① Topological group $(G, \cdot, e)$

$G\times G \to G \qquad G \to G^{-1}$ continuous

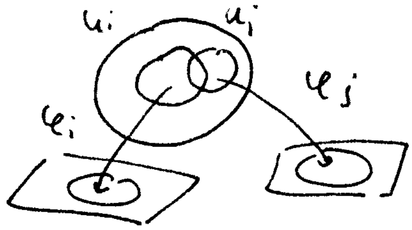
Equivalent definition

$G\times G \to G\times G$
$(g,h) \mapsto (g, gh)$ is a homeomorphism
$(g, g^{-1}k) \leftarrow (g,k)$

If G topological space has a group structure, then it is homogeneous: $\forall x,y\in G$ $\exists\, U\ni x$ $V \ni y$ neighbourhoods and homeo $(U,x) \underset{\varphi}{\simeq} (V,y)$ (e.g. $U=V$ $\varphi = L_{yx^{-1}}$ left ~~~~ translation
$L_{yx^{-1}}(z) = yx^{-1}z$

__Def__ G is a Lie group $\iff$ G is locally $\mathbb{R}^n$
(i.e. G is a manifold)

__Theorem__[?] If a topological group is a topological manifold then it admits a unique smooth structure (i.e. it is a differential manifold)



$\varphi_i : U_i \xrightarrow[\text{open}]{} \mathbb{R}^n$

$\varphi_j \varphi_i^{-1} : \varphi_i(U_i\cap U_j) \to \varphi_j(U_i \cap U_j)$

__Examples__ $(\mathbb{R}, +, 0) = \mathbb{R}_+$ $\qquad (\mathbb{R}-\{0\}, \cdot, 1) = \mathbb{R}^*$
$(\mathbb{C}, +, 0) = \mathbb{C}_+$ $\qquad (\mathbb{C}-\{0\}, \cdot, 1) = \mathbb{C}^*$
$S^1 = \{z\in\mathbb{C} : |z| = 1\}$
discrete groups (assume: countable)
$\mathbb{R}^* \simeq \mathbb{R}_+ \times \mathbb{Z}_2$ $\qquad \mathbb{C}^* \simeq \mathbb{R}_+ \times S^1$
$x \mapsto \left(\ln|x|, \frac{x}{|x|}\right)$ $\qquad z \mapsto \left(\ln|z|, \frac{z}{|z|}\right)$

Homomorphisms $\mathbb{R}_+ \xrightarrow[\varphi]{} S^1$ $\qquad \mathbb{C}_+ \to \mathbb{C}^*$
$t \mapsto \exp(2\pi i t)$ $\qquad z \longmapsto \exp z$

$\ker \varphi = \mathbb{Z}$ $\quad \mathbb{R}_+/\mathbb{Z} \simeq S^1$ $\qquad \ker \exp = 2\pi i\mathbb{Z}$ $\quad \mathbb{C}_+/2\pi i\mathbb{Z} \simeq \mathbb{C}^*$

(2) Noncommutative examples

Matrix groups $Gl_n(\mathbb{R}) = Gl(n, \mathbb{R}) = Gl(\mathbb{R}^n)$
$\mathbb{C}$

invertible matrices
General Linear

Subgroups: $O(n)$ ortogonal
- $SO(n)$ special ortogonal
- $U(n)$ unitary
- $SU(n)$ special unitary
- $Sp(n)$ symplectic (compact)

- every closed subgroup of a ~~Lie~~ Gl_n ~~group~~ is a Lie group
- every Lie group ~~embedds~~ admits "almost embedding"

Special role of compact groups
- every Lie group $G \underset{\text{diffeo}}{\cong} K \times \mathbb{R}^n$ ← maximal compact

- every compact, connected L.G. is of the form

$$\underbrace{G_1 \times G_2 \times \dots \times G_n}_{\text{simple Lie groups}} \times (S^1)^k / A$$ ← discrete abelian

- classification of simple Lie groups
Moral: Study Matrix Groups

③ Start systematic study with review of division algebras

$\mathbb{R} \subsetneq \mathbb{C}$ the only nontrivial field extension of finite degree

$\mathbb{H}$ quaternions: the only division $\mathbb{R}$-algebra bigger than $\mathbb{C}$ (noncommutative, associative, with unit containing $\mathbb{R}$, everything is commuting with $\mathbb{R}$, s.t. $\forall a \neq 0$ exists a inverse)

$\mathbb{C} = \mathbb{R} + \mathbb{R}i$

$H = \mathbb{C} + \mathbb{C}j \qquad j^2 = -1 \qquad k := ij = -ji \qquad H = \underbrace{R}_{\text{real}} + \underbrace{\mathbb{R}i + \mathbb{R}j + \mathbb{R}k}_{\text{imaginary = pure quaternion}}$

$k^2 = ij(-ji) = -ij^2i = -1$

$jk = i = -kj \qquad ki = j = -ik$

Matrix representation: $i \leftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad j \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad k \leftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$H \ni a + bj \longleftrightarrow \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$

conjugation $\bar{i} = -i \quad \bar{j} = -j \quad \bar{k} = -k$ anti-automorphism

$$A^* = \bar{A}^T$$

There are at least 3 copies of $\mathbb{C}$ in $\mathbb{H}$ $\left(\mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}, \mathbb{H}) \simeq S^2\right)$

