Representation theory of Lie group and algebras

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Bibliography:

- [FH] Fulton, W.; Harris, J. - Representation theory. A first course.
- [Segal] Segal's part in: Carter, R.; Segal, G.; Macdonald, I. - Lectures on Lie groups and Lie algebras. London Mathematical Society Student Texts, 32.
- [Adams] Adams, J.F. - Lectures on Lie groups.
- [Knapp] Knapp, A. W. - Representation theory of semisimple groups. An overview based on examples.
- [Bröcker-tom Dieck] Bröcker, Theodor; tom Dieck, Tammo - Representations of compact Lie groups. Graduate Texts in Mathematics, 98.

1 Mainly examples and overview

1.1 Topological groups [Bredon: Introduction to compact transformation groups, chapter 0]

- multiplication and taking the inverse are continuous
- equivalently $\phi : G^2 \to G^2$, $\phi(g,h) = (g, gh)$ is a homeomorphism
- „group object" in the category of topological spaces

1.2 Examples

- discrete groups
- $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{K}^*$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$
- compact torus $(S^1)^r$
- complex torus $(\mathbb{C}^*)^r$
- S^3 as a subgroup of $\mathbb{H}^*$
- $U(n)$, $SU(n)$ subgroups of $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$
- $O(n)$, $SO(n)$ subgroups of $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$
- $Sp(n)$ the subgroup of $GL_n(\mathbb{H})$ preserving the norm $|v|^2 = \sum_{i=1}^n |v_i|^2$
- Matrix groups defined by some identity e.g. preserving a given quadratic form (or trilinear form, octonionic multiplication etc)
- $O(m,n)$, the subgroup of $GL_{m+n}(\mathbb{R})$ preserving a nondegenerate symmetric form of the type (m,n).
- groups of isometries of a compact Riemannian manifold (can be realized as a matrix group)
- Heisenberg group N/Z where

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

cannot be realized as a matrix group

1.3 Exercise: $U(n)$, $SU(N)$, $SO(n)$, $Sp(n)$ are connected, $O(n)$ has two components

1.4 Exercise: $\pi_1(U(n)) = \mathbb{Z}$, $\pi_1(SU(n)) = 1$, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$ (long exact sequence of homotopy groups needed)

1.5 Exercise: Elements of $Sp(n)$ preserve the form $\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ given by $(v, w) = \sum_{i=1}^n v_i \overline{w}_i$.

1.6 Two approaches to Lie groups
- study of compact Lie groups
- study of complex algebraic reductive groups (definition later)

1.7 Noncompact or nonreductive groups are more difficult; theory of nilpotent or solvable groups is a separate subject.

1.8 But any Lie group G contains a maximal compact subgroup K (which is unique up to a conjugation) and as a topological space $G \simeq K \times \mathbb{R}^n$. (Cartan-Iwasawa-Malcev Theorem)

1.9 For every connected linear and semisimple (to be defined later) group we have

$$G = K \times A \times N$$

where K is maximal compact, $A \simeq \mathbb{R}^k$, N is a nilpotent group, $\simeq \mathbb{R}^\ell$ as a topological space. This is Iwasawa decomposition. The special case is the Gram-Schmidt ortogonalization process

$$GL_n(\mathbb{R}) = O(n) \times (\mathbb{R}_{>0})^n \times N$$

where N uppertriangular with 1's at the diagonal.

1.10 Every compact Lie group can be embedded into $U(n)$ as a closed subgroup.

1.11 Classification of compact connected groups [Cartan]: every such G is of the form $\tilde{G}/A$, where A is a finite abelian group and $\tilde{G} = \prod_{i=1}^k H_i$ and H_i is a torus $(S^1)^r$ or a simple* simply-connected group, which is of the form

- $SU(n)$ (Type A_{n-1})
- $\widetilde{SO(n)} = Spin(n)$ (Type B_m for $n = 2m+1$ or Type D_n for $n = 2m$). Here $\widetilde{SO(n)}$ means the two-fold cover.
- $Sp(n)$ (Type C_n)
- Exceptional group of the type E_6, E_7, E_8, G_2 or F_4

1.12 * Simple Lie group means that the every proper normal subgroups is finite.

1.13 The definitions of the simple Lie groups have common pattern, except that the field over which the definition is realized changes:

- $\mathbb{C}$ – Type A_n
- $\mathbb{R}$ – Type B_n and D_n

- $\mathbb{H}$ – Type C_n
- octonions – exceptional groups, eg $G_2 = Aut(Octonions)$

1.14 For each compact Lie group G there exists a complex Lie group $G_{\mathbb{C}}$, the complexification of G, in which G is the maximal compact subgroup. The group $G_{\mathbb{C}}$ is defined by a polynomial formula in $GL_N(\mathbb{C})$ for some N

- $SL(n, \mathbb{C})$ (Type A_{n-1})
- $\widetilde{SO_n(\mathbb{C})} = Spin_n(\mathbb{C})$ (Type B_m for $n = 2m+1$ or Type D_n for $n - 2m$), where $SO_n(\mathbb{C})$ is a subgroup of $SL_n(\mathbb{C})$ preserving a fixed nondegenerate symmetric form.
- $Sp_n(\mathbb{C})$ (Type C_n), where $Sp_n(\mathbb{C})$ is a subgroup of $GL_{2n}(\mathbb{C})$ preserving a fixed nondegenerate antisymmetric form.
- Complex exceptional group of the type E_6, E_7, E_8, G_2 or F_4, eg. $(G_2)_{\mathbb{C}} \subset GL_7(\mathbb{C})$ is the group preserving certain exterior 3-form.

1.15 Remark for future: Any complex reductive group is of the form $\left((\mathbb{C}^*)^r \times \prod_{i=1}^k (G_i)_{\mathbb{C}}\right)/A$, where A is a finite abelian group.

1.16 Exercise: The real symplectic group $Sp_n(\mathbb{R}) \subset GL_{2n}(\mathbb{R})$ (appears in real symplectic geometry or in classical mechanics) is noncompact and its maximal compact subgroup is equal to $U(n)$.

2 Basic notions: exp et al.

2.1 Recollection of quaternions: let

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \qquad \mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \qquad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \qquad \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}.$$

$$\mathbb{H} = lin_{\mathbb{R}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

Each quaternion is of the form

$$x = a\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$. Let x^* denote $\overline{x}^T$. For $x, y \in \mathbb{H}$ we have

- $(xy)^* = x^* y^*$ (it holds in $M_{2\times 2}(\mathbb{C})$).
- $x \cdot x^* \in \mathbb{R}_+$, $||x|| := \sqrt{x \cdot x^*}$.
- For $x = a\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, $a, b, c, d \in \mathbb{R}$ we have $||x|| = \sqrt{a^2 + b^2 + c^2 + d^2}$.
- $||xy|| = ||x||\,||y||$.

2.2 We have

$$\mathbb{H} = \{A \in M_{2\times 2}(\mathbb{C}) \,:\, \mathbf{j}A = \overline{A}\mathbf{j}\}.$$

Indeed, for $A = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$ we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} u & v \\ -s & -t \end{pmatrix} \qquad \begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -t & s \\ -v & u \end{pmatrix}$$

Hence $t = -\overline{u}$, $s = \overline{v}$.

2.3 The quaternion matrices understood as complex matrices of double size

$$M_{n\times n}(\mathbb{H}) \subset M_{2n\times 2n}(\mathbb{C})$$

satisfy the equation $J_nA = \overline{A}J_n$, where J_n is the block-diagonal matrix with $\mathbf{j}$'s on the diagonal. After reordering of coordinates J_n has a block form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the identity $n \times n$ matrix.

2.4 $Sp(n)$ defined as $GL_n(\mathbb{H}) \cap U(2n)$ consists of $2n \times 2n$ complex matrices satisfying

$$J_nA = \overline{A}J_n, \qquad A\,\overline{A}^T = I_{2n}\,.$$

- From these condition follows $J_nA = (A^T)^{-1}J_n$, hence $A^TJ_nA = J_n$ (i.e. A preserves the symplectic form J_n).
- On the other hand $A^TJ_nA = J_n$ and $\overline{A} = (A^T)^{-1}$ implies $J_nA = \overline{A}J_n$. This shows that $Sp(n) = Sp(n,\mathbb{C}) \cap U(2n)$

2.5 The basic tool to study Lie groups are Lie algebras. One considers the vector fields which are invariant with respect to the left translation $L_g : G \to G$, $L_g(h) = gh$.
The Lie algebra of a Lie group G usually is denoted by the $\mathfrak{gothic}$ letter $\mathfrak{g}$. It is a vector space equipped with

- antisymmetric binary operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the operation of commutator of vector fields)
- the commutator satisfies the Jacobi identity = Leibniz rule

2.6 Examples of abstract Lie algebras

- vector fields on differential manifold
- $M_{n\times n}(\mathbb{C})$ (a.k.a. $\mathfrak{gl}_n(\mathbb{C})$) - matrices with the standard matrix commutator operation
- associative k-algebra with commutator operation (there exists a forgetting functor *Algebras* $\to$ *Lie Algebras*),
- the Lie algebra generated by differential operators $\frac{\partial}{\partial x}$ and x
- derivations of any k-algebra

2.7 Classification of Lie groups is based on two steps:

- Up to taking a cover the Lie group is characterized by its Lie algebra
- Classification of Lie algebras.

2.8 Ado Theorem: every finite dimensional Lie algebra over a field $\mathbb{K}$ of characteristic 0 can be embedded into $M_{n\times n}(\mathbb{K})$.

Elementary theory of Lie groups, details in [Adams, chapter 2]:

2.9 Def: One parameter subgroup is a homomorphism of Lie groups $\mathbb{R} \to G$

2.10 1-1 correspondence between:

- left invariant vector fields
- tangent space T_eG
- 1-parameter subgroups

2.11 Exponential map $\mathfrak{g} = T_eG \to G$. Exp is smooth, diffeo of a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $e \in G$, it is natural with respect to maps of Lie groups.

2.12 $exp : M_{n\times n}(\mathbb{K}) = \mathfrak{gl}_n(\mathbb{K}) \to GL_n(\mathbb{K})$ is the standard exp of matrices given by the well known series. ($\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$)

2.13 Exercise: Show that for $X, Y \in M_{n\times n}(\mathbb{C})$ the $exp(tX)exp(tY)) = exp(\sum_{n=0}^{\infty} i^n A_n)$, where A_n is a Lie polynomial in X, Y, ie. can be expressed by X, Y, $+$, $-$, $[\ ,\ ]$ and scalar multiplication.

