and $Sq^2: H^3(Y) \to H^5(Y)$ is trivial, X and Y are not of the same homotopy type.

Further applications of the Steenrod squares will be given in the next chapter and in Chap. 8.

It is obvious that cohomology operations of the same type can be added and that the sum is again a cohomology operation of the same type. Given cohomology operations θ of type $(p,q;\ G,G')$ and θ' of type $(q,r;\ G',G'')$, their composite $\theta'\theta$ (of natural transformations) is a cohomology operation of type $(p,r;\ G,G'')$. In this way the Steenrod squares can be added and multiplied, and they generate an algebra of cohomology operations called the *modulo 2 Steenrod algebra.*

In this algebra the following *Adem relations*[1] hold:

$$Sq^i Sq^j = \sum_{0 \le k \le [i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k \qquad 0 < i < 2j$$

where $[i/2]$ denotes as usual the largest integer $\le i/2$ and the binomial coefficient $\binom{j-k-1}{i-2k}$ is reduced modulo 2. Using these relations, it is easily shown that the algebra of cohomology operations generated by Sq^i, where i is a power of 2, contains all the Steenrod squares. This implies that the only spheres that can be H spaces have dimension $2^n - 1$ for some n. By using deeper properties of the algebra of cohomology operations Adams[2] has shown that the only spheres that can be H spaces are the spheres S^0, S^1, S^3, and S^7. Each of these is, in fact, an H space, with multiplication defined to be the multiplication of the reals, complex numbers, quaternions, or Cayley numbers, respectively, of norm 1.

EXERCISES

A DISSECTIONS

Let C be a graded module over R. A *filtration* (*increasing*) of C is a sequence $\{F_sC\}$ of graded submodules of C such that $F_sC \subset F_{s+1}C$ for all s. It is said to be *bounded below* if for any t there is $s(t)$ such that $F_{s(t)}C_t = 0$, and it is *convergent above* if $\cup F_sC = C$.

1 If $\{F_sC\}$ is a filtration of a chain complex C by subcomplexes, there is an increasing filtration of $H_*(C)$ defined by $F_sH_*(C) = \operatorname{im}[H_*(F_sC) \to H_*(C)]$. If the original filtration on C is bounded below or convergent above, prove that the same is true of the induced filtration on $H_*(C)$.

An increasing filtration $\{F_sC\}$ of a chain complex C by subcomplexes is called a *dissection* if it is bounded below, convergent above, and if

$$H_q(F_{s+1}C, F_sC) = 0 \qquad q \ne s+1$$

[1] See J. Adem, The iteration of the Steenrod squares in algebraic topology, *Proceedings of the National Academy of Sciences, USA*, vol. 38, pp. 720–726, 1952, or H. Cartan, Sur l'iteration des operations de Steenrod, *Commentarii Mathematici Helvetici*, vol. 29, pp. 40–58, 1955.

[2] See J. F. Adams, On the non-existence of elements of Hopf invariant one, *Annals of Mathematics*, vol. 72, pp. 20–104, 1960.

Given a dissection $\{F_sC\}$ of a chain complex C, the sequence

$$\cdots \to H_{q+1}(F_{q+1}C, F_qC) \xrightarrow{\partial} H_q(F_qC, F_{q-1}C) \xrightarrow{\partial} H_{q-1}(F_{q-1}C, F_{q-2}C) \to \cdots$$

is a chain complex $\bar{C}$, called the *chain complex associated to the dissection.*

2 If $\bar{C}$ is the chain complex associated to a dissection of C, prove that $H_*(\bar{C}) \approx H_*(C)$.

3 Let $\{F_sC\}$ be a dissection of a free chain complex C by free subcomplexes such that $F_{s+1}C/F_sC$ is free for all s. If $\bar{C}$ is the chain complex associated to the dissection, prove that $\bar{C}$ and C have isomorphic homology and cohomology for all coefficient modules. [*Hint:* The freeness hypotheses ensure that the universal-coefficient theorems hold for both homology and cohomology. Then $\{F_sC \otimes G\}$ is a dissection of $C \otimes G$ whose associated chain complex is isomorphic to $\bar{C} \otimes G$. Dual considerations apply to $\{\operatorname{Hom}(F_sC,G)\}$ and $\operatorname{Hom}(C,G)$.]

A *block dissection* of a chain complex C is a collection of subcomplexes $\{E_j^q\}$, called *blocks*, where q varies over the set of integers and for each q, j varies over a set J_q, such that if F_sC is the subcomplex of C generated by $\{E_j^q\}_{q \le s}$ and if $\dot{E}_j^q = E_j^q \cap F_{s-1}C$, then

$$E_j^q \cap E_k^q \subset F_{q-1}C \qquad j \ne k$$

$$E_j^q = 0 \qquad q \text{ sufficiently small}$$

$$\cup F_sC = C$$

$$H_i(E_j^q, \dot{E}_j^q) \approx \begin{cases} 0 & i \ne q \\ R & i = q \end{cases}$$

4 If $\{E_j^q\}$ is a block dissection of a chain complex C, prove that the corresponding collection $\{F_sC\}$ is a dissection of C whose associated chain complex $\bar{C}$ is free with generators for $\bar{C}_q$ in one-to-one correspondence with the set J_q.

A *block dissection* of a simplicial complex K is a collection of subcomplexes $\{K_j^q\}$, where q varies over the set of integers and for each q, j varies over some indexing set J_q, such that if $F_sK = \cup_{j \le s} K_j^q$ and $\dot{K}_j^q = F_{s-1}K \cap K_j^q$, then

$$K_j^q \cap K_k^q \subset F_{q-1}K \qquad j \ne k$$

$$K_j^q = 0 \qquad q \text{ sufficiently small}$$

$$\cup F_sK = K$$

$$H_i(K_j^q, \dot{K}_j^q) \approx \begin{cases} 0 & i \ne q \\ Z & i = q \end{cases}$$

5 If $\{K_j^q\}$ is a block dissection of K, prove that $\{C(K_j^q)\}$ is a block dissection of the chain complex $C(K)$ by free subcomplexes. If $\bar{C}$ is the chain complex associated to the dissection, prove that $\bar{C}$ and $C(K)$ have isomorphic homology and cohomology with any coefficient group.

B HOMOLOGY MANIFOLDS

A *homology n-manifold* is a locally compact Hausdorff space X such that for all $x \in X$, $H_q(X, X - x) = 0$ for $q \ne n$ and either $H_n(X, X - x) = 0$ or $H_n(X, X - x) \approx Z$. Furthermore, if the *boundary* $\dot{X}$ of X is defined to be the subset

$$\dot{X} = \{x \in X \mid H_n(X, X - x) = 0\}$$

then we also assume that $X - \dot{X}$ is a nonempty connected set. If $\dot{X} = \varnothing$, X is said to be *without boundary*.

1 If X is a homology n-manifold and Y is a homology m-manifold, prove that $X \times Y$ is a homology $(n + m)$-manifold whose boundary equals $\dot{X} \times Y \cup X \times \dot{Y}$.

2 Prove that if a polyhedron is a homology n-manifold, its boundary is a subpolyhedron.

3 If K is a simplicial complex triangulating a homology n-manifold X, prove that K is an n-dimensional pseudomanifold and $\dot{K}$ triangulates $\dot{X}$. (A polyhedral homology n-manifold is said to be *orientable* or *nonorientable*, according to whether any triangulation of it is orientable or nonorientable as a pseudomanifold.)

