8

RIEMANN'S MEMOIR

In his epoch-making memoir of 1860 (his only paper on the theory of numbers) Riemann showed that the key to the deeper investigation of the distribution of the primes lies in the study of $\zeta(s)$ as a function of the complex variable s. More than 30 years were to elapse, however, before any of Riemann's conjectures were proved, or any specific results about primes were established on the lines which he had indicated.

Riemann proved two main results:

(a) The function $\zeta(s)$ can be continued analytically over the whole plane and is then meromorphic, its only pole being a simple pole at $s = 1$ with residue 1. In other words, $\zeta(s) - (s-1)^{-1}$ is an integral function.

(b) $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma[\tfrac{1}{2}(1-s)]\zeta(1-s),$$

which can be expressed by saying that the function on the left is an even function of $s - \frac{1}{2}$. The functional equation allows the properties of $\zeta(s)$ for $\sigma < 0$ to be inferred from its properties for $\sigma > 1$. In particular, the only zeros of $\zeta(s)$ for $\sigma < 0$ are at the poles of $\Gamma(\frac{1}{2}s)$, that is, at the points $s = -2, -4, -6, \ldots$. These are called the *trivial zeros*. The remainder of the plane, where $0 \le \sigma \le 1$, is called *the critical strip*.

Riemann further made a number of remarkable conjectures.

(a′) $\zeta(s)$ has infinitely many zeros in the critical strip. These will necessarily be placed symmetrically with respect to the real axis, and also with respect to the central line $\sigma = \frac{1}{2}$ (the latter because of the functional equation).

(b′) The number $N(T)$ of zeros of $\zeta(s)$ in the critical strip with $0 < t \le T$ satisfies the asymptotic relation

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1}$$

This was proved by von Mangoldt, first in 1895 with a slightly less good error term and then fully in 1905. We shall come to the proof in §15.

(c′) The integral function $\xi(s)$ defined by

$$\xi(s) = \tfrac{1}{2}s(s - 1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$$

(integral because it has no pole for $\sigma \geq \frac{1}{2}$ and is an even function of $s - \frac{1}{2}$) has the product representation

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}, \tag{2}$$

where A and B are constants and ρ runs through the zeros of $\zeta(s)$ in the critical strip. This was proved by Hadamard in 1893, as also was (a′) above. It played an important part in the proofs of the prime number theorem by Hadamard and de la Vallée Poussin. We shall come to the proof in §§ 11 and 12.

(d′) There is an explicit formula for $\pi(x) - \operatorname{li} x$, valid for $x > 1$, the most important part of which consists of a sum over the complex zeros ρ of $\zeta(s)$. As this is somewhat complicated to state, we give instead the closely related but somewhat simpler formula for $\psi(x) - x$, where

$$\psi(x) = \sum_{n \leq x} \Lambda(n). \tag{3}$$

It is:

$$\psi(x) - x = -\sum_{\rho}\frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \tfrac{1}{2}\log(1 - x^{-2}). \tag{4}$$

This was proved by von Mangoldt in 1895 (as was Riemann's original formula), and we give the proof in §17. In interpreting (4) two conventions have to be observed: first, in the sum over ρ the terms ρ and $\bar{\rho}$ are to be taken together, and second, if x is an integer, the last term $\Lambda(x)$ in the sum (3) defining $\psi(x)$ is to be replaced by $\frac{1}{2}\Lambda(x)$.

(e′) The famous Riemann Hypothesis, still undecided: that the zeros of $\zeta(s)$ in the critical strip all lie on the central line $\sigma = \frac{1}{2}$. It was proved by Hardy in 1914 that infinitely many of the zeros lie on the line, and by A. Selberg in 1942 that a positive proportion at least of all the zeros lie on the line.

There is very little indication of how Riemann was led to some of these conjectures. In 1932 Siegel[1] published an asymptotic expan-

[1] *Quellen und Studien zur Geschichte der Mathematik*, **2**, 45–80 (1932).

sion for $\zeta(s)$, valid in the critical strip, which had its origin in notes of Riemann preserved in the Göttingen University Library. From Siegel's description of the notes, it is plain that Riemann had more knowledge about $\zeta(s)$ than is apparent from his published memoir; but there is no reason to think that he had proofs of any of his conjectures.

In the present section we shall prove what Riemann proved, that is (in effect) the functional equation, and we shall follow one of his two methods. Many other proofs have since been given,[2] but this one is still the most elegant.