2.14 Remark: Let $U \subset \mathfrak{g}$ be a neighbourhood of 0 on which exp is a diffeomorphism. The multiplication in $exp(U)$ is determined by $[-,-]$. There is an explicit formula: the Baker-Campbell-Hausdorff formula

$$exp(X)exp(Y) = exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]) - \frac{1}{24}[Y,[X,[X,Y]]] - \frac{1}{720}??? \ldots\right)$$

For $G \subset GL_n(\mathbb{C})$ the remark follows from the exercise.

2.15 If G is connected, then the homomorphism $G \to H$ is determined by $\mathfrak{g} \to \mathfrak{h}$.
Pf: G in generated by image of exp (in fact by any neighbourhood of e).

2.16 Corollary: a homomorphism of Lie groups induces a map of Lie algebras (see (3.2)). The functor

connected Lie groups $\to$ *Lie algebras*

is faithful. Moreover, [Lie Theorem, see Segal §5, not easy; also (3.10))] the functor

connected simpliconnected Lie groups $\to$ *Lie algebras*

is an equivalence of categories.)

3 Elementary theory of Lie groups (cont.)

3.1 The commutator of vector fields can be computed as commutator of flows: let X and Y be vector fields on $U \subset \mathbb{R}^n$, $p \in U$ and let ϕ_t, ψ_t for $t \in (-\varepsilon, \varepsilon)$ be the flows satisfying $\phi_0 = \psi_0 = Id$, $\dot{\phi} = X$, $\dot{\psi} = Y$. Then

$$\phi_s\psi_t\phi_{-s}\psi_{-t}(p) = p + st[X,Y] + \mathcal{O}(||(s,t)||^3).$$

3.2 Corollary: a map of Lie groups induces a map of Lie algebras.
Pf: Any map of Lie groups preserves 1-parameter subgroups.

3.3 The commutator of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ is the matrix commutator.

$$exp(sX)exp(tY)exp(-sX)exp(-tY) =$$

$$= (1 + sX + \frac{s^2}{2}X^2)(1 + tY + \frac{t^2}{2}Y^2)(1 - sX + \frac{s^2}{2}X^2)(1 - tY + \frac{t^2}{2}Y^2) + \mathcal{O}(||(s,t)||^3)$$

$$= (1 + sX + tY + \frac{s^2}{2}X^2 + \frac{t^2}{2}Y^2 + stXY))(1 - sX - tY + \frac{s^2}{2}X^2 + \frac{t^2}{2}Y^2 + stXY) + \mathcal{O}(||(s,t)||^3)$$

$$= (1 - (sX + tY)^2 + 2(\frac{s^2}{2}X^2 + \frac{t^2}{2}Y^2 + stXY)) + \mathcal{O}(||(s,t)||^3) = 1 + st[X,Y] + \mathcal{O}(||(s,t)||^3)$$

3.4 Abelian connected Lie groups are of the form $(S^1)^n \times \mathbb{R}^m$.
Pf: Exp is a homomorphism of Lie groups.

3.5 Theorem: closed subgroup H of a Lie group G is submanifold and a Lie group
Pf: Sketch: Let $W = \{v \in \mathfrak{g} | exp(tv) \in H\}$, this is a linear subspace, and $exp(W)$ is a neighbourhood of e in H. See [Adams 2.27-2.30].

3.6 Lie algebras of the subgroups $O(n)$, $U(n)$, $Sp(n)$ in $GL_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) consists of matrices satisfying $A + \overline{A}^T = 0$ in $M_{n\times n}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

3.7 Let $H \subset G$ be a Lie subgroup, then $X = G/H$ is Hausdorff [Bredon Prop. 1.4], it is a differential manifold and $T_{[e]}X \simeq \mathfrak{g}/\mathfrak{h}$.

3.8 If a Lie group acts on a set X in a transitive way, then X is of the form G/G_x, where G_x is a stabilizer of a chosen point. If G_x is closed, then X admits a structure of a manifold.
Important homogeneous spaces

- $\mathbb{P}^{n-1}$, Grassmanians $U(n)/(U(k) \times U(n-k)) = GL_n(\mathbb{C})/\binom{**}{0*}$,
- lagrangian/ortogonal Grassmanians, isotropic Grassmanians,
- flag varieties,
- the space of scalar products $GL_n(\mathbb{R})/O(n)$,
- the space of complex structures $GL_{2n}(\mathbb{R})/GL_n(\mathbb{C})$,
- the space of complex structures adapted to a given symplectic structure $Sp_n(\mathbb{R})/U(n)$, etc.

3.9 Not every Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponds to a closed subgroup, but rather to an immersed subgroup.

3.10 Faragment of a proof of Lie Theorem 2.16. For any Lie algebra there exists a Lie groups with the given Lie algebra.
Pf: Using Ado Theorem we embed $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$. We obtain a distribution in $V \subset T(GL_n(\mathbb{R}))$. Since $\mathfrak{g}$ is a Lie algebra V is involutive, hence integrable (Frobenius theorem). This means there exists a foliation of $GL_n(\mathbb{R})$ tangent to V. The leaf passing through the identity is the subgroup with Lie algebra $\mathfrak{g}$.
To see why a map of Lie algebras induce a map of Lie group [Segal, Theorem 5.4]

4 Adjoint representation, reductive groups

4.1 A representation of a Lie group G is a homomorphism $G \to Aut(V)$, where V is a vector space. Equivalently we can say that a linear action of G on V is given.

4.2 A representation of a Lie algebra $\mathfrak{g}$ is a homomorphism of Lie algebras $\mathfrak{g} \to End(V)$, where V is a vector space, or equivalently a linear action of $\mathfrak{g}$ on V, that is for any $X, Y \in \mathfrak{g}$ and any $v \in V$

$$X(Y(v)) - Y(X(v)) = [X, Y](v)$$

4.3 Any lie Group has the adjoint representation: the action by conjugation of G on G fixes e, hence we get $Ad : G \to Aut(\mathfrak{g})$.

4.4 Example: The adjoint action of $SU(2) = S^3 \subset \mathbb{H}$ on $\mathfrak{su}(2) = im(\mathbb{H})$ preserves the norm. We obtain an injective map $SU(2)/\mathbb{Z}_2 \to O(3)$. Since the dimensions are the same and $SU(2)$ is connected this map is an isomorphism.

4.5 If G is connected, then $ker(Ad) = Z(G)$.

4.6 The differential of Ad, i.e. $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the commutator $ad_X(Y) = [X, Y]$.
Pf:

$$ad_X(Y) = \frac{d}{dt}\left(\frac{d}{ds} exp(tX)exp(sY)exp(-tX)\right)_{s=t=0}$$

$$ad_X(Y) = \frac{d}{dt}\left(\frac{d}{ds} exp(sY)exp(-sY)exp(tX)exp(sY)exp(-tX)\right)_{s=t=0}$$

$$= \frac{d}{dt}\left(Y(exp(-sY)exp(tX)exp(sY)exp(-tX))\right)_{s=t=0} + \frac{d}{dt}\left(\frac{d}{ds}(exp(-sY)exp(tX)exp(sY)exp(-tX))\right)_{s=t=0}$$

$$Y\frac{d}{dt}L_{exp(sY)}(st[-Y, X] + higher\ terms)_{s=t=0} + \frac{d^2}{dtds}L_{exp(sY)}(st[-Y, X] + higher\ terms)_{s=t=0}$$

$$\frac{d^2}{dtds}(Id + \mathcal{O}(s))(st[-Y, X] + higher\ terms)_{s=t=0} = [-Y, X]$$

4.7 Note that $ker(ad) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g}\ [X, Y] = 0\}$. This is the center of a Lie algebra $Z(\mathfrak{g})$. If $Z(\mathfrak{g}) = 0$, then Ado theorem 2.8 is for free; $\mathfrak{g}$ embeds in $End(\mathfrak{g})$.

4.8 Complex groups = complex manifolds with holomorphic multiplication and inverse

4.9 Compact complex groups have to be of the form $\mathbb{C}^n/Latice$.
Pf $Ad : G \to Aut(\mathfrak{g})$ is constant. Hence the conjugation by g induces the identity on $\mathfrak{g}$. Therefore G is commutative.

4.10 Complex linear algebraic group = subgroup of $GL_n(\mathbb{C})$, given by some polynomial equations in the entries of the matrix and det^{-1}.

4.11 We can assume that G is a closed algebraic set in $M_{n\times n}(\mathbb{C})$.

4.12 Our definition of a **reductive group**: Complex group G is reductive if there exists an embedding into $GL_n(\mathbb{C})$, such that the image is invariant with respect to the Cartan involution: $\Theta : A \mapsto (\overline{A}^T)^{-1}$.

4.13 A reductive group for algebraic geometers is a linear algebraic group G (over an algebraically closed field k) such that *the largest connected solvable normal subgroup (the radical) is an algebraic torus* $\simeq (k^*)^r$. This is an equivalent definition to ours for $k = \mathbb{C}$, but we will not discuss it.

4.14 The groups $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ are reductive.

4.15 Properties of the Cartan involution $\Theta : G \to G$ and $\theta : \mathfrak{g} \to \mathfrak{g}$ $\theta(A) = -\overline{A}^T$ [Knapp §1]

- θ is a homomorphism of Lie algebras
- the fixed points is a compact subgroup $K := G^\theta = G \cap U(n)$
- the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces of θ: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$

– $\mathfrak{k} := \mathfrak{g}_1$ is the Lie algebra of K (this is the $\mathfrak{gothic}$ k).
– $\mathfrak{p} := \mathfrak{g}_{-1}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$,
– $\mathfrak{p} = i\mathfrak{k}$ and $\mathfrak{g} \simeq \mathfrak{k} \otimes \mathbb{C}$ as complex Lie algebras.

4.16 For $G = GL_n(\mathbb{C})$ the space $\mathfrak{p}$ consists of the hermitian (or self-adjoint) matrices matrices $A = \overline{A}^T$.

4.17 Corollary: let $\phi, \psi : G \to H$ homomorphism of complex Lie groups, G reductive, connected. If $\phi_{|K} = \psi_{|K}$ then $\phi = \psi$.

4.18 The map $K \times \mathfrak{p} \to G$ given by $(g, X) \mapsto g \cdot exp(X)$ is a diffeomorphism.

4.19 Proof of 4.18 for $G = GL_n(C)$: by polar decomposition every invertible matrix A can be written uniquely as $A = Q\,P$, where $Q \in U(n)$ and $P = \theta(P)$ is positive definite (for $P = (A^*A)^{\frac{1}{2}}$, $Q = AP^{-1}$ we check $QQ^* = (AP^{-1})(P^{-1}A^*) = A(A^*A)^{-1}A^* = I$). Any positive definite matrix P has logarithm.