4 Let $(K,\dot{K})$ be a simplicial pair triangulating a polyhedral homology n-manifold $(X,\dot{X})$ and let L be the subcomplex of the barycentric subdivision K' consisting of all simplexes disjoint from $\dot{K}'$. If s^q is a q-simplex of $K - \dot{K}$, let $E^{n-q}(s^q)$ be the subcomplex of L generated by the star of the barycenter $b(s^q)$. Prove that $\{E^{n-q}(s^q)\}_{s^q \in K-\dot{K}}$ is a block dissection of L and that if $\bar{C}$ is the chain complex associated to this block dissection, then $\bar{C}$ has homology and cohomology isomorphic to that of $X - \dot{X}$. (*Hint:* let st $s^q = s^q * B(s^q)$, where $B(s^q)$ is a subcomplex of K. Then $E^{n-q}(s^q) = b(s^q) * [B(s^q)]'$ and $\dot{E}^{n-q}(s^q) = [B(s^q)]'$. Also note that $|L|$ is a strong deformation retract of $|K| - |\dot{K}|$.)

5 *Lefschetz duality theorem.* Let $(K,\dot{K})$ be a simplicial pair triangulating a compact homology n-manifold $(X,\dot{X})$ and assume that $z \in H_n(K,\dot{K})$ is an orientation of K. For each q-simplex s^q of $K - \dot{K}$ let $z(s^q) \in H_n(K, K - \text{st } s^q)$ be the image of z, and assume an orientation σ^q of s^q chosen once and for all. Then $z(s^q) = \sigma^q * \bar{z}(\sigma^q)$, where $\bar{z}(\sigma^q) \in H_{n-q-1}(B(s^q))$. Define $z'(\sigma^q) \in H_{n-q}(E^{n-q}(s^q),\dot{E}^{n-q}(s^q))$ to correspond to $\bar{z}(\sigma^q)$ under the isomorphisms

$$H_{n-q-1}(B(s^q)) \approx H_{n-q-1}(\dot{E}^{n-q}(s^q)) \approx H_{n-q}(E^{n-q}(s^q),\dot{E}^{n-q}(s^q))$$

Let φ: Hom $(C_q(K,\dot{K}), G) \to \bar{C}_{n-q} \otimes G$ be the homomorphism defined by

$$\varphi(u) = \sum_{\sigma^q} z'(\sigma^q) \otimes u(\sigma^q) \qquad u \in \text{Hom}\,(C_q(K,\dot{K}), G)$$

Prove that φ is an isomorphism and that it commutes up to sign with the respective coboundary and boundary operators. Deduce isomorphisms

$$H^q(X,\dot{X}; G) \approx H_{n-q}(X - \dot{X}; G) \qquad \text{and} \qquad H_q(X,\dot{X}; G) \approx H^{n-q}(X - \dot{X}; G)$$

C PROPERTIES OF THE TORSION PRODUCT AND EXT

In this group of exercises all modules will be over a principal ideal domain R.

1 Prove that the torsion product is associative.

2 If A, B, and C are modules, prove that

$$A \otimes (B * C) \oplus A * (B \otimes C)$$

is symmetric in A, B, and C.

3 Given a module A and a short exact sequence of modules

$$0 \to B' \to B \to B'' \to 0$$

prove there is an exact sequence

$$0 \to \text{Hom}\,(A,B') \to \text{Hom}\,(A,B) \to \text{Hom}\,(A,B'') \to \text{Ext}\,(A,B') \to \text{Ext}\,(A,B) \to \text{Ext}\,(A,B'') \to 0$$

4 Given a short exact sequence of modules

$$0 \to A' \to A \to A'' \to 0$$

and given a module B, prove there is an exact sequence

$$0 \to \text{Hom}\,(A'',B) \to \text{Hom}\,(A,B) \to \text{Hom}\,(A',B) \to \text{Ext}\,(A'',B) \to \text{Ext}\,(A,B) \to \text{Ext}\,(A',B) \to 0$$

If $C = \{C_i\}$ and $C^* = \{C^j\}$ are graded modules, there is a graded module Hom $(C,C^*) = \{\text{Hom}^q\,(C,C^*)\}$, where $\text{Hom}^q\,(C,C^*) = \times_{i+j=q} \text{Hom}\,(C_i,C^j)$ [thus an element of $\text{Hom}^q\,(C,C^*)$ is an indexed family $\{\varphi_i: C_i \to C^{q-i}\}_i$]. Similarly, there is a graded module Ext $(C,C^*) = \{\text{Ext}^q\,(C,C^*)\}$, where $\text{Ext}^q\,(C,C^*) = \times_{i+j=q} \text{Ext}\,(C_i,C^j)$.

5 If C is a chain complex and C^* is a cochain complex, prove that Hom (C,C^*) is a cochain complex, with

$$(\delta\varphi)_{i,j} = \varphi_{i-1,j} \circ \partial_i + (-1)^i \delta^{j-1} \circ \varphi_{i,j-1} \qquad \varphi = \{\varphi_{i,j}\} \in \text{Hom}^q\,(C,C^*)$$

and that Ext (C,C^*) is a cochain complex with

$$(\delta\psi)_{i,j} = \text{Ext}\,(\partial_i,1)(\psi_{i-1,j}) + (-1)^i\, \text{Ext}\,(1,\delta^{j-1})(\psi_{i,j-1}) \qquad \psi = \{\psi_{i,j}\} \in \text{Ext}^q\,(C,C^*)$$

6 If C is a chain complex and C^* is a cochain complex such that Ext (C,C^*) is acyclic, prove that there is a split short exact sequence

$$0 \to \text{Ext}^{q-1}\,(H_*(C),H^*(C^*)) \to H^q(\text{Hom}\,(C,C^*)) \to \text{Hom}^q\,(H_*(C),H^*(C^*)) \to 0$$

7 If C and C' are chain complexes and C^* is a cochain complex, prove that the exponential correspondence is an isomorphism

$$\text{Hom}\,(C, \text{Hom}\,(C',C^*)) \approx \text{Hom}\,(C \otimes C', C^*)$$

8 Let (X,A) and (Y,B) be topological pairs such that $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$. For any module G prove that there is a split short exact sequence

$$0 \to \text{Ext}^{q-1}\,(H_*,H^*) \to H^q((X,A) \times (Y,B); G) \to \text{Hom}^q\,(H_*,H^*) \to 0$$

where $H_* = H_*(X,A; R)$ and $H^* = H^*(Y,B; G)$.

D CATEGORY

A topological space X is said to have *category* $\leq n$, denoted as cat $X \leq n$, if X is the union of n closed sets, each deformable to a point in X.

1 If X is a connected polyhedron of dimension n, prove that cat $X \leq n + 1$.

2 If X is any space, prove that cat $(SX) \leq 2$.

3 If cat $X \leq n$, prove that all n-fold cup products of positive-dimensional cohomology classes of X vanish.

4 Prove that cat $P^n = n + 1$ and cat $(P^{n_1} \times \cdots \times P^{n_k}) = n_1 + \cdots + n_k + 1$.

E HOMOLOGY OF FIBER BUNDLES

1 Let $p: E \to B$ be a fiber-bundle pair, with total pair $(E,\dot{E})$ and fiber pair $(F,\dot{F})$, such that $H_*(F,\dot{F}) = 0$. Prove that $H_*(E,\dot{E}) = 0$.