Scalar product $\langle x, y \rangle = \mathrm{Re}\,(x\bar{y})$

- $\langle xy, xy \rangle = \langle x x \rangle \langle y, y \rangle$

  $\|xy\| = \|x\| \cdot \|y\|$

$x = a + bi + cj + dk$

- $x \cdot \bar{x} = \|x\|^2 = a^2+b^2+c^2+d^2 \qquad x^{-1} = \dfrac{\bar{x}}{\|x\|^2}$

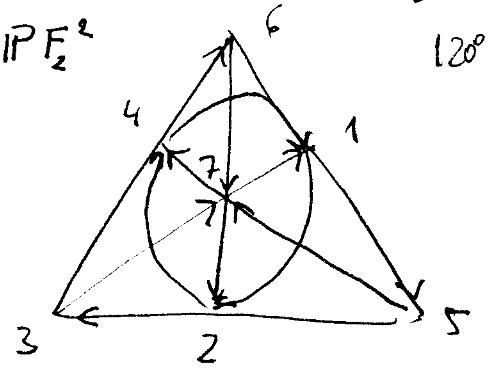
- $\omega(x,y,z) = \langle x \cdot y, z \rangle$ 3-form, antisymmetric on im

  ω contains all information about multiplication of $\mathbb{H}$

Further extension: nonassociative, admits $\langle , \rangle$, conjugation

Octonions ("alternative" ie subalgebra generated by 2 elements is associative). $\mathbb{O}$ unique with this property

$\mathbb{P}\mathbb{F}_2^2$



$120°$ ↻ doubling of indices mod 7

$e_i^2 = -1 \qquad$ e.g. $e_3 \cdot e_7 = e_1$ cycl.

line - copy of $H \subset \mathbb{O}$

vertex copy of $\mathbb{C} \subset \mathbb{O}$

④ For $k = \mathbb{R}\ \mathbb{C}\ \mathbb{H}$ define groups of units

$\{ z \in \mathbb{K} : \|z\| = 1 \}$ $\quad \mathbb{Z}_2 = S^0 \quad S^1 \quad S^3$

$$S^3 = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in Gl_2(\mathbb{C}) : |a|^2 + |b|^2 = 1 \right\} \simeq SU_2$$

$$= \{ A \in Gl_2(\mathbb{C}) : \det A = 1 \quad \bar{A}^T \cdot A = 1 \}$$

Another group $\quad Aut(H) \supset Inn(H)$ = inner automorphism

$x \mapsto h x h^{-1}$

Prop $\quad Aut(H) = Inn(H) = H^*/\mathbb{R}^* = S^3/\{\pm 1\} = SO(3)$

How to get something from $\mathbb{O}$

$Aut(\mathbb{O}) := G_2$ exceptional group (one of the list)

- G_2 preserves $\operatorname{Im} \mathbb{O}$

$G_2 \hookrightarrow GL(\operatorname{Im}\mathbb{O}) = Gl_7(\mathbb{R})$

- G_2 preserves $\langle , \rangle$ $\quad G_2 \subset SO(7)$.

<u>General linear groups</u>

$Gl_n(K) = Aut_K(K^n)$ as left k-modules

Embedding $\quad Gl_n(\mathbb{C}) \longrightarrow Gl_{2n}(\mathbb{R})$

$$\mathbb{C}^n = \underset{a}{\mathbb{R}^n} + \underset{b}{\mathbb{R}^n} \cdot i \qquad i(a+bi) = -b + ai$$

$\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$-linear $\iff$ φ is $\mathbb{R}$-linear & commutes with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & -A \\ D & -C \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}$$

$$B = -C \qquad D = A \qquad \begin{pmatrix} A & -C \\ C & A \end{pmatrix}$$

If K is noncommutative one has to be careful

$$Gl_1(H) = (H^*)^{op}$$

$\varphi_a : H \to H$ $\qquad \varphi_a \varphi_b(1) = \varphi_a(b) = ba$

$1 \to a$

$x \to xa$ $\qquad \varphi_a \circ \varphi_b = \varphi_{ba}$

$$Gl_1(H) \simeq H^*$$

$$\varphi_{a^{-1}} \leftarrow a$$

(5) Description of $Gl_n(\mathbb{H})$

$H^n = \underset{u}{\mathbb{C}^n} \oplus \underset{v}{\mathbb{C}^n} j \qquad u + vj \qquad u, v \in \mathbb{C} \qquad ji = -ij$

$j : H^n \to H^n \qquad ju + jvj = \bar{u} j - \bar{v} j^2 = -\bar{v} + \bar{u} j \qquad (j i = \bar{i} j)$

(not $\mathbb{C}$-linear) $\quad j = J \circ c = c \circ J \qquad c(u,v) = (\bar{u}, \bar{v})$ conjugation