Riemann started from the classical definition of the Γ function:

$$\Gamma(\tfrac{1}{2}s) = \int_0^\infty e^{-t} t^{\frac{1}{2}s-1}\, dt,$$

valid for $\sigma > 0$. Putting $t = n^2\pi x$, we get

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)n^{-s} = \int_0^\infty x^{\frac{1}{2}s-1}e^{-n^2\pi x}\, dx.$$

Hence, for $\sigma > 1$,

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\left(\sum_1^\infty e^{-n^2\pi x}\right) dx,$$

the inversion of order being justified by the convergence of

$$\sum_1^\infty \int_0^\infty x^{\frac{1}{2}\sigma-1}e^{-n^2\pi x}\, dx.$$

Writing

$$\omega(x) = \sum_1^\infty e^{-n^2\pi x},$$

we have

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) &= \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\, dx \\ &= \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)\, dx + \int_1^\infty x^{-\frac{1}{2}s-1}\omega(1/x)\, dx.\end{aligned}$$

Plainly

$$2\omega(x) = \theta(x) - 1,$$

[2] See Titchmarsh, Chap. 2.

where

$$\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}. \tag{5}$$

This function satisfies the simple functional equation

$$\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x) \qquad \text{for } x > 0, \tag{6}$$

as we shall prove below; this equation is a special case of those satisfied by the ϑ functions of Jacobi. It follows that

$$\omega(x^{-1}) = -\tfrac{1}{2} + \tfrac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x).$$

Hence

$$\int_1^\infty x^{-\frac{1}{2}s-1}\omega(x^{-1})\,dx = \int_1^\infty x^{-\frac{1}{2}s-1}[-\tfrac{1}{2} + \tfrac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\,\omega(x)]\,dx$$
$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}}\omega(x)\,dx.$$

We have therefore proved that

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}})\omega(x)\,dx. \tag{7}$$

This holds for $\sigma > 1$. But the integral on the right converges absolutely for any s, and converges uniformly with respect to s in any bounded part of the plane, since

$$\omega(x) = O(e^{-\pi x})$$

as $x \to +\infty$. Hence the integral represents an everywhere regular function of s, and the above formula gives the analytic continuation of $\zeta(s)$ over the whole plane. It also gives the functional equation, since the right side is unchanged when s is replaced by $1 - s$.

We note that the function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$$

is regular everywhere. Since $\frac{1}{2}s\Gamma(\frac{1}{2}s)$ has no zeros, the only possible pole of $\zeta(s)$ is at $s = 1$, and we have already seen (p. 32) that this is in fact a simple pole with residue 1.

Since $\Gamma(\frac{1}{2}s) \sim (\frac{1}{2}s)^{-1}$ as $s \to 0$, we deduce from (7) that $\zeta(0) = -\frac{1}{2}$. It is easily verified that

$$\omega(x) = e^{-\pi x} + e^{-4\pi x} + e^{-9\pi x} + \cdots < \tfrac{1}{2}x^{-\frac{1}{2}} \qquad \text{for } x > 1,$$

so if $0 < s < 1$ the integral in (7) is less than $\{s(1 - s)\}^{-1}$. Hence $\zeta(s) < 0$ for $0 < s < 1$. [The same conclusion may be drawn, more simply, from (7) of §4.]

It remains to prove the functional equation (6) of the θ function. We shall prove this in the more general form

$$(8) \qquad \sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = x^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-n^2\pi x + 2\pi i n\alpha},$$

which reduces to (6) when $\alpha = 0$, since we shall need this in the next section. It is supposed in (8) that $x > 0$ and that α is any real number (though actually the equation holds for complex x and α, provided $\Re x > 0$, with the value of $x^{-\frac{1}{2}}$ which has argument between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$).

By Poisson's summation formula (§2),

$$\sum_{n=-N}^{N}{}' e^{-(n+\alpha)^2\pi/x} = \sum_{\nu=-\infty}^{\infty} \int_{-N}^{N} e^{-(t+\alpha)^2\pi/x + 2\pi i\nu t}\,dt.$$

Here we can replace N by ∞, since

$$\int_N^\infty e^{-(t+\alpha)^2\pi/x} \cos 2\pi\nu t\,dt = -\frac{1}{2\pi\nu}\int_N^\infty \sin 2\pi\nu t\, d[e^{-(t+\alpha)^2\pi/x}]$$

by integration by parts, and therefore

$$\left|\sum_{\nu\neq 0} \int_N^\infty e^{-(t+\alpha)^2\pi/x} \cos 2\pi\nu t\,dt\right| < Ce^{-(N+\alpha)^2\pi/x},$$

where C is a constant. Since this disappears as $N \to \infty$, the limit operation is justified. Thus

$$\begin{aligned}\sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} &= \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(t+\alpha)^2\pi/x + 2\pi i\nu t}\,dt \\ &= x \sum_{\nu=-\infty}^{\infty} e^{-2\pi i\nu\alpha} \int_{-\infty}^{\infty} e^{-\pi x u^2 + 2\pi i\nu x u}\,du.\end{aligned}$$

The quadratic in the exponent is

$$-\pi x(u - i\nu)^2 - \pi x\nu^2.$$

Now

$$\int_{-\infty}^{\infty} e^{-\pi x(u+\beta)^2}\,du = \int_{-\infty}^{\infty} e^{-\pi x v^2}\,dv = Ax^{-\frac{1}{2}},$$

where A is a positive constant; this holds for any β (real or complex) and simply expresses a movement in the path of integration from the real axis to another line parallel to it. Hence

$$\sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = Ax^{\frac{1}{2}} \sum_{\nu=-\infty}^{\infty} e^{-\pi x\nu^2 - 2\pi i\nu\alpha}.$$

If we now take $\alpha = 0$ and apply this formula twice, we get $A^2 = 1$, whence $A = 1$. This proves (8), on replacing ν by $-\nu$ on the right.