2 If $p: E \to B$ is a fiber-bundle pair over a path-connected base space B, prove that a homomorphism $\theta: H^*(F,\dot{F}; R) \to H^*(E,\dot{E}; R)$ is a cohomology extension of the fiber if and only if for some $b \in B$ the composite

$$H^*(F,\dot{F}; R) \xrightarrow{\theta} H^*(E,\dot{E}; R) \to H^*(E_b,\dot{E}_b; R)$$

is an isomorphism.

3 Let $p: E \to B$ be a fiber-bundle pair over a path-connected base space. If for some $b \in B$ the pair $(E_b,\dot{E}_b)$ is a weak retract of $(E,\dot{E})$, prove there exists a cohomology extension of the fiber.

4 Prove that a q-sphere bundle ξ with base space B is orientable over R if and only if for every map $\alpha: S^1 \to B$ the bundle $\alpha^*(\xi)$ is orientable over R.

5 Prove that a q-sphere bundle ξ is orientable over $\mathbf{Z}$ if and only if there is an element $U \in H^{q+1}(E_\xi,\dot{E}_\xi;\, \mathbf{Z}_4)$ whose image in $H^{q+1}(E_\xi,\dot{E}_\xi;\, \mathbf{Z}_2)$ is the unique orientation class of ξ over $\mathbf{Z}_2$. (*Hint:* Show that there is such an element U if and only if for every closed path ω in the base space, $h[\omega]^*$ is the identity map of $H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\, \mathbf{Z}_4)$, and this, in turn, is equivalent to the condition that $h[\omega]^*$ is the identity map of $H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\, \mathbf{Z})$.)

6 Let ξ be a q-sphere bundle with base space B and with orientation class $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\, R)$ and let $\Omega_\xi \in H^{q+1}(B;R)$ be the corresponding characteristic class. Prove that $\Phi_\xi^*(\Omega_\xi) = U_\xi \smile U_\xi$.

7 Prove that the characteristic class Ω_ξ of an even-dimensional sphere bundle ξ oriented over $\mathbf{Z}$ has order 2.

8 Let ξ be a sphere bundle oriented over R, with base space B. If ξ has a section in $\dot{E}_\xi$ (that is, if the map $\dot{p}_\xi: \dot{E}_\xi \to B$ has a left inverse), prove that its characteristic class $\Omega_\xi = 0$. [*Hint:* Any two sections $B \to E_\xi$ are homotopic in E_ξ. Since E_ξ is the mapping cylinder of $\dot{p}_\xi: \dot{E}_\xi \to B$, there is an inclusion map $k: B \subset E_\xi$ which is a section. There is a section in $\dot{E}_\xi$ if and only if k is homotopic to a map $B \to \dot{E}_\xi$, in which case the composite

$$H^{q+1}(E_\xi,\dot{E}_\xi;\, R) \xrightarrow{j^*} H^{q+1}(E_\xi;R) \xrightarrow{p^{*-1}} H^{q+1}(B;R)$$

is trivial, because $p^{*-1} = k^*$.]

F HOPF ALGEBRAS

1 Prove that the tensor product of connected Hopf algebras is a connected Hopf algebra.

2 If B is a connected Hopf algebra of finite type over a field R, prove that $B^* = \operatorname{Hom}(B;R)$ is a connected Hopf algebra over R whose product and coproduct are dual, respectively, to the coproduct and product of B.

3 Let B be a connected Hopf algebra over a field of characteristic $p \neq 0$ and assume that B has an associative and commutative product and is generated as an algebra by a single element x of positive degree. Prove that if $\deg x$ is odd and $p \neq 2$, then $B = E(x)$, and if $\deg x$ is even or $p = 2$, then either $B = S_{\deg x}(x)$ or $B = T_{\deg x, h}(x)$, where $h = p^k$ for some $k \geq 1$.

4 Let B be a connected Hopf algebra of finite type over a field of finite characteristic $p \neq 0$ and assume that B has an associative and commutative product. If the pth power of every element of positive degree of B is 0, prove that B is the tensor product of exterior algebras (with generators of odd degree if $p \neq 2$) and truncated polynomial algebras of height p (with generators of even degree if $p \neq 2$).

G THE BOCKSTEIN HOMOMORPHISM

1 Show that the Bockstein homomorphism in homology (or cohomology) anticommutes with the boundary homomorphism (or coboundary homomorphism) of a pair.

For any prime p let β_p be the Bockstein homomorphism in either homology or cohomology for the short exact sequence of abelian groups

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$$

Let $\tilde{\beta}_p$ be the Bockstein homomorphism for the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\lambda_p} \mathbf{Z} \xrightarrow{\mu_p} \mathbf{Z}_p \to 0$$

where $\lambda_p(n) = pn$ and μ_p is reduction modulo p.

2 Prove that $\beta_p = (\mu_p)_* \circ \tilde{\beta}_p$.

3 Prove that $\beta_p \circ \beta_p = 0$.

4 Prove that $\beta_p(u \smile v) = \beta_p(u) \smile v + (-1)^{\deg u}\, u \smile \beta_p(v)$.

5 Prove that $Sq^{2i+1} = \beta_2 \circ Sq^{2i}$ for $i \geq 0$. [*Hint:* Show that there exist functorial homomorphisms $\{D_j\}_{j\geq 0}$, with D_j of degree j from the integral singular chain complex $\Delta(X)$ to $\Delta(X) \otimes \Delta(X)$, such that D_0 is a chain map commuting with augmentation and

$$\partial D_{2j-1} + D_{2j-1}\partial = D_{2j} - TD_{2j} \qquad j \geq 0$$
$$\partial D_{2j} - D_{2j}\partial = D_{2j-1} + TD_{2j-1} \qquad j > 0$$

where $T(\sigma_1 \otimes \sigma_2) = (-1)^{\deg \sigma_1 \deg \sigma_2}\, \sigma_2 \otimes \sigma_1$.]

6 Let ξ be a q-sphere bundle and let $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\, \mathbf{Z}_2)$ be its unique orientation over $\mathbf{Z}_2$. Prove that ξ is orientable over $\mathbf{Z}$ if and only if $\beta_2(U_\xi) = 0$.

H STIEFEL-WHITNEY CHARACTERISTIC CLASSES

Let ξ be a q-sphere bundle, with base space B, and let $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\, \mathbf{Z}_2)$ be its orientation class over $\mathbf{Z}_2$. The ith *Stiefel-Whitney characteristic class* $w_i(\xi) \in H^i(B;\mathbf{Z}_2)$ for $i \geq 0$ is defined by

$$\Phi_\xi^*(w_i(\xi)) = Sq^i(U_\xi)$$

1 Let $f: B' \to B$ be continuous. Prove that $f^*(w_i(\xi)) = w_i(f^*\xi)$.

2 If ξ is a product bundle, prove that $w_i(\xi) = 0$ for $i > 0$.

3 Prove the following:

(*a*) $w_0(\xi)$ is the unit class of $H^0(B;\mathbf{Z}_2)$.
(*b*) $\beta_2(w_{2i}(\xi)) = w_{2i+1}(\xi) + w_1(\xi) \smile w_{2i}(\xi)$ for $i \geq 0$.
(*c*) If ξ is a q-sphere bundle, then $w_i(\xi) = 0$ for $i > q + 1$, and $w_{q+1}(\xi)$ is the characteristic class of ξ over $\mathbf{Z}_2$.
(*d*) ξ is orientable over $\mathbf{Z}$ if and only if $w_1(\xi) = 0$.