5 Invariant scalar product

5.1 Proof of 4.18 for arbitrary G (after [Knapp, I§2]). It show that if $P = exp(X) \in G$ for X hermitian, then $X \in \mathfrak{p}$. After a linear change of coordinates we can assume that X is diagonal $X = diag(a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{R}$ and $P = diag(b_1, b_2, \ldots, b_n)$, $b_i = e^{a_i} \in \mathbb{R}_+$. Then $P^k = (b_1^k, b_2^k, \ldots, b_n^k) \in G$ for all $k \in \mathbb{Z}$. We will use the fact: *G is defined by polynomial equations*, the polynomials f_i defining G vanish on $(b_1^k, b_2^k, \ldots, b_n^k)$. We rewrite the equations for G and we get expressions of the form $\phi(k) = \sum \alpha_j c_j^k$ with c_i equal to some products of b_i's. (c_j can be 1 as well). The function $\phi(k)$ vanishes for $k \in \mathbb{Z}$. It follows that it vanishes for any $k \in \mathbb{R}$.
The last step in the proof. If $QP \in G$ then $\Theta(PQ) = Q^{-1}P \in G$. Hence $P^2 \in G$ and $P^2 = exp(X)$ for $X \in \mathfrak{u}(n)$. By the previous step we conclude that $X \in \mathfrak{p}$, so $P = exp(\frac{1}{2}X) \in G$.

5.2 Trace form defined for $\mathfrak{gl}_n(\mathbb{C})$:

$$B_0(X, Y) = Tr(XY).$$

- B_0 is Ad-invariant
- We have

$$B_0([X, Y], Z) + B_0(Y, [X, Z]) = 0$$

• B_0 is nondegenerate since $B_0(X, \theta(X))$ is real and < 0 for $X \neq 0$

• the form $(X, Y) = Re(-B_0(X, \theta(Y)))$ is a scalar product on $\mathfrak{g}$.

• the form (X, Y) for $X, Y \in \mathfrak{k}$ is equal to $Re(-Tr(X, Y))$, hence it is *ad*-invariant. It follows that for any ideal $\mathfrak{h} \subset \mathfrak{k}$ the orthogonal complement is a Lie subalgebra and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ as Lie algebras.

5.3 If $H \subset G$ is normal, then $\mathfrak{h} \subset \mathfrak{g}$ is an ideal: $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$ implies $[X, Y] \in \mathfrak{h}$. The quotient $\mathfrak{g}/\mathfrak{h}$ has a structure of a Lie algebra and is isomorphic to the Lie algebra of G/H.

5.4 Definition: $\mathfrak{g}$ is simple if it is not abelian (i.e. has nontrivial commutator) and does not admit any proper quotient Lie algebra. A semisimple Lie algebra is a Lie algebra which is the direct sum of Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ and each $\mathfrak{g}_i$ is simple.

5.5 If G is a complex reductive group, then $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.
Pf. Use invariant scalar product in $\mathfrak{k}$ and complexify.

5.6 The same proof shows that the Lie algebra has the above form provided that we have an *ad*-invariant scalar product in $\mathfrak{g}$. This is the case for Lie algebras of compact groups, which admit an invariant integral as will be shown. Then to construct an invariant scalar product it is enough to take any and average it.

5.7 Classification shows, that every compact connected Lie group admits a reductive Lie group in which it is the fixed point set of Θ.

5.8 Construction of invariant integration [Bröcker-tom Dieck I§5]. There exists a measure dg on G such that for any $f \in C(G)$ and $h \in G$

$$\int_G f(hg)dg = \int_G f(g)dg.$$

The measure is given by a differential form of top degree

$$\int_G f(g)dg = \int_G f(g)\omega.$$

The invariance means that

$$\int_G f(hg)dg = \int_G L^*_{h^{-1}}(f)\,\omega = \int_G f\,L^*_h\,\omega$$

for any f and h, which is equivalent to: $L^*_h\omega = \omega$. Construction: Take $0 \neq \omega_0 \in \Lambda^{\dim(G)}T^*_eG$, $\omega(h) = L^*_h\omega_0$. If we assume that $\int_G \omega = 1$ then ω is unique.

5.9 The left-invariant integral is right-invariant: $\int_G f(gh)dg = \int_G f(g)dg$.
Pf: G acts by conjugation on $\Lambda^{\dim(G)}T^*_eG \simeq \mathbb{R}$. Since G is compact $R^*_{h^{-1}}L^*_h\omega = \pm\omega$, sign depends whether conjugation by h changes the orientation of G.

5.10 Corollary: for any real/complex representation of a compact Lie group there exists an invariant scalar/hermitin product.
Pf: take any and average: $(v, w)' := \int_G (gv, gw)$.

6 In future: torus representations

6.1 How to construct an ad-invariant scalar product in $\mathfrak{g}$

• for a complex reductive group G the form $-Tr(XY)$ is positive definite on $\mathfrak{g}^{\theta}$.

• if G is compact: take any scalar product in $\mathfrak{g}$ and average it

• if G is compact and $Z(\mathfrak{g}) = 0$ there is a canonical choice: – Killing form, see below.

6.2 Killing form: $\psi(X, Y) = Tr(ad_X \circ ad_Y)$. This form is symmetric and G-invariant (hence also ad-invariant).

6.3 If G is compact, then ψ is nonpositive definite:
Pf. for $X \in \mathfrak{g}$ let H be the closed group generated by $Ad(exp(tX)) \subset Aut(\mathfrak{g})$. The group $H \simeq (S^1)^k$ is a compact torus, choosing some coordinates in $\mathfrak{g}$ we can assume that H acts diagonally (see 6.20). Hence $h \in H$ has a form $diag(e^{it_1}, e^{it_1}, \ldots, e^{it_{\dim \mathfrak{g}}})$, so $ad_X = diag(it_1, it_1, \ldots, it_{\dim \mathfrak{g}})$. Therefore $\psi(X, X) = -\sum t_j^2$.

6.4 Killing form is nondegenerate on $\mathfrak{g}/Z(\mathfrak{g})$.

6.5 Exercise: Let G be a reductive group. Define a hermitian product in $\mathfrak{g} \subset M_{n\times n}(\mathbb{C})$ by the formula $\langle\langle X, Y\rangle\rangle = Tr(X \circ \overline{Y}^T)$. The hermitian product in $\mathfrak{g}$ allows to define the Cartan involution Θ in $Aut(\mathfrak{g})$. Show that $Ad(G) \subset Aut(\mathfrak{g})$ is Θ-invariant. (Hint: Show that $(ad_X)^* = ad_{X^*}$.)

6.6 Torus = compact abelian connected group = $(S^1)^r$. Every compact Lie group contains a maximal torus.

6.7 Main idea of representation theory of reductive/compact connected Lie groups

• Complex holomorphic representations of a reductive group G

• = Complex representations of its maximal compact subgroup

• (can be understood by studying) Representation of the maximal torus

• (description via combinatorial data) Every representation is determined by its ,,set of weight”, a choice of lattice points in $\mathfrak{t}^*$

6.8 Definition: Let G be a (topological) group.

• A representation V is irreducible (or simple) if it does not have any subrepresentation other than 0 and V.

• A representation V is indecomposable if V cannot be presented as $V_1 \oplus V_2$, where V_i are subrepresentations (ie. G-subspaces), with $V_i \neq 0$

6.9 $\mathbb{R} \to GL_2(\mathbb{R})$ given by $t \mapsto \binom{1\,t}{0\,1}$ is a representation which is indecomposable, but not simple.

6.10 Main advantage of compact groups: every representation admits an invariant scalar/hermitian product.

6.11 Corollary. Representations of compact groups are direct sums of irreducible representations.
Pf: any subrepresentation has ortogonal complement

6.12 Corollary: Holomorphic representations of reductive groups are direct sums of irreducible representations.

6.13 The above is a characterization of reductive groups among complex linear groups.

Characters

6.14 Character of a representation $(V, \rho : G \to Aut(V))$

$$\chi_V(g) := Tr(\rho(g) : V \to V)$$

6.15 General properties of characters:

- χ_V is a function on conjugacy classes of G (i.e. it is a class function)
- $\chi_{V^*}(g) = \chi_V(g^{-1})$
- $\chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$ for compact groups
- $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$
- $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$
- Corollary: $\chi_{Hom(W,V)}(g) = \chi_V(g)\overline{\chi_W(g)}$ for compact groups

6.16 If G is compact, then $\int_G \chi_V(g)dg = \dim(V^G)$
Pf: Let $p : V \to V$ be given by $v \mapsto \int_G g \cdot v\, dg$. It is a projector onto V^G:

$$\dim(V^G) = Tr(p) = \int_G Tr(v \mapsto g \cdot v)$$

6.17 Cor. Let V, W be irreducible representations, then $(\chi_V, \chi_W) := \int_G \chi_V(g)\overline{\chi_W(g)}dg = 0$ if $W \not\simeq V$ or 1 for $W \simeq V$.
Pf. $\dim Hom(W, V) = 0$ or 1.

6.18 Peter-Weyl for arbitrary compact group: characters of irreducible representations form an ortonormal basis of the Hilbert space of class functions $\subset L^2(G)$. They are also dense in $C^0(G/conjugation)$.

6.19 Corollary: Let G be a compact group. If V is a direct sum of irreducible representations $V \simeq \sum(V_\alpha)^{\oplus a_\alpha}$, then a_α depend only on V. Two representations are isomorphic if and only if their characters are equal. *Proof.* $a_\alpha = (\chi_V, \chi_{V_\alpha})$.

6.20 Complex irreducible representation of compact abelian groups are of dimension 1:
Pf: $\rho : G \to GL_n(\mathbb{C})$, can assume $\rho : G \to U(n)$. Every element of $U(n)$ is diagonalizable. Every family of commuting operators have common eigenvector v ; $lin(v)$ is a subrepresentation.

6.21 Let T be a torus, let $\Lambda = ker(exp : \mathfrak{t} \to T)$. The irreducible representations of T are in bijection with $\Lambda^* = Hom(\Lambda, \mathbb{Z})$.

$$\begin{array}{ccccc} \Lambda = ker(exp) & \longrightarrow & 2\pi i\mathbb{Z} & \simeq & \mathbb{Z} \\ \downarrow & & \downarrow & & \\ \mathfrak{t} & \longrightarrow & \mathfrak{s}^1 & = & i\mathbb{R} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} & & \\ T & \longrightarrow & S^1 & \subset & \mathbb{C}^* \end{array}$$

Pf. The image $Hom(T, S^1) \hookrightarrow Hom(\mathfrak{t}, \mathfrak{s}^1)$ consists of the linear maps preserving latices.