$\mathbb{C}$-linear $\quad J(u,v) = (-v, u) \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

$\varphi : H^n \to H^n$ is $\mathbb{H}$-linear $\iff$ $\mathbb{C}$-linear & commutes with j

$M(\varphi) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\begin{array}{ccc} (u,v) & \longmapsto & (Au + Bv,\ Cu + Dv) \\ j \downarrow & & \downarrow j \\ & & (-\bar{C}\bar{u} + \bar{D}\bar{v},\ \bar{A}\bar{u} + \bar{B}\bar{v}) \\ (-\bar{v}, \bar{u}) & \longrightarrow & (-A\bar{v} + B\bar{u},\ -C\bar{v} + D\bar{u}) \end{array}$$

$A = \bar{D} \qquad B = -\bar{C}$

$$\begin{pmatrix} A & -\bar{C} \\ C & \bar{A} \end{pmatrix} \in Gl_n(\mathbb{H}) \subset Gl_{2n}(\mathbb{C})$$

Compact groups: subgroups of $Gl_n(\mathbb{K})$ preserving $\langle\, ,\, \rangle$

$Gl_n(\mathbb{R}) \supset O_n$ ~~$(\mathbb{R})$~~ closed in $(S^{n-1})^n$

$\forall v, w \in \mathbb{R}^n \quad \langle Av, Aw \rangle = \langle v, w \rangle$

$(Av)^T Aw = v^T A^T A w = v^T w \implies A^T A = I \quad \boxed{A^{-1} = A^T}$

Unitary $\quad Gl_n(\mathbb{C}) \supset U(n) \qquad \bar{A}^T A = I \qquad A^{-1} = \bar{A}^T$

$U(n) = Gl_n(\mathbb{C}) \cap O_{2n}$

(Hermitian product $\langle\langle u, v \rangle\rangle = u^T \bar{v} = \langle u, v \rangle + \omega(u,v) \cdot i$

$\omega(u,v) = \langle u, iv \rangle = \langle -iu, v$

$Sp(n) = Gl_n(\mathbb{H}) \cap U(2n) = Gl_n(\mathbb{H}) \cap O(4n)$

$A \in U(2n)$ is symplectic if $jA = Aj \iff$ ~~$JcA = AJc$~~

$\iff cJA = AcJ \iff cJA = c\bar{A}J \iff JA = \bar{A}J$

A unitary $A^T = \bar{A}^{-1}$ get $A^T J A = J$.

Get $Sp(n) = \{ A \in U(2n) : A$ preserves the bilinear antisymmetric form $J \}$

Other incarnations: $\underset{\text{complex}}{Sp_n(\mathbb{C})}$ – without unitary condition

$Sp_n(\mathbb{C}) \supset Sp_n(\mathbb{R})$ real, noncompact $\supset U(n)$ max compact

$Sp_n(\mathbb{C}) \supset Sp(n)$ compact

(6) Special groups: $\det A = 1$ $Sl_n(\underset{\mathbb{C}}{\mathbb{R}})$

$$SO_{2n} \subset O_{2n} \subset Gl_{2n}(\mathbb{R})$$
$$\cup \qquad \cup$$
$$U_n \subset GL_n(\mathbb{C})$$

$$SU_{2n} \subset U_{2n} \subset Gl_{2n}(\mathbb{C})$$
$$\cup \qquad \cup$$
Exercise
$$Sp(n) \subset Gl_n(\mathbb{H})$$

$Sp(n)$ is simple. (no normal subgroups) det for H does not exist

Every compact Lie group G admits a complexification $G_{\mathbb{C}}$ in which G is a maximal compact, $G_{\mathbb{C}}/G \simeq \mathbb{R}^{\dim G}$

$O_n \subset O_n(\mathbb{C}) \subset Gl_n(\mathbb{C})$ preserving a symmetric non deg 2 form

$SO_n \subset SO_n(\mathbb{C})$ complex matrices $A^T \cdot A = I$ $(\det A = 1)$

$U(n) \subset Gl_n(\mathbb{C})$

$SU(n) \subset Sl_n(\mathbb{C})$

$Sp(n) \subset Sp_n(\mathbb{C})$ the group preserving the antisymmetric form

complex matrices $A^T J A = J$

$G_2 \subset G_2(\mathbb{C}) \subset Gl_7(\mathbb{C})$ the group preserving 3-form

in fact $G_2 \subset SO_7(\mathbb{C})$

For complex groups compact Lie groups ~~have no importance~~ plays different role:

<u>Theorem</u> Every complex ~~and~~ compact connected group is abelian. ($\Rightarrow$ iso $\mathbb{C}^n/\Lambda$ Λ - lattice $\Lambda \simeq \mathbb{Z}^{2n}$)

Instead of compact one considers reductive groups. They are of the form $G_{1\mathbb{C}} \times \dots G_{n\mathbb{C}} \times (\mathbb{C}^*)^n/A$ where $G_{i\mathbb{C}}$ are complexifications of simple compact groups.

Algebraic characterization: Maximal normal solvable subgroup is a complex torus $(\mathbb{C}^*)^n$.