If ξ is a q-sphere bundle over B and ξ' is a q'-sphere bundle over B', their *cross product* $\xi \times \xi'$ is a $(q + q' + 1)$-sphere bundle with $E_{\xi\times\xi'} = E_\xi \times E_{\xi'}$, $\dot{E}_{\xi\times\xi'} = E_\xi \times \dot{E}_{\xi'} \cup \dot{E}_\xi \times E_{\xi'}$ and $p_{\xi\times\xi'} = p_\xi \times p_{\xi'}$.

4 If $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\, \mathbf{Z}_2)$ and $U_{\xi'} \in H^{q'+1}(E_{\xi'},\dot{E}_{\xi'};\, \mathbf{Z}_2)$ are respective orientation classes, prove that

$$U_\xi \times U_{\xi'} \in H^{q+q'+2}(E_{\xi\times\xi'},\dot{E}_{\xi\times\xi'};\, \mathbf{Z}_2)$$

is the orientation class of $\xi \times \xi'$.

5 Prove that $w_k(\xi \times \xi') = \sum_{i+j=k} w_i(\xi) \times w_j(\xi')$.

If ξ and ξ' are sphere bundles with the same base space B, their *Whitney sum* $\xi \oplus \xi'$ is the sphere bundle over B induced from $\xi \times \xi'$ by the diagonal map $B \to B \times B$.

6 *Whitney duality theorem.* Prove that

$$w_k(\xi \oplus \xi') = \sum_{i+j=k} w_i(\xi) \smile w_j(\xi')$$

I HOMOLOGY WITH LOCAL COEFFICIENTS

If $\sigma: \Delta^q \to X$ is a singular q-simplex of X, with $q \geq 1$, let ω_σ be the path in X obtained by composing the linear path in Δ^q from v_0 to v_1 with σ. Given a local system Γ of

R modules on X, define $\Delta_q(X;\Gamma)$ to be the R module of finitely nonzero formal sums $\sum \alpha_\sigma \sigma$ in which σ varies over the set of singular q-simplexes of X and $\alpha_\sigma \in \Gamma(\sigma(v_0))$ is zero except for a finite set of σ. For $q > 0$ define a homomorphism $\partial: \Delta_q(X;\Gamma) \to \Delta_{q-1}(X;\Gamma)$ by

$$\partial(\alpha\sigma) = \sum_{0<i\leq q} (-1)^i \alpha\sigma^{(i)} + \Gamma(\omega_\sigma)(\alpha)\sigma^{(0)}$$

1 Prove that $\Delta(X;\Gamma) = \{\Delta_q(X;\Gamma), \partial\}$ is a chain complex which is free (or torsion free) if Γ is a local system of free (or torsion free) R modules, and if $A \subset X$, show that $\Delta(A; \Gamma \mid A)$ is a subcomplex of $\Delta(X;\Gamma)$.

The *homology of* (X,A) *with local coefficients* Γ, denoted by $H_*(X,A; \Gamma)$, is defined to be the graded homology module of $\Delta(X,A; \Gamma) = \Delta(X;\Gamma)/\Delta(A; \Gamma \mid A)$.

2 For a fixed ring R let $\mathcal{C}$ be the category whose objects are topological pairs (X,A), together with local systems Γ of R modules on X, and whose morphisms from (X,A) and Γ to (Y,B) and Γ' are continuous maps $f: (X,A) \to (Y,B)$, together with indexed families of homomorphisms $\{f_x: \Gamma(x) \to \Gamma'(f(x))\}_{x\in X}$. Prove that $H_*(X,A; \Gamma)$ is a covariant functor from $\mathcal{C}$ to the category of graded R modules.

3 *Exactness.* Given $A \subset B \subset X$ and a local system Γ of R modules on X, prove that there is an exact sequence

$$\cdots \to H_q(B,A; \Gamma \mid B) \to H_q(X,A; \Gamma) \to H_q(X,B; \Gamma) \to H_{q-1}(B,A; \Gamma \mid B) \to \cdots$$

4 *Excision.* Let X_1 and X_2 be subsets of a space X such that $X_1 \cup X_2 = \operatorname{int} X_1 \cup \operatorname{int} X_2$. For any local system Γ of R modules on X prove that the excision map j_1 from $(X_1, X_1 \cap X_2)$ and $\Gamma \mid X_1$ to $(X_1 \cup X_2, X_2)$ and $\Gamma \mid (X_1 \cup X_2)$ induces an isomorphism

$$j_{1*}: H_*(X_1, X_1 \cap X_2; \Gamma \mid X_1) \approx H_*(X_1 \cup X_2, X_2; \Gamma \mid (X_1 \cup X_2))$$

5 Two morphisms f and g in $\mathcal{C}$ from (X,A) and Γ to (Y,B) and Γ' are said to be *homotopic* in $\mathcal{C}$ if there is a homotopy $F: (X,A) \times I \to (Y,B)$ from f to g and an indexed family of homomorphisms $\{F_{(x,t)}: \Gamma(x) \to \Gamma'(F(x,t))\}_{(x,t)\in X\times I}$ such that $F_{(x,0)} = f_x$ and $F_{(x,1)} = g_x$. Prove that homotopy is an equivalence relation in the set of morphisms from (X,A) and Γ to (Y,B) and Γ' and that the composites of homotopic morphisms are homotopic (so that the homotopy category of $\mathcal{C}$ can be defined).

6 *Homotopy.* If f and g are morphisms from (X,A) and Γ to (Y,B) and Γ' and f is homotopic to g in $\mathcal{C}$, prove that $f_* = g_*: H_*(X,A; \Gamma) \to H_*(Y,B; \Gamma')$.

7 If Γ and Γ' are local systems of R modules on X, there is a local system $\Gamma \otimes \Gamma'$ on X with $(\Gamma \otimes \Gamma')(x) = \Gamma(x) \otimes \Gamma'(x)$ and $(\Gamma \otimes \Gamma')(\omega) = \Gamma(\omega) \otimes \Gamma'(\omega)$. In case Γ' is the constant local system equal to G, then prove that

$$\Delta(X,A; \Gamma \otimes G) \approx \Delta(X,A; \Gamma) \otimes G$$

Deduce a universal-coefficient formula for homology with local coefficients.

8 If Γ and Γ' are local systems of R modules on X and Y, respectively, let $\Gamma \times \Gamma' = p^*(\Gamma) \otimes p'^*(\Gamma')$ be the local system on $X \times Y$, where $p^*(\Gamma)$ and $p'^*(\Gamma')$ are induced from Γ and Γ', respectively, by the projections $p: X \times Y \to X$ and $p': X \times Y \to Y$. Prove that there is a natural chain equivalence of $\Delta(X;\Gamma) \otimes \Delta(Y;\Gamma')$ with $\Delta(X \times Y; \Gamma \times \Gamma')$. Deduce a Künneth formula for homology with local coefficients.