6.22 An element of Λ^* is a called weight. Consider the irreducible representation $\mathbb{C}_w$ of weight w. For $X \in \mathfrak{t}$ the action of $exp(X) \in T$ is given by the multiplication by the number $e^{i\langle w,X\rangle} \in S^1 \subset \mathbb{C}^*$. After identification $T = (S^1)^r$ we have $\Lambda = \mathbb{Z}^r$ and $(e^{it_1}, e^{it_2}, \dots, e^{it_r}) \in (S^1)^r$ acts as multiplication by the scalar $e^{i(w_1t_1+w_2t_2+\cdots+w_rt_r)} = \prod(e^{it_j})^{w_j}$.

6.23 Representations of T are determined by formal combinations weights. The formal sum $\sum a_w[\mathbb{C}_w]$ can be treated as an element of the group ring $\mathbb{Z}[\Lambda^*]$ with nonnegative coefficients. After identification $T = (S^1)^r$ we have $\mathbb{Z}[\Lambda^*] = \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_r, x_r^{-1}]$.

- $V \oplus W$ corresponds to addition of Laurent polynomials,
- $V \otimes W$ corresponds to multiplication of Laurent polynomials.

6.24 For a torus $T = (S^1)^r$ and the representation $V \simeq \bigoplus(\mathbb{C}_w)^{\oplus a_w}$, in coordinates $\xi_j = e^{it_j}$

$$\chi_V(\xi_1, \xi_2, \dots, \xi_r) = \sum_{w\in\Lambda^*} a_w \xi_1^{w_1}\xi_2^{w_2}\dots\xi_r^{w_r}$$

6.25 Peter-Weyl theorem for torus $T = S^1$ is equivalent to the Fourier theorem: the functions $\xi \mapsto \xi^n$ form an ortonormal basis of $L^2(S^1)$.

6.26 For $T = S^1$ the multiplicity of the representation of weight n in V is the coefficient a_n in the Fourier expansion of the character $\chi_V(\xi) = \sum_{n\in\mathbb{Z}} a_n\xi^n$.

7 Maximal tori

7.1 Remark about 6.19, without compactness assumption for G: if $V \simeq \sum(V_\alpha)^{\oplus a_\alpha}$ happens to be a sum of irreducible representations, then $a_\alpha = \dim(Hom(V_\alpha, V)^G)$ does not depend on the presentation.

7.2 Isomorphism classes of representations form a semiring. The associated ring (Grothendieck construction) is called representation ring, $R(G)$. For $G = T$ we have $R(T) = \mathbb{Z}[\Lambda^*] \simeq \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_r, x_r^{-1}]$

7.3 General remark: For a category $\mathcal{C}$ with exact sequences and (with monoidal structure $\otimes$) define $K(\mathcal{C})$ as an abelian group generated by isomorphism classes of objects and relations being a consequence of $[x] + [z] = [y]$ whenever we have an exact sequence $0 \to x \to y \to z \to 0$. (Ring structure is given by $\otimes$.) For compact G we have $K(\mathbb{C}[G] - mod) = R(G)$, where $\mathbb{C}[G] - mod$ denotes the category of complex vector spaces with G-action.

7.4 Maximal tori (examples in $SU(n)$, $SO(n)$).

7.5 T decomposes $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ into weight spaces

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Lambda^*\setminus 0} \mathfrak{g}_\alpha.$$

(α's are called roots.) If α is a root, then $-\alpha$ is a root ($e^{i\alpha(t)}$ is an eigenvalue, then $e^{-i\alpha(t)}$ is).

7.6 For $U(n)$ let $L_i : \Lambda \simeq \mathbb{Z}^n \to \mathbb{Z}$ be the i-th component. The roots of $U(n)$ are $L_i - L_j$, $i \neq j$.

7.7 For $SU(n)$ the same roots, but there is a relation $\sum L_i = 0$

7.8 For $SO(2n)$ we complexify $\mathbb{R}^{2n}$ and consider an equivalent quadratic form $\sum_{i=1}^{n} x_i x_{2n-i+1}$. The roots are $L_i \pm L_j$, $i \neq j$

7.9 For $SO(2n+1)$ we complexify $\mathbb{R}^{2n+1}$ and consider an equivalent quadratic form $\sum_{i=1}^{n+1} x_i x_{2n-i+2}$. The roots are $\pm L_i \pm L_j$, $i \neq j$, $\pm L_i$.

7.10 Weyl group: for a maximal torus $T \in G$, G compact, $W = NT/T$ is finite. Pf. NT acts on T, $Aut(T)$ discrete.

7.11 Let $f : M \to M$ selfmap of orientable manifold. Lefschetz number $\Lambda(f)$: intersection of cycles $[graph(f)]$ and $[\Delta] = [graph(id)]$ in $M \times M$

- if no fixed points then $\Lambda(f) = 0$
- $\Lambda(f) = \Lambda(g)$ if f and g homotopic
- If the fixed point set is finite and the intersection is transverse, then $\Lambda(f)$ is the sum of fixed points with signs. The sign at the point p is equal to $sgn\, det \begin{pmatrix} I & I \\ Df(p) & I \end{pmatrix} = sgn\, det(I - Df(p))$
- $\Lambda(f) = \sum_{i=0}^{\dim M} (-1)^i Tr(f_* : H_i(M;\mathbb{R}) \to H_i(M;\mathbb{R}))$

7.12 Any $g \in G$ is contained in a conjugate of T.
Pf [Adams, p 90-92]. Equivalently $L_g : G/T \to G/T$ has a fixed point. Enough to compute Lefshetz number of L_g; can replace g by t, a generator of T. Fixed points of L_t are NT/T. Local contributions to the Lefschetz number coming from the points of NT/T are equal. It is enough to compute for $eT \in NT/T$. Local computation: action of $\xi = exp(t) \in T$ on $T(G/T)$ has eigenvalues $e^{i\alpha(t)}$, hence $det(I - DL_\xi(eT)) = \prod_{roots}(1 - e^{i\alpha(t)}) = \prod_{\pm roots} |1 - e^{i\alpha(t)}|^2 > 0$. Hence $\Lambda(L_g) = \Lambda(L_\xi) = |N(T)/T| \neq 0$, hence there is a fixed point.

7.13 Special cases (for $U(n)$ and $SO(n)$) are the classical theorems from linear algebra.

8 Representations of $\mathfrak{sl}_2(\mathbb{C})$

8.1 Cor. Any two maximal tori are conjugate.

8.2 Cor. Euler characteristic $\chi(G/T) = |W|$.

8.3 Let V a representation. $(\chi_V)_{|T} : T \to \mathbb{C}$ is W-invariant.

8.4 The character χ_V is determined by its restriction to the maximal torus.

8.5 Cor. $R(G) \to R(T)^W$ is mono.

8.6 Generalities about Lie algebra representations: dual, $\otimes$. (Exercise: Hom, Sym^k and Λ^k.)

8.7 Groups $SU(2)$, $SL_2(\mathbb{C})$, $SL_2(\mathbb{R})$ and relations between their representations. Representations of Lie algebras $\mathfrak{su}(2)$, $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{R})$.

8.8 $\mathfrak{sl}_2(\mathbb{C})$ is spanned by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have $[H, X] = 2X$, $[H, Y] = 2Y$, $[X, Y] = H$.

8.9 H decomposes any representation V into weight spaces $V = \bigoplus_{k\in\mathbb{Z}} V_k$. For $v \in V_k$:

- $Hv = kv$
- $Xv \in V_{k+2}$
- $Yv \in V_{k-2}$

In general: if $Z \in \mathfrak{g}_\alpha$, $v \in V_\beta$ then $Zv \in V_{\alpha+\beta}$.

8.10 Examples of representations of $\mathfrak{sl}_2(\mathbb{C})$: symmetric powers of the natural representations $Sym^k(\mathbb{C}^2)$

8.11 The algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the subalgebra of differential operators in 2 variables generated by $X = x\frac{\partial}{\partial y}$ and $Y = y\frac{\partial}{\partial x}$, $H = [X, Y] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. The natural representation: linear forms, $Sym^k(\mathbb{C}^2) \simeq \{k - \text{linear forma}\}$.

8.12 Highest weight vectors in the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

8.13 [Fulton-Harris, §11] Theorem: irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ (or $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{sl}_2(\mathbb{Z})$) are isomorphic to $Sym^k(\mathbb{C}^2)$. They are characterized by the weight of the vector $v \in ker(X)$ (highest weight vector).

8.14 Lemma: if $v \in V_n$, $Xv = 0$ then $XY^m v = m(n - m + 1)Y^{m-1}$.

8.15 Corollary: if $\dim V < \infty$ then $n \in \mathbb{N}$.

9 Examples of SL_3-representations, rank one groups

9.1 Every complex representation of $\mathfrak{s}l_2(\mathbb{C})$ extends to a representation of $SL_2(\mathbb{C})$. (since $\pi_1(SL_2(\mathbb{C}) = 1)$

9.2 Corollary: Every complex/real representation of $\mathfrak{s}l_2(\mathbb{R})$ extends to a representation of $SL_2(\mathbb{R})$. (via complexification)

9.3 If $V = \bigoplus_{n\in\mathbb{N}} Sym^n(\mathbb{C}^2)^{\oplus a_n}$ as a $\mathfrak{s}l_2(\mathbb{C})$-representation, then $a_n = \dim V_n - \dim V_{n+2}$.

9.4 Some examples of representations of $\mathfrak{sl}_3(\mathbb{C})$.In paricular $Sym^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$, see Fulton-Harris §12-13. Claim: every irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ is isomorphic to the subrepresentation of $Sym^a(\mathbb{C}^3) \otimes Sym^b((\mathbb{C}^3)^*)$ generated by $v = (e_1)^a \otimes (e_3^*)^b$. The vector v is ,,the highest weight vector". The remaining vectors are obtained by application of the operators E_{21}, E_{31}, E_{32} given by the action of elementary matrices

9.5 Rank of the Lie group $r(G) := \dim(T)$, where T is a maximal torus.

9.6 Theorem: Compact connected Lie group of rank 1 is isomorphic to $SU(2)$ or $SO(3)$ or S^1.
Pf: Let $n = \dim(G)$. G acts on $S^{n-1} \subset \mathfrak{g}$ via Ad. The tangent action has the kernel $= \mathfrak{t}$. Therefore $G/T \to S^{n-1}$ is a covering, so it has to be a homeomorphism. We get a fibration $S^1 = T \to G \to G/T = S^{n-1}$. If $n > 3$ the $\pi_1(T) \to \pi_1(G)$ is a monomorphism. The group G contains a subgroup H isomorphic to $SU(2)$ or $SO(3)$ with the Lie algebra $\mathfrak{t} \oplus \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{-\alpha_0}$, where α_0 the shortest root. There is an element $g \in N(T) \subset H$ such that $Ad(g)_{|\mathfrak{t}} = -Id : \mathfrak{t} \to \mathfrak{t}$, so $gtg^{-1} = t^-1$. But in G the conjugation by g is homotopic to Id. Contradiction. † Hence $n \leq 3$.

