. This requires the construction of a "nice set" on which we shall integrate the first return time with respect to such a hypothetical measure. At the seminar we managed to conclude the proofs of lemmas 2,3,4 and 5, which allow the construction of U and give us bounds for the first return time to U. The rest was left for next week.

Non-existence of absolutely continuous invariant probabilities for exponential maps, Part II (31/3/17)

Krzysztof Lech

We finished the proof from last week. This required some comments on lemma 5, specifically a construction from the famous McMullen paper ("Area and Hausdorff dimension of Julia sets of entire functions") which allows to estimate the Lebesque measure of some sets. After that we apply the "Folklore theorem" to prove the existence of a sigma-finite measure. At the seminar there was a lively discussion about the validity of how we use the folklore theorem, Bock's theorem, and the assumption that the omega limit set is not contained in the postsingular set. In particular the proof as presented at the seminar hides the way we use the nice sets, since that is necessary for the folklore theorem which was not proven. We left these details for future seminars, as Łukasz Pawelec enthusiastically volunteered to present them. After all of that the proof follows from Kac's lemma and lemma 5. Together they give us a contradiction by saying that the integral of the first return time with respect to the lebesque measure would have to be finite on one hand (Kac's lemma), but infinite on the other (lemma 5).

Non-uniform Bernoulli convolutions (7/4/17)

Adam Śpiewak

For $\beta_1, \beta_2 \in (0,1)$ consider the transformations $T_1, T_2 : \mathbb{R} \to \mathbb{R}$

$$T_1 x = \beta_1 x, \ T_2 x = \beta_2 x + 1.$$

Probabilistic IFS $\left(\{T_1, T_2\}, (\frac{1}{2}, \frac{1}{2})\right)$ admits a unique invariant probability μ_{β_1,β_2}. This measure is called the (non-uniform) Bernoulli convolution and is always either purely singular or absolutely continuous. The major open problem is the characterization of parameters for which μ_{β_1,β_2} is singular. Even the uniform case $\beta_1 = \beta_2$ is still unresolved and has been intensively studied since 1930's. During the talk I presented a survey on the known results for both uniform and non-uniform case. I also proved a result on the existence of a family of algebraic curves in the parameter space $(0,1)^2$ on which the Bernoulli convolution is singular. This is based on the paper *A family of exceptional parameters for non-uniform self-similar measures*, J. Neunhäuserer, Electron. Commun. Probab. 16 (2011), 192–199.

Bock's Theorem and its application to finding the invariant measures for entire maps (21/4/17)

Łukasz Pawelec

We looked more closely at the problem of the existence of invariant measures for entire maps, most notably the exponential family. We were looking for measures absolutely continuous with respect to the Lebesgue measure. We commented on the so-called 'Folklore Theorem', which proves the existence of a suitable measure, if (simplifying slightly) the map is ergodic and recurrent w.r.t the Lebesgue measure. For many entire, and meromorphic, maps this may by proved using a result of H. Bock. We stated and proved his theorem.

Inner functions and accesses to infinity from Fatou components (28/4/17)

Vasiliki Evdoridou

We first gave an introduction to accesses to infinity from a simply connected invariant Fatou component U of a transcendental entire function f. Then we defined the inner function g which is associated to $f|_U$. The biggest part of the talk was devoted to presenting and proving a result by Evdoridou, Fagella and Jarque which relates the order of growth of f to the number of singularities of g on $\partial\mathbb{D}$. In particular, we showed that whenever the set of singular values of f is contained in a topological disc whose closure belongs to $F(f)$ then the number of singularities of g is at most two times the order of f.

Random conformal snowflakes (5/5/17)

Jacek Gałęski

We discussed a paper written by Dimitri Beliaev and Stanislaw Smirnov, titled "Random conformal snowflakes" published in the Annals of Mathematics in 2010. The aim of the work was to construct sets such that the "spectrum of harmonic measure" is as high as possible.