J COHOMOLOGY WITH LOCAL COEFFICIENTS

If Γ is a local system of R modules on X, define $\Delta^q(X;\Gamma)$ to be the module of functions φ assigning to every singular q-simplex σ of X an element $\varphi(\sigma) \in \Gamma(\sigma(v_0))$. Define a homomorphism $\delta: \Delta^q(X;\Gamma) \to \Delta^{q+1}(X;\Gamma)$ by

$$(\delta\varphi)(\sigma) = \sum_{0<i\leq q+1} (-1)^i \varphi(\sigma^{(i)}) + \Gamma(\omega_\sigma^{-1})(\varphi(\sigma^{(0)}))$$

1 Prove that $\Delta^*(X;\Gamma) = \{\Delta^q(X;\Gamma), \delta\}$ is a cochain complex and that if $A \subset X$, the restriction map $\Delta^*(X;\Gamma) \to \Delta^*(A; \Gamma \mid A)$ is an epimorphism.

The *cohomology of* (X,A) *with local coefficients* Γ, denoted by $H^*(X,A; \Gamma)$, is defined to be the graded cohomology module of

$$\Delta^*(X,A; \Gamma) = \ker [\Delta^*(X;\Gamma) \to \Delta^*(A; \Gamma \mid A)]$$

2 For a fixed ring R let $\mathcal{C}'$ be the category whose objects are topological pairs (X,A), together with local systems Γ of R modules on X, and whose morphisms from (X,A) and Γ to (Y,B) and Γ' are continuous maps $f: (X,A) \to (Y,B)$, together with indexed families of homomorphisms $\{f^x: \Gamma'(f(x)) \to \Gamma(x)\}_{x\in X}$. Prove that $H^*(X,A; \Gamma)$ is a contravariant functor from $\mathcal{C}'$ to the category of graded R modules.

3 Prove that the cohomology with local coefficients has exactness, excision, and homotopy properties analogous to those of the homology with local coefficients.

4 If Γ is a local system of R modules on X and G is an R module, there is a local system $\operatorname{Hom}(\Gamma,G)$ of R modules on X which assigns to $x \in X$ the module $\operatorname{Hom}(\Gamma(x),G)$. Prove that

$$\Delta^*(X,A; \operatorname{Hom}(\Gamma,G)) \approx \operatorname{Hom}(\Delta(X,A; \Gamma), G)$$

Deduce a universal-coefficient formula for cohomology with local coefficients.

Let ξ be a q-sphere bundle with base space B and let Γ_ξ be the local system on B such that $\Gamma_\xi(b) = H_{q+1}(E_b,\dot{E}_b)$. Let $p_\xi^*(\Gamma_\xi)$ be the local system on E_ξ induced from Γ_ξ by $p_\xi: E_\xi \to B$. A *Thom class* of ξ is an element $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi; p_\xi^*(\Gamma_\xi))$ such that for every $b \in B$ the element

$$U_\xi \mid (E_b,\dot{E}_b) \in H^{q+1}(E_b,\dot{E}_b; p_\xi^*(\Gamma_\xi) \mid E_b) = H^{q+1}(E_b,\dot{E}_b; H_{q+1}(E_b,\dot{E}_b))$$

corresponds to the identity map of $H_{q+1}(E_b,\dot{E}_b)$ under the universal-coefficient isomorphism

$$H^{q+1}(E_b,\dot{E}_b; H_{q+1}(E_b,\dot{E}_b)) \approx \operatorname{Hom}(H_{q+1}(E_b,\dot{E}_b), H_{q+1}(E_b,\dot{E}_b))$$

5 Prove that every q-sphere bundle has a unique Thom class. (*Hint:* Prove the result first for a product bundle, and then use Mayer-Vietoris sequences to extend the result to arbitrary bundles.)

6 Let ξ be a q-sphere bundle with a base space B and let U_ξ be its Thom class. If Γ is any local system of abelian groups on X, prove that the homomorphism

$$\Phi_\xi: H_n(E_\xi,\dot{E}_\xi; p^*(\Gamma)) \to H_{n-q-1}(B; \Gamma_\xi \otimes \Gamma)$$

such that $\Phi_\xi(z) = p_*(U_\xi \frown z)$, where $U_\xi \frown z$ is an element of $H_{n-q-1}(E; p^*(\Gamma_\xi \otimes \Gamma))$, is an isomorphism. If B is compact, prove that the homomorphism

$$\Phi_\xi^*: H^r(B;\Gamma) \to H^{r+q+1}(E_\xi,\dot{E}_\xi; p^*(\Gamma \otimes \Gamma_\xi))$$

such that $\Phi_\xi^*(v) = p^*(v) \smile U_\xi$ is an isomorphism.

By the Thom isomorphism theorem, this implies the result. ∎

In case Y is Euclidean space, $w_k(Y) = 0$ for $k > 0$, and theorem 22 shows that $\bar{w}_i$ and $w_j(X)$ determine each other recursively. In particular, the classes $\bar{w}_i$ are independent of the imbedding of X in the Euclidean space. If X is a compact n-manifold imbedded in $\mathbf{R}^{n+d}$, it follows from example 19 and 20 and from the fact that the Euler class of X in $\mathbf{R}^{n+d}$ is zero that $\bar{w}_i = 0$ for $i \geq d$. This gives the following necessary condition for imbeddability of X in $\mathbf{R}^{n+d}$.

23 COROLLARY *Let X be a compact n-manifold imbedded in $\mathbf{R}^{n+d}$ and let $\bar{w}_i \in H^i(X;\mathbf{Z}_2)$ be defined by*

$$\sum_{i+j=k} \bar{w}_i \smile w_j(X) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

Then $\bar{w}_i = 0$ for $i \geq d$. ∎

We present some examples.

24 For P^2, $\bar{w}_1(P^2) = w$ and $\bar{w}_2(P^2) = 0$, so P^2 cannot be imbedded in $\mathbf{R}^3$.

25 For P^3, $\bar{w}_i(P^3) = 0$ for $i > 0$.

26 For P^4, $\bar{w}_1(P^4) = w$, $\bar{w}_2(P^4) = w^2$, $\bar{w}_3(P^4) = w^3$, and $\bar{w}_4(P^4) = 0$. Therefore P^4 cannot be imbedded in $\mathbf{R}^7$.

27 For P^5, $\bar{w}_1(P^5) = 0$, $\bar{w}_2(P^5) = w^2$, $\bar{w}_3(P^5) = 0$, $\bar{w}_4(P^5) = 0$, and $\bar{w}_5(P^5) = 0$. Hence P^5 cannot be imbedded in $\mathbf{R}^7$ (which is also a consequence of example 26).

The last examples show the importance of calculating $w_i(P^n)$, which we now do.

28 THEOREM *Let $\binom{n}{i}_2$ be the binomial coefficient $\binom{n}{i} = n!/i!(n-i)!$ reduced modulo 2. Then*

$$w_i(P^n) = \binom{n+1}{i}_2 w^i$$

PROOF Since $\binom{n+1}{n}_2 \equiv n + 1 = \chi(P^n)$, the result is true for $i = n$. For $i < n$, where $n > 1$, we suppose P^{n-1} linearly imbedded in P^n. Then $P^n - P^{n-1}$ is an affine space, hence $\tilde{H}^*(P^n - P^{n-1}) = 0$ and $H^q(P^n, P^n - P^{n-1}) \approx H^q(P^n)$. Then the normal Thom class $\theta(1) \in H^1(P^n, P^n - P^{n-1})$ maps to w in $H^q(P^n)$, so $\bar{w}_1 = w$. By theorem 22, $\bar{w}_i(P^n) \mid P^{n-1} = w_i(P^{n-1}) + w \smile w_{i-1}(P^{n-1})$. Since $H^q(P^n) \approx H^q(P^{n-1})$ for $q < n$, it follows by induction on n that

$$\bar{w}_i(P^n) = [\binom{n}{i-1}_2 + \binom{n}{i}_2]w^i = \binom{n+1}{i}_2 w^i \quad ∎$$

EXERCISES

A MANIFOLDS

1 If X is an n-manifold with boundary $\dot{X}$, prove that X is a homology n-manifold whose boundary, as a homology manifold, equals $\dot{X}$.