10 Systems of roots

10.1 System of roots of rank 2 $SU(3)$, $SO(4)$, $SO(5)$, $Sp(2)$) see [FH.§21]

10.2 Theorem. Ler R be the set of roots of a semisimple compact/complex reductive Lie group.

1. the roots R span $\mathfrak{t}^*$
2. the action of $W = NT/T$ on $\mathfrak{t}^*$ preserves R
3. $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in R$
4. $\alpha \in \Sigma \Longrightarrow -\alpha \in \Sigma$ and no other multiplicity of α belongs to R.

10.3 In general 10.2.1 holds, if $Z(G)_0 = 1$. More general we have $\bigcap_{\alpha\in R} ker(\alpha) = T(Z(G))$.

10.4 For any compact group 10.2.2 holds, because if $n \in NT$, then the action by $Ad(n)$ shuffles $\mathfrak{g}_\alpha$'s.

10.5 Lemma 1: for any root α there exists a subgroup $H_\alpha \subset G$ with Lie algebra

$$\mathfrak{h}_\alpha = \mathfrak{t} \oplus \bigoplus_{\beta \text{ proportional to } \alpha} \mathfrak{g}_\beta$$

Pf. H generated by $exp(\mathfrak{h}_\alpha)$; the roots of the closure have the same kernel as α, hence H closed.

10.6 Cor. 10.2.3-4.
Pf. $Z(H_\alpha)_0 = exp(ker(\alpha))$, $H_\alpha/Z(H_\alpha)_0$ is a rank 1 group, see 9.6.

10.7 Lemma 2. There exist a subgroup $K_\alpha \subset H_\alpha$ with the Lie algebra $ker(\alpha)^\perp \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ where $\perp$ taken with respect to an invariant scalar product in $\mathfrak{g}$.
Pf. $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, then for $Z \in \ker(\alpha)$ we have $Z \perp [X,Y]$. (since $0 = X(Y,Z) = ([X,Y],Z) + (Y,[X,Z]) = ([X,Y],Z) + (Y, -\alpha(Z)X)$.)

10.8 Fix a G-invariant metric. For any root α the subspace $\alpha^\perp \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$.

10.9 Cor: for any root α there exists a map $f_\alpha : SU(2) \to G$, such that the image of $\mathfrak{su}(2)_\mathbb{C} = \alpha^\perp \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

10.10 The action of the element of Weyl group of $\left[f_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$ is the reflection in $ker(\alpha)$ denoted by s_α.

10.11 Abstract system of roots: V a finite dimensional real vector space with a scalar product, $R \subset V$ a finite subset, called roots:

1. the roots R spans V
2. $\alpha \in \Sigma \Longrightarrow -\alpha \in \Sigma$ and no other multiplicity of α belongs to R.
3. s_α preserves R

4. For a pair of roots the Cartan number $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer (the number $n_{\alpha\beta}$ satisfies $s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha$)

10.12 Pf.A0of (4) for roots of a compact Lie group (see ([B-tD,§V.2.7]) $\alpha^* \in V^* = \mathfrak{t}$ coroots: defined by the property $s_\alpha(\beta) = \beta - \beta(\alpha)\alpha$ (under the identification $\mathfrak{t} = \mathfrak{t}^*$ we have $\alpha^* = \frac{2}{(\alpha,\alpha)}\alpha$).

- We show that $\alpha* \in \Lambda$: let $x = \frac{1}{2}\alpha^*$,

– $\alpha(x) = 1$ so $exp(x) \in ker(\bar{\alpha} : T \to S^1)$

– $exp(x) \in T$ is invariant with $\bar{s}_\alpha : T \to T$ and on the other hand $\bar{s}_\alpha(exp(x)) = exp(s_\alpha(x)) = exp(-x)$. So $exp(x)^2 = 1$,hence $2x \in \Lambda$. Therefore $\beta(\alpha^*) = \beta(2x) \in \mathbb{Z}$.

10.13 Definition of Weyl chambers for a Lie group

10.14 Theorem:
1. W acts transitively and freely on the set of chambers
2. W is generated by s_α's

Pf see [B-tD, Th 2.12]: Let W_0 be generated by s_α's. Claim 1: W_0 acts transitively. Claim 2: W acts freely. Claims 1 and 2 imply Theorem.
Pf of Claim 1. geometric proof,
Pf of Claim 2 follows from the following Lemma:

10.15 Lemma. Suppose X lies in the interior of some chamber in $\mathfrak{t} \simeq \mathfrak{t}^*$, then $exp(tX)$ is contained only in one maximal torus.
Pf: X acts nontrivially on each $\mathfrak{g}_\alpha$, so $Z(exp(tX))_0 = T$.

10.16 Pf of Claim 2 cont: if $g \in NT$ acts trivially on a chamber K, one may assume $g(X) = X$ for some $X \in K$. The group topologically generated by $exp(tX)$ and g abelian, $\simeq torus \times \mathbb{Z}_n$ can be topologically generated by one element, so it is contained in a maximal torus. This torus has to be T. Hence $[g] = 1 \in NT/T$.

10.17 Let V be a representation of G. Then the multiset in $\mathfrak{t}^*$ representing the weights of V is preserved by the action of Weyl group.
Pf:

$$\begin{array}{cccc} & G \times V & \longrightarrow & V \\ {\scriptstyle (g \bullet g^{-1}) \times g} & \downarrow & & \downarrow {\scriptstyle g} \\ & G \times V & \longrightarrow & V \end{array}$$

Therefore for $g \in NT$ we have $gV_\alpha = V_{[g]\alpha}$.

11 Classification of irreducible representation by highest weight vectors

11.1 The Cartan numbers $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ satisfy
- $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2(\angle(\alpha,\beta))$
- $n_{\alpha\beta}n_{\beta\alpha} \in \mathbb{Z}$

Therefore $n_{\alpha\beta}n_{\beta\alpha} \in \{0,1,2,3\}$ for $\alpha \neq \pm\beta$.

11.2 Positive roots: we chose a linear function $\phi : \mathfrak{t}^* \to \mathbb{R}$, such that no root belongs to $ker(\phi)$. Get division of roots into $R = R_+ \sqcup R_-$. • Positive roots for $\mathfrak{sl}(n)$: $L_i - L_j$, $i < j$
e.g. $\phi(\alpha) = (\beta_0, \alpha)$ where $\beta_0 = \frac{1}{2}\sum_{i<j}(L_i - L_j)$
for $n = 3$: $\frac{1}{2}((L_1 - L_2) + (L_1 - L_3) + (L_2 - L_3)) = L_1 - L_3 = (3L_1 + 2L_2 + L_3)$;
in general $\sum_{i=1}^{n}(n - i + 1)L_i$

- for $\mathfrak{so}(2n+1)$: $L_i - L_j$, L_i for $1 \le i < j \le n$
- for $\mathfrak{sp}(n)$: $L_i - L_j$, $2L_i$ for $1 \le i < j \le n$
- for $\mathfrak{so}(2n+1)$: $L_i - L_j$, $L_i + L_j$ for $1 \le i < j \le n$

11.3 A positive root is simple if it cannot be written as a sum of positive roots.

- for $\mathfrak{sl}(n)$: $L_k - L_{k+1}$ for $1 \le k < n$
- for $\mathfrak{so}(2n+1)$: $L_k - L_{k+1}$ for $1 \le k < n$ and L_n
- for $\mathfrak{sp}(n)$: $L_k - L_{k+1}$ for $1 \le k < n$ and $2L_n$
- for $\mathfrak{so}(2n+1)$: $L_k - L_{k+1}$ for $1 \le k < n$ and $L_{n-1} + L_n$

11.4 Exercise: every positive root can be written as a sum of simple roots. Simple roots are linearly independent.

11.5 Having chosen positive roots there is a canonical choice for a new ϕ: $\phi(-) = (\beta_0, -)$, where $\beta_0 = \frac{1}{2}\sum_\alpha \alpha$ is the $\frac{1}{2}$sum of positive roots. (Exercise: Then for a positive root α: $(\beta_0, \alpha) > 0$.)

11.6 We chose a distinguished Weyl chamber $K_0 = \bigcap\{\beta \in \mathfrak{t}^* : (\beta, \alpha) > 0\}$. (Nonempty because $\beta_0 \in K_0$.)

11.7 Theorem: W is generated be reflections in the walls of K_0.

11.8 Let S be the set of walls of the distinguished Weyl chamber. Dynkin diagram: vertices = S, number of edges is equal to $n_{\alpha\beta}n_{\beta\alpha}$ (encode the angles $\angle(\alpha\beta)$). Hence the relations for the corresponding relations $s, t \in S$

- $s^2 = 1$
- no edge: s and t commute $\iff (st)^2 = 1$
- one edge $(st)^3 = 1$
- double edge $(st)^4 = 1$
- triple edge $(st)^6 = 1$

11.9 Exercise: (Theorem) These are the relations defining W. (This is an example of a Coxeter group)

11.10 Additionally we draw ,,<" to denote which root is longer. This does not effect the Weyl group.

11.11 In what follows we fix a distinguished Weyl chamber K_0, $\beta_0 \in K_0$ and we say that $\alpha > 0$ if $(\alpha, \beta_0) > 0$. We have $\overline{K}_0 = \{w \in \mathfrak{t}^* \mid \forall\, \alpha \in R_+\ (w, \alpha) \ge 0\}$.

11.12 Enveloping algebra of a Lie algebra $U(\mathfrak{g})$: it the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $XY - YX - [X,Y]$. Let $\mathfrak{g}$ be a reductive Lie algebra (a complexification of a Lie algebra of a compact group). A representation of $\mathfrak{g}$ is the same as $U(\mathfrak{g})$-module.

- Let $\mathfrak{b}_+ = \mathfrak{t} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. Similarly $\mathfrak{b}_-$. (It is a solvable Lie algebra.)
- Let $\mathfrak{n}_+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. Similarly $\mathfrak{n}_-$. (It is a nilpotent Lie algebra.)

11.13 Let V be a representation, suppose that $v \in V_w$ for $w \in \overline{K}_0$, and suppose that $Xv = 0$ for $X \in \mathfrak{g}_\alpha$ with $\alpha > 0$ (we say that v is a highest weight vector). Then:

- the representation generated by v that is $U(\mathfrak{g})v$ is equal to $U(\mathfrak{n}_-)v$.
- $\dim((U(\mathfrak{g}))_w) = 1$.

Pf: one can replace each monomial in $U(\mathfrak{g})$ be a combination of monomials with increasing $\phi(\alpha)$.

- $U(\mathfrak{g})v$ is simple.

Pf. If $U(\mathfrak{g})v = \bigoplus W_i$ then for some i_0 the projection of v onto W_{i_0} does not vanish. The projection preserves weights, thus $v \in W_{i_0}$. Hence $U\mathfrak{g} = W_{i_0}$.