The dimension spectrum at the point α of the measure ω is the dimension of the set of points x such that $\omega(B(x,r)) \approx r^\alpha$ as the radius of the ball r goes to 0.

The random conformal snowflakes are exteriors of the unit disc with many "slits" added in random places and distorted a little bit as more and more slits are added. The randomness lies in places (angles) that those slits are added and it is governed by an infinite independent family of random variables θ with uniform distributions on interval $(0, 2\pi)$.

The notion of dimension spectrum is adapted to the random nature of snowflakes described in the paper and in the first three parts of the paper the authors describe the relations of the spectrum of conformal snowflake with properties of certain integral operator Q and its adjoint P. It turns out, through some scary looking but natural computations, that the maximal eigenvalue λ of the adjoint operator P gives the estimate on average spectrum $\overline{\beta}$ in the following way:

$$\overline{\beta}(t) \geq \log \lambda / \log k,$$

where k is an integer responsible for the number of slits we add at every step of the construction.

In the part 5 there is an explicit computation of the dimension spectrum for specific random snowflakes. The authors calculated the "almost eigenfunction of operator" P, and obtained the estimation

$$\beta(1) > 0.2308.$$

This is close to the conjectured value 0.25 from the Universal spectrum conjecture 1

$$\beta(t) = t^2/4 \text{ for } t \in (-2, 2)$$

obtained in the paper "Harmonic Measure on Fractal Sets" by the same authors in the European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 41–59.

The box and Haussdorf dimension of general Sierpiński carpets
(19/5/17)
Jan Kwapisz

The aim of my talk was to present how to calculate the box and Hausdorff dimension of general Sierpiński carpets. The talk was based on the famous McMullen's paper "The Hausdorff dimension of general Sierpiński carpets" and on the Simon semester note: "Bedford-McMullen carpets - an example in dimension theory of self affine sets" In the talk I focused on the techniques such as the pressure function or invariant measure. The talk was divided into four parts. Firstly, I introduced the notion of general Sierpiński carpets, then I showed how to get the upper bound for the box counting dimension from the pressure function. In the third part I calculated the box dimension, and finally the Hausdorff dimension.

Generalized Darling-Kac Theorem (26/5/17)
Łukasz Treszczotko

During the talk I formulated a fundamental distributional result in IET, which is the general version of Darling-Kac Theorem. I have presented a proof due to Owada and Samorodnitsky, which establishes a strong distributional convergence of partial ergodic sums in the Skorochod space to the Mittag-Leffler process.
https://projecteuclid.org/euclid.aop/1415801557

Continuity of Hausdorff Dimension in Hyperbolic Polynomials
(2/6/17)
Timothy Wilson (University of North Texas)

We considered a family of hyperbolic polynomials $P_\lambda : \mathbb{C} \to \mathbb{C}$ defined by $P_\lambda(z) = \lambda z^{d+1} + P(z)$ where $P(z)$ is hyperbolic polynomial of degree d. Let $J(P_\lambda)$ be the Julia set of P_λ. Our main result is that the function $\lambda \mapsto h_\lambda := HD(J(P_\lambda)), \lambda \in [0, \epsilon)$ is continuous at the point 0. The main tools we used were the associated conformal measures.

Lyapunov exponents (9/6/17)
Reza Mohammadpour Bejargafsheh

In this seminar we discussed the lecture "Lyapunov exponents" by Artur Avila and Jairo Bochi, presented in Trieste, Italy. In this seminar we talked about Lyapunov exponents and cocycles and the relation between them. Moreover, we presented some important theorems like Furstenberg-kesten for $SL(2, R)$ and Oseledets theorem for one side and two sides for describing the asymptotic behavior of the vector $A^n(x).v$. One of our tools was smooth dynamics, for example, we used uniform hyperbolicity and hyperbolic decomposition. Finally, I talked about zero Lyapunov exponent because the Furstenberg theorem leads to the impression that very few cocycles with zero Lyapunov exponent exist.