In the rest of the exercises of this group, X will be an n-manifold without boundary and R will be a fixed principal ideal domain.

2 If Γ is a local system of R modules on X, prove that ~~for~~ any $A \subset X$

$$H_q(A \times X, A \times X - \delta(A); R \times \Gamma) = 0 \qquad q < n$$

(*Hint:* Prove this first for $\bar{A}$ contained in a coordinate neighborhood of X. Prove it next for compact $\bar{A}$ by using the Mayer-Vietoris technique. Then prove it for arbitrary A by taking direct limits over the family of compact subsets of A.)

3 Prove that there is a local system Γ_X of R modules on X such that $\Gamma_X(x) = H^n(X, X - x; R)$ for $x \in X$.

For $x \in X$ let $z_x \in H_n(X, X - x; \Gamma_X)$ be the generator corresponding under the isomorphism

$$H_n(X, X - x; \Gamma_X) \approx \operatorname{Hom}(H^n(X, X - x; R), H^n(X, X - x; R))$$

to the identity homomorphism of $H^n(X, X - x; R)$. A *Thom class* of X is an element

$$U \in H^n(X \times X, X \times X - \delta(X); R \times \operatorname{Hom}(\Gamma_X, R))$$

such that $(U \mid [x \times (X, X - x)])/z_x = 1 \in H^0(x;R)$ for all $x \in X$.

4 If V is an open subset of X and U is a Thom class of X, prove that $U \mid (V \times V, V \times V - \delta(V))$ is a Thom class of V.

5 Prove that $\mathbf{R}^n$ has a unique Thom class.

6 Prove that X has a unique Thom class. [*Hint:* Use exercise 2 to show that

$$H^n(X \times X, X \times X - \delta(X); R \times \operatorname{Hom}(\Gamma_X, R)) \approx \lim_{\leftarrow} \{H^n(V \times X, V \times X - \delta(V); R \times \operatorname{Hom}(\Gamma_X, R))\}$$

where V varies over finite unions of coordinate neighborhoods. Then the result follows from exercises 4 and 5 by Mayer-Vietoris techniques.]

If (A,B) is a pair in X and G is an R module, define

$$\gamma: H_q(X - B, X - A; \Gamma_X \otimes G) \to H^{n-q}(A,B; G)$$

by $\gamma(z) = [U \mid (A,B) \times (X - B, X - A)]/z$, where U is the Thom class of X. As (V,W) varies over neighborhoods of a closed pair (A,B) in X, there are isomorphisms

$$\lim_{\leftarrow} \{H_q(X - W, X - V; \Gamma_X \otimes G)\} \approx H_q(X - B, X - A; \Gamma_X \otimes G)$$

and

$$\lim_{\rightarrow} \{H^{n-q}(V,W; G)\} \approx \bar{H}^{n-q}(A,B; G)$$

and a homomorphism

$$\bar{\gamma}: H_q(X - B, X - A; \Gamma_X \otimes G) \to \bar{H}^{n-q}(A,B; G)$$

is defined by passing to the limit with γ.

7 *Duality theorem.* Prove that for a compact pair (A,B) in X, $\bar{\gamma}$ is an isomorphism.

B THE INDEX OF A MANIFOLD

1 Let X be a compact n-manifold, with boundary $\dot{X}$ oriented over a field R, and let $[X] \in H_n(X,\dot{X}; R)$ be the corresponding fundamental class. For $u \in H^q(X,\dot{X}; R)$ and $v \in H^{n-q}(X;R)$ prove that $\varphi_X(u,v) = \langle u \smile v, [X]\rangle \in R$ is a nonsingular bilinear form from $H^q(X,\dot{X}) \times H^{n-q}(X)$ to R [that is, $u = 0$ if and only if $\varphi_X(u,v) = 0$ for all v].

2 With the same hypotheses as above, let $[\dot{X}] = \partial[X] \in H_{n-1}(\dot{X};R)$ and let $\varphi_{\dot{X}}$ be the corresponding bilinear form from $H^{q-1}(\dot{X};R) \times H^{n-q}(\dot{X};R)$ to R. Let $j: \dot{X} \subset X$, and if $u \in H^{q-1}(\dot{X};R)$ and $v \in H^{n-q}(X;R)$, prove that

$$\varphi_{\dot X}(u, j^*(v)) = \varphi_X(\delta(u), v)$$

3 Prove that the Euler characteristic of any odd-dimensional compact manifold is 0 and the Euler characteristic of an even-dimensional compact manifold which is a boundary is even. (*Hint:* If $\dot X$ is the boundary of a $(2n+1)$-manifold X, then, with Z_2 coefficients,

$$\dim \operatorname{im} [j^*: H^n(X) \to H^n(\dot X)] = \dim \operatorname{im} [\delta: H^n(\dot X) \to H^{n+1}(X,\dot X)]$$

and their sum equals $\dim H^n(\dot X)$.)

Let Y be a compact $4m$-manifold, without boundary oriented over R, and define the *index* of Y to be the index of the nonsingular bilinear form φ_Y from $H^{2m}(Y;R) \times H^{2m}(Y;R)$ to R (when φ_Y is represented as a sum of k squares minus a sum of j squares, the index of φ_Y is $k - j$).

4 If Y is oppositely oriented, prove that its index changes sign. Show that the index of the product of oriented manifolds is the product of their indices.

5 If X is a compact $(4m+1)$-manifold, with boundary $\dot X$ oriented over R, prove that the index of $\dot X$ is 0. [*Hint:* Prove that $j^*(H^{2m}(X;R))$ is a subspace of $H^{2m}(\dot X;R)$ whose dimension equals one-half the dimension of $H^{2m}(\dot X;R)$ and on which $\varphi_{\dot X}$ is identically zero. This implies the result.]

C CONTINUITY

1 Let $\{(X_j,A_j), \pi_j^k\}_{j\in J}$ be an inverse system of compact Hausdorff pairs and let $(X,A) = \lim_{\leftarrow} \{(X_j,A_j)\}$. Prove that (X,A) can be imbedded in a space in which it is a directed intersection of compact Hausdorff pairs $\{(X_j',A_j')\}_{j\in J}$, where (X_j',A_j') has the same homotopy type as (X_j,A_j). [*Hint:* For each $j \in J$ imbed X_j in a contractible compact Hausdorff space Y_j, for example, a cube, and let $(X_k',A_k') \subset \times_{j\in J} Y_j$ be defined as the pair of all points (y_j) with y_k in X_k or in A_k, respectively, such that if $j \le k$, then $y_j = \pi_j^k(y_k)$, and if $j \not\le k$, then y_j is arbitrary.]

2 Prove that a cohomology theory has the continuity property if and only if it has the weak continuity property.

3 The *p-adic solenoid* is defined to be the inverse limit of the sequence

$$S^1 \xleftarrow{f} S^1 \leftarrow \cdots \leftarrow S^1 \xleftarrow{f} S^1 \leftarrow \cdots$$

where $f(z) = z^p$. Compute the Alexander cohomology groups of the p-adic solenoid for coefficients Z, Z_p, and R.