11.14 Cor. Any two irreducible representations with v satisfying the assumptions above are isomorphic.
Pf. Consider the product representation an the subrepresentatation generated by (v, v') and the projections to V and V'.

11.15 Let $w \in \overline{K}_0$. There is a map $U(\mathfrak{b}_+) \to U(\mathfrak{t})$ which allows to treat $\mathbb{C}_w$ as a $U(\mathfrak{b}_+)$-module. The induced representation $M(w) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_w$ is called the Verma module. It is of infinite dimension, but irreducible representation of the weight w is a quotient of $M(w)$. [R. Carter, Lie Algebras of Finite and Affine Type, §10 (2005)]

11.16 Theorem: Let $\Lambda^*_{coroots} = \{w \in \mathfrak{t}^* \mid \forall \alpha \in R \; w(\alpha^*) \in \mathbb{Z}\}$. There is a bijection between irreducible representations of $\mathfrak{g}$ (reductive/compact) and the lattice points $\Lambda^*_{coroots} \cap \overline{K}_0$

- Uniqueness is given by 11.14
- existence:

– quotients of Verma modules
(one needs to prove that in $M(w)$ there is a largest proper submodule and it is of finite codimension).
– effective method case by case.

11.17 Example. $Sp(2)$.

- Roots $\pm L_i \pm L_j$, $2L_i$
- Coroots $\pm L_i \pm L_j$, L_i
- $\Lambda^*_{coroots} = \langle L_1, L_2 \rangle$
- Simple roots $-2L_2$, $L_1 - L_2$,
- $\overline{K}_0 = \{(x,y) \in \mathbb{R}^2 \mid x > 0, \; x > y\} = Conv(t(0,1), s(1,1))$.
- $\beta_0 = \frac{1}{2}(2L_1 + 2L_2 + (L_1 + L_2) + (L_1 - L_2)) = 2L_1 + 1L_2$
- $\Lambda^*_{coroots} \cap \overline{K}_0 = \{mL_1 + nL_2 \mid m, n \in \mathbb{Z}, \; m \geq 0, \; n \in [0, m]\}$
- The natural representation has the highest weight L_1

• In the second exterior power $\Lambda^2\mathbb{C}^4$ there are weights: $\pm L_1 \pm L_2$, twice 0. This representation is not irreducible, since $\Lambda^2\mathbb{C}^4 = \Lambda^2(\mathbb{C}^4)^*$ contains the invariant symplectic form ω. The kernel of

$$\omega : \Lambda^2\mathbb{C}^4 \to \mathbb{C}$$

is irreducible (all weights without multiplicities).

• The irreducible representation of $\mathfrak{sp}(2)$ with highest weight $mL_1 + nL_2$ is the subrepresentation of $Sym^{m-n}\mathbb{C}^4 \otimes Sym^n\Lambda^2\mathbb{C}^4$ generated by the vector $e_1^{n-m} \otimes (e_1 \wedge e_2)^m$

12 π_1, center, rank 2 examples

12.1 • Roots and coroots: $\Lambda^*_{roots} := \langle R \rangle \subset \Lambda^*$,

• We have shown in 10.12 that $\alpha^* \in ker(exp : \mathfrak{t} \to T) = \Lambda$ i.e. $\Lambda_{coroots} := \langle R^* \rangle \subset \Lambda$

• Thus $(\alpha^*, w) \in \mathbb{Z}$ for $w \in \Lambda^*$, hence $\Lambda^* \subset \Lambda^*_{coroots}$.

$$\Lambda^*_{roots} \subset \Lambda^* \subset \Lambda^*_{coroots}$$

• Dually

$$\Lambda_{coroots} \subset \Lambda \subset \Lambda_{roots},$$

where $\Lambda_{roots} = \{X \in \mathfrak{t} : \forall\, \alpha \in R\ \alpha(X) \in \mathbb{Z}\}$.

• Fact:

$$\Lambda_{roots}/\Lambda = Z(G)\,, \qquad \Lambda/\Lambda_{coroots} = \pi_1(G).$$

12.2 The proof is long and based on the exact sequence of homotopy groups of the fibration $T \hookrightarrow G \twoheadrightarrow G/T$, which gives the surjection $\pi_1(T) \twoheadrightarrow \pi_1(G)$:

• one shows that $\pi_2(G) = 0$ and $\pi_1(G/T) =$ (this e.g. follows from the fact, that $G/T = (G_{\mathbb{C}})/B_+$ can be decomposed into even-dimensional cells.)

• There is an exact sequence

$$0 \to H_2(G/T) = \pi_2(G/T) \to \pi_1(T) \to \pi_1(G) \to 0$$

• We have $\pi_1(T) = \Lambda = Hom(T, S^1)$. To see that each coroot is in the kernel check it for $SU(2)$ and $SO(3)$ and use the diagram for each root:

$$\begin{array}{ccccc} exp(t\alpha^*) & = & S^1 & \subset & T \\ & & \cap & & \cap \\ & & G_\alpha & \subset & G \end{array}$$

(Here $G_\alpha \simeq SU(2)$ or $SO(3)$.)

• for the proof that $ker(\pi_1(T) \to \pi_1(G)) = \Lambda_{coroots}$ see [B-tD,§7].

12.3 If for all roots $\alpha(X) \in \mathbb{Z}$, then $exp(X)$ acts trivially on $\mathfrak{g}_\alpha$. This give a map $exp : \Lambda_{roots} \to Z(G)$ with the kernel Λ. The map is surjective: if $g \in Z(G)$, then $g \in T$, hence $g = exp(X)$. For any root α the value $\alpha(X)$ has to be integral.

12.4 Exercise: Compute the lattices of roots an coroots for $GL(n)$ (roots will be of smaller rank and Λ_{roots} will not be discrete). Check that the formulas for $Z(G)$ and for $\pi_1(G)$ work.

12.5 Example. $SO(5)$.

- Roots $\pm L_i \pm L_j$, L_i
- Coroots $\pm L_i \pm L_j$, $2L_i$
- $\Lambda^*_{coroots} = \{xL_1 + yL_2 \mid 2x \in \mathbb{Z},\ 2y \in \mathbb{Z},\ x + y \in \mathbb{Z}\rangle\}$
- Simple roots $-2L_2$, $L_1 - L_2$,
- $\overline{K}_0 = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ x > y\} = Conv(t(0, 1), s(1, 1))$.
- $\beta_0 = \frac{1}{2}(L_1 + L_2 + (L_1 + L_2) + (L_1 - L_2)) = \frac{3}{2}L_1 + \frac{1}{2}L_2$ (Note: $\beta_0 \in \Lambda^*_{coroots}$)
- $\Lambda^*_{coroots} \cap \overline{K}_0 = \{mL_1 + nL_2 \mid m, n \in \frac{1}{2}\mathbb{Z},\ m \geq 0,\ n \in [0, m],$ same fractional part$\}$
- The natural representation has the highest weight L_1
- In the second exterior power $\Lambda^2\mathbb{C}^5$ the vector $e_1 \wedge e_2$ has weight: $L_1 + L_2$ but no way to get a half weight
- Missing the representation with the highest weight $\frac{1}{2}(L_1 + L_2)$. TBA **Spinor representation**. But here one can use the isomorphism $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$. Then $ker(\Lambda^2\mathbb{C}^4 \to \mathbb{C})$ is switched to the natural representation of $SO(5)$ and the spinor representation corresponds to the natural representation of $Sp(2)$.

12.6 Exercise: $SO(4)$

- Roots $\pm L_i \pm L_j$,
- Coroots $\pm L_i \pm L_j$,
- $\Lambda^*_{coroots} = \{xL_1 + yL_2 \mid x - y \in \mathbb{Z},\ x + y \in \mathbb{Z}\rangle\}$
- Simple roots $L_1 + L_2$, $L_1 - L_2$,
- $\overline{K}_0 = \{(x, y) \in \mathbb{R}^2 \mid x > |y|\} = Conv(t(1, -1), s(1, 1))$.
- $\beta_0 = \frac{1}{2}((L_1 + L_2) + (L_1 - L_2)) = L_1$
- $\Lambda^*_{coroots} \cap \overline{K}_0 = \{mL_1 \pm nL_2 \mid m, n \in \frac{1}{2}\mathbb{Z},\ m \geq n \geq 0,$ same fractional part$\}$
- The natural representation has the highest weight L_1
- In the second exterior power $\Lambda^2\mathbb{C}^4$ the vector $e_1 \wedge e_2$ has weight: $L_1 + L_2$
- Missing half weight representations $\frac{1}{2}(L_1 \pm L_2)$, the spinor representations S_+ and S_-. They do not come from representations of $SO(4)$ since $\frac{1}{2}(L_1 \pm L_2) \notin \Lambda^*$. But they come form the representation of the universal cover $\widetilde{SO}(4) = SU(2) \times SU(2)$ a.k.a. $Spin(4)$.

13 Representations of $SL(n)$

This lecture contains no proof, but only an algorithm how to compute the irreducible representations of $SL(n)$ and $GL(n)$. The universal example is the representation of $SL(3)$ of the highest weight $3L_1 + L_2$. We summarize four important theorems

1) Every irreducible representation of $SL(n)$ is given by the value of the Schur functor $\mathbb{S}_\lambda(\mathbb{C}^n)$

2) Characters of irreducible representations are the Schur functions S_λ

3) One can compute Schur function by determinant of a matrix with symmetric functions

4) Combinatorial methods of computing coefficients of Schur functions, i.e. Kostka numbers

5) How to multiply irreducible representations (or Schur functions) : $\mathbb{S}_\lambda(\mathbb{C}^n) \otimes \mathbb{S}_\mu(\mathbb{C}^n) =?$

– **Pieri rule**

– **Littlewood-Richardson rule.**

13.1 $SL(n)$

- Roots $L_i - L_j$,
- Coroots $L_i - L_j$,
- $\Lambda^*_{coroots}$ spanned by L_i
- Simple roots $L_1 - L_2$, $L_2 - L_3$,... , $L_{n-1} - L_n$
- $\overline{K}_0$ is given by $\sum a_i L_i \in \overline{K}_0$ if and only if $a_i \geq a_{i+1}$ for all $i = 1, 2, \dots, n-1$. It is spanned by
$$L_1,\ L_1 + L_2,\ \dots\ ,\ L_1 + L_2 + \cdots + L_n$$
- (exercise) $\beta_0 = \frac{1}{2}(nL_1 + (n-1)L_2 + \dots L_n$
- $\Lambda^*_{coroots} \cap \overline{K}_0 =$ integral points of $\overline{K}_0$
- The natural representation has the highest weight L_1
- the k-th exterior power $\Lambda^k \mathbb{C}^n$ has the highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ of weight: $L_1 + L_2$
- $\Lambda^k \mathbb{C}^n$ is irreducible, since all the weight spaces are of codimension one.
- To construct a representation of the weight $\sum a_i L_i$ with $a_i \geq a_{i+1}$ for all $i = 1, 2, \dots, n-1$ write $b_i = a_i - a_{i+1}$, $b_n = a_n$ (can assume $b_n = a_n = 0$). Then the representation

$$\bigotimes_{i=1}^{n-1} Sym^{b_i} \Lambda^i \mathbb{C}^n$$

has the highest weight vector of the weight

$$\sum_{i=1}^{n-1} b_i (L_1 + L_2 + \cdots + L_i) = \sum_{i=1}^{n-1} a_i L_i\,.$$

The subrepresentation generated by

$$e_1^{b_1} \otimes (e_1 \wedge e_2)^{b_2} \otimes \cdots \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1})^{b_{n-1}}$$

has the desired weight.