4 Generalize the solenoid of the preceding example to the case where there is a sequence of integers $n_1, n_2, \ldots$ such that the mth map of S^1 to S^1 sends z to z^{n_m}. Compute the integral Alexander cohomology groups of the resulting space.

5 Find a compact Hausdorff space X such that $\bar{\bar H}{}^q(X;Z) = 0$ if $q \ne 1$ and $\bar H^1(X;Z) \approx R$.

D ČECH COHOMOLOGY THEORY

1 Let $(\mathfrak{U},\mathfrak{U}')$ be an open covering of (X,A) ($\mathfrak{U}$ is an open covering of X and $\mathfrak{U}' \subset \mathfrak{U}$ is a covering of A) and let $K(\mathfrak{U})$ be the nerve of $\mathfrak{U}$ and $K'(\mathfrak{U}')$ the subcomplex of $K(\mathfrak{U})$ which is the nerve of $\mathfrak{U}' \cap A = \{U' \cap A \mid U' \in \mathfrak{U}'\}$. Prove that the chain complexes $(C(K(\mathfrak{U})),C(K'(\mathfrak{U}')))$ and $(C(X(\mathfrak{U})),C(A(\mathfrak{U}')))$ are canonically chain equivalent. (*Hint:* If $s = \{U_0, \ldots, U_q\}$ is a simplex of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$], let $\lambda(s)$ be the subcomplex of $X(\mathfrak{U})$ [or of $A(\mathfrak{U}')$] generated by all simplexes of $X(\mathfrak{U})$ [or of $A(\mathfrak{U}')$] in $\cap U_i$. If $s' = \{x_0, \ldots, x_q\}$ is a simplex of $X(\mathfrak{U})$ [or of $A'(\mathfrak{U}')$], let $\mu(s')$ be the subcomplex of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$] generated by all simplexes $\{U_0, \ldots, U_r\}$ of $K(\mathfrak{U})$ [or of $K'(\mathfrak{U}')$] such that U_i contains s' for $0 \le i \le r$. Then $C(\lambda(s))$ and $C(\mu(s'))$ are acyclic, and the method of acyclic models can be applied to prove the existence of chain maps

$$\tau: (C(K(\mathfrak{U})),C(K'(\mathfrak{U}'))) \to (C(X(\mathfrak{U})),C(A(\mathfrak{U}')))$$
$$\tau': (C(X(\mathfrak{U})),C(A(\mathfrak{U}))) \to (C(K(\mathfrak{U})),C(K'(\mathfrak{U}')))$$

such that $\tau(C(s)) \subset C(\lambda(s))$ and $\tau'(C(s')) \subset C(\mu(s'))$. Similarly, the method of acyclic models shows that τ and τ' are chain homotopy inverses of each other.[1])

2 Let $(\mathfrak{V},\mathfrak{V}')$ be a refinement of $(\mathfrak{U},\mathfrak{U}')$, let $\pi: (K(\mathfrak{V}),K'(\mathfrak{V}')) \to (K(\mathfrak{U}),K'(\mathfrak{U}'))$ be a projection map, and let $j: (X(\mathfrak{V}),A(\mathfrak{V}')) \subset (X(\mathfrak{U}),A(\mathfrak{U}'))$. For any abelian group G prove that there is a commutative diagram

$$\begin{array}{ccc} H^*(K(\mathfrak{U}),K'(\mathfrak{U}'); G) & \approx & H^*(X(\mathfrak{U}),A(\mathfrak{U}'); G) \\ \pi^*\downarrow & & \downarrow j^* \\ H^*(K(\mathfrak{V}),K'(\mathfrak{V}'); G) & \approx & H^*(X(\mathfrak{V}),A(\mathfrak{V}'); G) \end{array}$$

where the horizontal maps are induced by the canonical chain equivalences of exercise 1 above.

3 The *Čech cohomology group* of (X,A) with coefficients G is defined by $\check H^*(X,A; G) = \lim_{\to} \{H^*(K(\mathfrak{U}),K'(\mathfrak{U}'); G)\}$. Prove that there is a natural isomorphism

$$\check H^*(X,A; G) \approx \bar H^*(X,A; G).$$

4 If $\dim (X - A) \le n$, prove that $\check H^q(X,A; G) = 0$ for all $q > n$ and all G.

E THE KÜNNETH FORMULA FOR ČECH COHOMOLOGY

If K_1 and K_2 are simplicial complexes, their *simplicial product* $K_1 \Delta K_2$ is the simplicial complex whose vertex set is the cartesian product of the vertex sets of K_1 and of K_2 and whose simplexes are sets $\{(v_0,w_0), \ldots, (v_q,w_q)\}$, where $v_0, \ldots, v_q$ are vertices of some simplex of K_1 and $w_0, \ldots, w_q$ are vertices of some simplex of K_2.

1 Prove that $K_1 \Delta K_2$ is a simplicial complex, and if $L_1 \subset K_1$ and $L_2 \subset K_2$, then $L_1 \Delta L_2 \subset K_1 \Delta K_2$.

2 For simplicial pairs (K_1,L_1) and (K_2,L_2) define

$$(K_1,L_1) \Delta (K_2,L_2) = (K_1 \Delta K_2, K_1 \Delta L_2 \cup L_1 \Delta K_2)$$

Prove that $C((K_1,L_1) \Delta (K_2,L_2))$ is canonically chain equivalent to $C(K_1,L_1) \otimes C(K_2,L_2)$. (*Hint:* Use the method of acyclic models.)

3 If $(\mathfrak{U},\mathfrak{U}')$ is an open covering of (X,A) and $(\mathfrak{V},\mathfrak{V}')$ is an open covering of (Y,B), let $(\mathfrak{U},\mathfrak{U}') \times (\mathfrak{V},\mathfrak{V}') = (\mathfrak{W},\mathfrak{W}')$ be the open covering of $(X,A) \times (Y,B)$, where

$$\mathfrak{W} = \{U \times V \mid U \in \mathfrak{U}, V \in \mathfrak{V}\}$$

and $\mathfrak{W}' = \{U \times V \mid U \in \mathfrak{U}', V \in \mathfrak{V}'\}$. Prove that

$$(K(\mathfrak{W}),K'(\mathfrak{W}')) = (K(\mathfrak{U}),K'(\mathfrak{U}')) \Delta (K(\mathfrak{V}),K'(\mathfrak{V}'))$$

4 If (X,A) and (Y,B) are compact Hausdorff pairs, prove that the family of coverings of $(X,A) \times (Y,B)$ of the form $(\mathfrak{U},\mathfrak{U}') \times (\mathfrak{V},\mathfrak{V}')$ is cofinal in the family of all open coverings of $(X,A) \times (Y,B)$.

[1] For details see C. H. Dowker, Homology groups of relations, *Annals of Mathematics*, vol. 56, pp. 84–95, 1956.

5 If (X,A) and (Y,B) are compact Hausdorff pairs and G and G' are modules such that $G * G' = 0$, prove that there is a short exact sequence

$$0 \to (\check{H}_1^* \otimes \check{H}_2^*)^q \to \check{H}^q((X,A) \times (Y,B); G \otimes G') \to (\check{H}_1^* * \check{H}_2^*)^{q+1} \to 0$$

where $\check{H}_1^* = \check{H}^*(X,A; G)$ and $\check{H}_2^* = \check{H}^*(Y,B; G')$.