13.2 For representation of $\mathfrak{gl}(n)$ the exterior power $\Lambda^n \mathbb{C}^n$ is nontrivial, $\sum_{i=1}^n L_i \neq 0 \in \mathfrak{t}^*_{\mathfrak{gl}(n)}$ and we cannot assume $b_n = 0$.

13.3 For $\mathfrak{gl}(n)$ the representations are indexed by ,,partitions", i.e. the sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. The highest weight of V_λ is $\sum \lambda_i L_i$. (*We omit zeros in the notation or write* k^b *if* k *repeats* b *times.*)

$$S_{1^k} = \Lambda^k \mathbb{C}^n\,, \qquad V_k = Sym^k(\mathbb{C}^n)$$

Remark: this way w do not obtain all representations of GL_n, since we never get $(\Lambda^n \mathbb{C}^n)^*$.

13.4 Explicite construction of V_λ via Schur functors, see [Fulton-Haris §6].

- Let $\lambda^\vee$ be the transpose partition: eg $(3,1)^\vee = (2,1,1)$. Then

$$V^{\otimes|\lambda|} = V^{\otimes \lambda_1} \otimes V^{\otimes \lambda_2} \otimes \cdots \otimes V^{\otimes \lambda_n} = V^{\otimes \lambda_1^\vee} \otimes V^{\otimes \lambda_2^\vee} \otimes \cdots \otimes V^{\otimes \lambda_m^\vee}$$

where m=number of rows in the Young diagram.

• Symmetrization w/r to rows, $a : V^{\otimes|\lambda|} \to Sym^{\lambda_1}V \otimes Sym^{\lambda_2}V \otimes \cdots \otimes Sym^{\lambda_n}V$

• Antisymmetrization w/r to columns $b : V^{\otimes|\lambda|} \to \Lambda^{\lambda_1^\vee}V \otimes \Lambda^{\lambda_2^\vee}V \otimes \cdots \otimes \Lambda^{\lambda_n^\vee}V$

• $V_\lambda = \mathbb{S}_\lambda(\mathbb{C}^n) = im(a \circ b)$

$$\mathbb{S}_\lambda(V) = im\Big(\bigotimes \Lambda^{\lambda_i^\vee}V \hookrightarrow V^{\otimes|\lambda|} \twoheadrightarrow \bigotimes Sym^{\lambda_i}V\Big)$$

• basis is numerated by filling of the Young tableau: non decreasing in rows, increasing in columns.

• Dimension of $\mathbb{S}_\lambda(\mathbb{C}^n)_\nu$ for $\lambda \in \overline{K}_0$, $\nu \in \Lambda$ is called the Kostka number $K_{\lambda\mu}$. This number is equal to the number of fillings of the shape λ with numbers: ν_1 of 1's, ν_2 of 2's etc. The filling has to be nondecreasing in rows and increasing in columns: eg $k_{(3,1),(2,1,1)} = 2$ because there are two fillings:

$$\begin{matrix} 1 & 1 & 2 \\ 3 & & \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 1 & 3 \\ 2 & & \end{matrix}$$

By symmetry it is enough to consider $\mu \in \overline{K}_0$.

13.5 (**Weyl character formula [F-H,§24]**) What is the character of the representation V_λ?

$$\chi(V_\lambda)_T = S_\lambda$$

where S_λ is the Schur function: Let $\rho = (n-1, n-2, \dots, 0)$. Define $S_\lambda(x_1, x_2, \dots, x_n) = W_{\lambda+\rho}/W_\rho$, where

$$W_{(a_1,a_2,\dots,a_n)} = det \begin{pmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} & \dots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_3} & \dots & x_2^{a_n} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} & \dots & x_3^{a_n} \\ & \vdots & & & \vdots \\ x_n^{a_1} & x_n^{a_2} & x_n^{a_3} & \dots & x_n^{a_n} \end{pmatrix}$$

In particular $W_\rho = \pm Vandermonde$.

13.6 In short

$$\chi_{\mathbb{S}_\lambda(\mathbb{C}^n)}(x_1, x_2, \dots, x_n) = S_\lambda(x_1, x_2, \dots, x_n) \sum_\mu K_{\lambda\mu}\, x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n} \,.$$

13.7 There are more efficient methods of computation of Schur functions: let $\lambda^\vee$ be the transpose partition: eg $(3,1)^\vee = (2,1,1)$. Then $S_\lambda = \det(\{e_{\lambda_i+j-i}\}_{i,j=1,\dots length(\lambda^\vee)})$, eg $S_{3,1} = \det \begin{pmatrix} e_2 & e_3 & e_4 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{pmatrix}$.
Here e_i is the elementary symmetric function, $e_0 = 1$, $e_{?negative} = 0$. See [Ian MacDonald-Symmetric Functions and Hall Polynomials,§I].

13.8 Exercise S_λ is a polynomial, and ,,does not depend" on the number of variables, provided that this number is sufficiently large.

13.9 Exercise: Show that $\dim V_\lambda = \prod_{1 \le i \le j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$

13.10 Exercise: Check the Weyl character formula for $\lambda = 1^k$ and $\lambda = k$

13.11 For $n = 2$, $\lambda = (a, b)$, the highest weight is $aL_1 + bL_2$.
$W_\rho = x_1 - x_2$, $W_{\lambda+\rho} = x_1^{a+1}x_2^b - x_2^{a+1}x_1^b = (x_1x_2)^b(x_1^{a-b+1} - x_2^{a-b+1})$
$S_{a,b} = (x_1x_2)^b \sum_{k+\ell=a-b} x_1^k x_2^\ell$
This specializes to the character of $SL(2)$ (substitution $x_1 = x$, $x_2 = x^{-1}$):
$S_{a,b} = \sum_{k+\ell=a-b} t^{k-\ell}$

13.12 For $n = 3$, $\lambda = (2, 1, 0)$ the highest weight is $3L_1 + L_2$ and $V_\lambda \subset \mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3$ (This is the representation considered before for SL_3, where we have $\Lambda^2\mathbb{C}^3 = (\mathbb{C}^3)^*$.)
$W_\rho = det[((x_1^2, x_1, 1), (x_2^2, x_2, 1), (x_3^2, x_3, 1))]$
$W_{\lambda+\rho} = det[((x_1^4, x_1^2, 1), (x_2^4, x_2^2, 1), (x_3^4, x_3^2, 1))]$

$$S_\lambda = \frac{(x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$$

Corrolary $\dim(V_{2,1}) = 8$. It is a proper subrepresentation of $\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3$. It is the kernel of the map to $\Lambda^3(\mathbb{C}^3)$.

13.13 It is also irreducible as representations of $SL(n)$: it is isomorphic to the adjoint representation.

13.14 Exercise: Irreducible representations of $GL(n)$ are irreducible as representations of $SL(n)$.

13.15 For $n = 3$, $\lambda = (3, 1, 0)$ the highest weight is $3L_1 + L_2$ and $V_\lambda \subset Sym^2\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3$ (This is the representation considered before for SL_3, where we have $\Lambda^2\mathbb{C}^3 = (\mathbb{C}^3)^*$.)
$W_\rho = det[((x_1^2, x_1, 1), (x_2^2, x_2, 1), (x_3^2, x_3, 1))]$
$W_{\lambda+\rho} = det[((x_1^5, x_1^2, 1), (x_2^5, x_2^2, 1), (x_3^5, x_3^2, 1))]$

$$S_\lambda = \frac{???}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} =$$

$x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 + x_2^2x_3^2 + x_2x_3^3$

13.16 Main problem: how to multiply the representations?

$$V_\lambda \otimes V_\mu = \sum c^\nu_{\lambda\mu} V_\nu$$

How to compute ν?

13.17 Pieri formula [Fulton-Harris, p.455]. Multiplication by V_k or by V_{1^k}

13.18 Littlewood-Richardson formula

13.19 All the representations V_λ lift to ,,polynomial representations of" $GL_n(\mathbb{C})$. Irreducible representations of $GL_n(\mathbb{C})$ are of the form $\mathbb{S}_\lambda(\mathbb{C}^n) \otimes (\Lambda^n\mathbb{C}^n)^{\otimes k}$ with λ partition, $\lambda_n = 0$ and $k \in \mathbb{Z}$

14 Weil character formula (cont.), representations of $SO(n)$, Clifford algebras

14.1 Let G any semisimple Lie group with fixed maximal torus. We fix a Weyl chamber K_0, in that way choosing positive roots, $\rho = \beta_0$=half-sum of positive roots. Let $w \in \overline{K}_0$, then the character of the irreducible representation restricted to the maximal torus is equal to $W(w + \rho)/W(\rho)$, where

$$W(w) = \sum_{\sigma \in \text{Weyl group}} x^w = \sum_{\sigma \in \Sigma_2} \sum_{\epsilon \in \{-1,1\}^2} (-1)^\epsilon t_1^{\epsilon_1 w_1} t_2^{\epsilon_2 w2}$$

Here $x^w = \exp(w(t))$ denotes the map $T \to \mathbb{C}^*$ given by the exponential of w.