6 Let (X,A) and (Y,B) be locally compact Hausdorff pairs with A and B closed in X and Y, respectively. If G and G' are modules such that $G * G' = 0$, prove that there is a short exact sequence

$$0 \to (\bar{H}_{c,1}^* \otimes \bar{H}_{c,2}^*)^q \to \bar{H}_c^q((X,A) \times (Y,B); G * G') \to (\bar{H}_{c,1}^* * \bar{H}_{c,2}^*)^{q+1} \to 0$$

where $\bar{H}_{c,1}^* = \bar{H}_c^*(X,A; G)$ and $\bar{H}_{c,2}^* = \bar{H}_c^*(Y,B; G')$.

F LOCAL SYSTEMS AND SHEAVES

Throughout this group of exercises we assume X to be a paracompact Hausdorff space.

1 If Γ is a local system on X, let $\hat{\Gamma}$ be the presheaf on X such that for an open set $V \subset X$, $\hat{\Gamma}(V)$ is the set of all functions f assigning to each $x \in X$ an element $f(x) \in \Gamma(x)$ with the property that for any path ω in V, $f(\omega(1)) = \Gamma(\omega)(f(\omega(0)))$. Prove that $\hat{\Gamma}$ is a sheaf on X and the association of $\hat{\Gamma}$ to Γ is a natural transformation from local systems to sheaves.

2 A presheaf Γ on X is said to be *locally constant* if there is an open covering $\mathfrak{U} = \{U\}$ of X such that if $U \in \mathfrak{U}$ and $x \in U$, then $\Gamma(U) \approx \lim_{\rightarrow} \{\Gamma(V)\}$, where V varies over open neighborhoods of x. If $U \in \mathfrak{U}$ and U' is a connected open subset of U, prove that the composite

$$\Gamma(U) \to \Gamma(U') \to \hat{\Gamma}(U')$$

is an isomorphism. Deduce that if Γ is a locally constant sheaf and U' is a connected open subset of $U \in \mathfrak{U}$, then $\Gamma(U) \approx \Gamma(U')$.

3 If X is locally path connected and Γ' is a locally constant sheaf on X, prove that there is a local system Γ on X such that $\hat{\Gamma} \approx \Gamma'$.

4 If X is locally path connected and semilocally 1-connected, prove that there is a one-to-one correspondence between equivalence classes of local systems on X and equivalence classes of locally constant sheaves on X.

5 If Γ is a local system of R modules on X, let $\Delta^q(\cdot;\Gamma)$ be the presheaf on X such that $\Delta^q(\cdot;\Gamma)(V) = \Delta^q(V;\Gamma \mid V)$ for V open in X. Prove that $\Delta^q(\cdot;\Gamma)$ is fine.

6 If Γ is a local system of R modules on X, let $\Delta^*(\cdot;\Gamma)$ be the cochain complex of presheaves $\Delta^q(\cdot;\Gamma)$ on X and let $\hat{\Delta}^*(\cdot;\Gamma)$ be the cochain complex of completions $\hat{\Delta}^q(\cdot;\Gamma)$. Prove that there is an isomorphism

$$H^*(\Delta^*(\cdot;\Gamma)(X)) \approx H^*(\hat{\Delta}^*(\cdot;\Gamma)(X))$$

7 Let Γ be a local system of R modules on X and assume that $H^q(\Delta^*(\cdot;\Gamma))$ is locally zero on X for all $q > 0$. Prove that there is an isomorphism

$$\bar{H}^*(X;\hat{\Gamma}) \approx H^*(X;\Gamma)$$

(*Hint:* Note that $\hat{\Gamma} = H^0(\Delta^*(\cdot;\Gamma))$ and apply theorem 6.8.7.)

G SOME PROPERTIES OF EUCLIDEAN SPACE

1 Find a compact subset X of $\mathbf{R}^2$ that is n-connected for all n and such that $\bar{H}^1(X;\mathbf{Z}) \approx \mathbf{Z}$.

2 If X is a compact subset of $\mathbf{R}^n$ and $\dim X < n - 1$, prove that $\mathbf{R}^n - X$ is connected.

Let A_1 and A_2 be disjoint closed subsets of $\mathbf{R}^n$ and let $z_1 \in H_p(A_1;R)$ and $z_2 \in H_q(A_2;R)$, with $p + q = n - 1$. If $z_1 \in \tilde{H}_p(A_1;R)$, let $z_1' \in H_{p+1}(\mathbf{R}^n,\mathbf{R}^n - A_2;R)$ be the image of z_1 under the composite

$$\tilde{H}_p(A_1) \to \tilde{H}_p(\mathbf{R}^n - A_2) \xrightarrow[\approx]{\partial^{-1}} H_{p+1}(\mathbf{R}^n, \mathbf{R}^n - A_2)$$

The *linking number* $\mathrm{Lk}\,(z_1,z_2) \in R$ is defined by

$$\mathrm{Lk}\,(z_1,z_2) = \langle \gamma_U(z_1'),z_2 \rangle$$

where U is an orientation class of $\mathbf{R}_n$ over R fixed once and for all.

3 Prove that $\mathrm{Lk}\,(z_1,z_2) = \langle U, i_*(z_2 \times z_1') \rangle$, where

$$i: A_2 \times (\mathbf{R}^n, \mathbf{R}^n - A_2) \subset (\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n - \delta(\mathbf{R}^n))$$

4 Assume that $\mathrm{Lk}\,(z_2,z_1)$ is also defined [that is, $z_2 \in \tilde{H}_q(A_2)$]. Prove that $\mathrm{Lk}\,(z_1,z_2) = (-1)^{pq+1}\,\mathrm{Lk}\,(z_2,z_1)$.

5 Let A_1 be a p-sphere and A_2 a q-sphere imbedded as disjoint subsets of $\mathbf{R}^n$, where $p + q = n + 1$. Prove that $H_p(A_1) \to H_p(\mathbf{R}^n - A_2)$ is trivial if and only if $H_q(A_2) \to H_q(\mathbf{R}^n - A_1)$ is trivial.

H IMBEDDINGS OF MANIFOLDS IN EUCLIDEAN SPACE

1 Prove that a compact n-manifold which is nonorientable over $\mathbf{Z}$ cannot be imbedded in $\mathbf{R}^{n+1}$.

2 Let X be a compact connected n-manifold imbedded in $\mathbf{R}^{n+1}$ and let U and V be the components of $\mathbf{R}^{n+1} - X$. Let $i: X \subset \mathbf{R}^{n+1} - U$ and $j: X \subset \mathbf{R}^{n+1} - V$ and prove that over any R, $i^*(\bar{H}^*(\mathbf{R}^{n+1} - U))$ and $j^*(\bar{H}^*(\mathbf{R}^{n+1} - V))$ are subalgebras of $\bar{H}_*(X)$ and there is a direct-sum representation

$$\{i^*,j^*\}: \bar{H}^q(\mathbf{R}^{n+1} - U) \oplus \bar{H}^q(\mathbf{R}^{n+1} - V) \approx \bar{H}^q(X) \qquad 0 < q < n$$

3 Prove that for $n \geq 2$ the real projective n-space P^n cannot be imbedded in $\mathbf{R}^{n+1}$.