14.2 Example for $Sp(2)$: $\rho = 2L_1 + L_2$, $W = \mathbb{Z}_2^2 \rtimes \mathbb{Z}^2$ (permutations with signs)

$$W(aL_1+bL_2) = (x_1^a x_2^b - x_1^{-a} x_2^b - x_1^a x_2^{-b} + x_1^{-a} x_2^{-b}) - (x_1^b x_2^a + \dots) = \sum_{\sigma \in \Sigma_2} (-1)^\sigma (x_{\sigma(1)}^a - x_{\sigma(1)}^{-a})(x_{\sigma(2)}^b - x_{\sigma(2)}^{-b})$$

(in general $det\big(\{x_j^{w_i} - x_j^{-w_i}\}_{1\le i,j\le n}\big)$)

$$W(\rho) = (x_1 - x_2)(x_1x_2 - 1)(x_1^2 - 1)(x_2^2 - 1)/(x_1x_2)^2$$

The natural representation has weight L_1

$$W(3L_1 + L_2) = (x_1^2 - x_2^2)((x_1x_2)^2 - 1)(x_1^2 - 1)(x_2^2 - 1)/(x_1^3x_2^3)$$

The character is equal to

$$(x_1 + x_2)({x_1}^{-1}{x_2}^{-1} + 1)$$

14.3 Exercise: Show that for $Sp(n)$ we have

$$W(\rho) = \Big(\prod_{i<j}(x_i - x_j)(x_ix_j - 1) \cdot \prod_i (x_i^2 - 1)\Big)/\prod_i x_i^n$$

14.4 Exercise: Compute the character for $w = L_1 + L_2$

14.5 [B-tD §I.6] Clifford algebra of $V \simeq \mathbb{K}^n$ with a quadratic form $Q : V \to \mathbb{K}$: $C(Q) = T(V)/(v \otimes v - Q(v))$.

- $\iota : V \hookrightarrow C(V)$ as generators
- $\iota(v)\iota(w) + \iota(w)\iota(v) = 2\phi(v, w)$, where $\phi(v, w) = W(v + w) - Q(v) - Q(w)$ is the associated bilinear form.
- $\mathbb{Z}_2$ gradation $C(Q) = C(Q)^{ev} \oplus C(Q)^{odd}$
- antihomomorphism $x \mapsto t(x) = x^t$, $\iota(v)^t = \iota(v)$ for $v \in V$
- canonical homomorphism $x \mapsto \alpha(x)$, $\alpha(\iota(v)) = -\iota(v)$ for $v \in V$
- $\overline{x} := t\alpha(x) = \alpha t(x)$

14.6 $\Gamma(Q) = \{x \in C(Q)^* \,|\, \alpha(x)vx^{-1} \in V\}$ acts on V

14.7 Norm map $N : C(Q) \to C(Q)$, $N(x) = x\overline{x}$, for $v \in V$ we have $N(\iota(v)) = -Q(v) \cdot 1$

14.8 $\mathbb{H} \simeq C(Q)^{ev}$ for $V = \mathbb{R}^3$, $Q(v) = -|v|$

14.9 Construction of Pin and Spin groups: $V = \mathbb{R}^n$, $Q(v) := -|v|^2$, $N(v) = |v|^2$.

15 Spinors

15.1 $ker(\Gamma_n \to Aut(\mathbb{R}^n)) = \mathbb{R}^*)$

15.2 For $x \in \Gamma_n$ we have $N(x) \in \mathbb{R}^*$.
Pf We check that for any $v \in \mathbb{R}^n$ the element $N(\overline{x}) = \overline{x}x = t\alpha(x)x$ acts trivially on $\mathbb{R}^n$.
By the definition of Γ_n we have

$$\alpha(x)vx^{-1} \in V.$$

t is constant on V, hence

$$t(x)^{-1}vt\alpha(x) = \alpha(x)vx^{-1}$$

hence

$$v = t(x)\alpha(x)v(t\alpha(x)x)^{-1},$$

hence

$$\overline{x}x \in ker = \mathbb{R}^*.$$

Remark: this implies $\frac{1}{N(\overline{x})}\overline{x} = x^{-1}$, so $N(x) = N(\overline{x})$.

15.3 N is homomorphism on Γ_n

15.4 The action of Γ_n is via isometries $ker : (\rho : \Gamma_n \to O(n)) = \mathbb{R}^*$.
Pf: We know that $v \in \Gamma_n$ (it defines the reflection in $v^\perp$) and check that $N(\overline{x}v) = N(xv)$ $(= N(\overline{x})N(v)$)

15.5 $Pin(n) := ker(N : \Gamma_n \to \mathbb{R}^*)$, there is an exact sequence

$$0 \to \mathbb{Z}_2 \to Pin(n) \to O(n) \to 0$$

Pf. $ker(N) \cap ker(\rho) = \mathbb{Z}_2$

15.6 $Spin(n) := Pin(n) \cap C(Q)^{ev} = \rho^{-1}(SO(n))$,

15.7 ρ is a nontrivial double covering.
Pf: the path $s \mapsto \gamma(s) = \cos(s) + \sin(s)e_1e_2$, $s \in [0, \pi]$ joins 1 with -1, $\rho(\gamma(s))$ =rotation in the plane $lin(e_1e_2)$ by the angle $2s$.

15.8 Spin representations: for $\mathfrak{so}(2n)$ there are highest vectors $\alpha = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-1} \pm L_n)$. The spinor representation $S_{2n} \subset \Lambda^* V$.
Taking the convex hull of $W \cdot \alpha \subset \Lambda^*_{coroot}$ we obtain the polytope spanned by $\frac{1}{2}(\pm L_1 \pm L_2 \pm \cdots \pm L_{n-1} \pm L_n)$ with even or odd number of $-$ depending whether it is S^+_{2n} or S^-_{2n}. 0 lies in the interior but cannot be a weight of $S^\pm_{2n}$. Hence $\dim S^\pm_{2n} = 2^{n-1}$.

15.9 Similarly: S_{2n+1} has the highest weight $\frac{1}{2}(L_1 + L_2 + \cdots + L_n)$, $\dim S_{2n} = 2^n$.

15.10 Construction via complexification: $V \otimes \mathbb{C} = W \oplus W^*$. Then $S^+_{2n} = \Lambda^{ev}W$, $S^-_{2n} = \Lambda^{odd}W$ with the action of $C(Q_n)$ given on the generators $w \cdot \xi = w \wedge \xi$ for $w \in W$, $\xi \in \Lambda^* W$ and the derivation $f \cdot \xi = D_{2f}\xi$ for $f \in W^*$.
[Fu-Ha §Lemma 20.9]. Weight of e_I is equal to $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{i \notin I} L_i)$

15.11 For $\mathfrak{so}(2n+1)$ the highest vectors is equal to $\frac{1}{2}(L_1 + L_2 + \cdots + L_n)$. The spinor representation is constrcted via complexification: $V \otimes \mathbb{C} = W \oplus W^* \oplus \mathbb{C}$. The additional unit vector acts as α on $S_{2n+1} = \Lambda^* W$. This action cannot be split into even and odd parts.

15.12 Special cases: by π we mean the natural action of $SO(n)$ on $\mathbb{R}^n$

Group	π	$S^{\pm}$
$Spin(3) = SU(2)$	$im\mathbb{H}$ $\pm 2L_1$, 0	$\mathbb{C}^2$ $\pm L_1$
$Spin(4)_{\mathbb{C}} = SL_2(\mathbb{C})^2$	$M(2\times 2)$ $\pm L_1 \pm L_1'$	two copies of $\mathbb{C}^2$ $\pm L_1$ and $\pm L_1'$
$Spin(5)_{\mathbb{C}} = Sp(2)$	$ker(\omega) \subset \Lambda^2(\mathbb{C}^4)$ $\pm L_1 \pm L_2$, 0	$\mathbb{C}^4$ $\pm L_1$, $\pm L_2$
$Spin(6)_{\mathbb{C}} = SL_4(\mathbb{C})$	$\Lambda^2(\mathbb{C}^4)$ $L_i + L_j$	$\mathbb{C}^4$ and $(\mathbb{C}^4)^*$ L_i and $-L_i$

15.13 Representation ring:[B-tD, §VI.6]

15.14 Special properties of $SO(8)$: triality. Note: $\dim(S_8^{\pm}) = 8$.
The basis of S_8^+

vector	weight	
1	$\frac{1}{2}(-L_1 - L_2 - L_3 - L_4)$	$-L_1'$
$\boxed{e_1 \wedge e_2}$	$\frac{1}{2}(+L_1 + L_2 - L_3 - L_4)$	L_2'
$\boxed{e_1 \wedge e_3}$	$\frac{1}{2}(+L_1 - L_2 + L_3 - L_4)$	L_3'
$\boxed{e_1 \wedge e_4}$	$\frac{1}{2}(+L_1 + L_2 - L_3 - L_4)$	L_4'
$e_2 \wedge e_3$	$\frac{1}{2}(-L_1 + L_2 + L_3 - L_4)$	$-L_4'$
$e_2 \wedge e_4$	$\frac{1}{2}(-L_1 + L_2 - L_3 + L_4)$	$-L_3'$
$e_3 \wedge e_4$	$\frac{1}{2}(-L_1 - L_2 + L_3 + L_4)$	$-L_2'$
$\boxed{e_1 \wedge e_2 \wedge e_3 \wedge e_4}$	$\frac{1}{2}(+L_1 + L_2 + L_3 + L_4)$	L_1'

The boxed vectors span an isotropic subspace of S_{2n}^+ (with isotropic to the quadratic form given by $\wedge$.) There exists an action of $Spin(8)$ such that S_8^+ becomes the representation π. Precisely, let us denote by ρ^+ the spinor representation.

15.15 Claim: There exist an automorphism $\phi : Spin(8) \to Spin(8)$ such that $\pi = \rho^+ \circ \phi$. Moreover $\rho^- = \rho^+ \circ \phi^2$ and $\phi^3 = id$. That is ϕ permutes cyclically $\mathbb{C}^8$, S_8^+ and S_-^8.

15.16 Triality map: given three vector spaces of equal dimensions with quadratic nondegenerate forms: (V_i, Q_i). Triality is a trilinear map

$$V_1 \times V_2 \times V_3 \to \mathbb{K}$$

such that the associated maps

$$f_k : V_i \times V_j \to V_k^* \simeq V_k$$

satisfy

$$Q_k(f_k(v,w)) = Q_i(v)Q_j(w).$$

Examples:
- Clifford multiplication $S_8^+ \times \mathbb{R}^8 \to S_8^-$
- $V_1 = V_2 = V_3 = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ or octonions
- there are no more trialities.

15.17 Construction of triality automorphism via permutation of weights, see Appendix.

15.18 Remark $S_8^{\pm}$ admits a real structure, i.e. $S_8^{\pm} \simeq S_{\mathbb{R},8}^{\pm} \otimes \mathbb{C}$ and triality can be realized as an automorphism of the real group $Spin(8)$

15.19 More about spinors and triality: A. Vistoli, Notes on Clifford algebra, Spin Groups and Triality, http://homepage.sns.it/vistoli/clifford.pdf

Exceptional group G_2

15.20 $so(8)^{\phi} = \mathfrak{g}_2$

15.21 $\mathfrak{g}_2$ acting on $\mathbb{R}^8$ fixes one direction, say e_1, hence $\mathfrak{g}_2$ acts on $e_1^{\perp} \simeq \mathbb{R}^7$

15.22 $\mathfrak{g}_2$ acting on $\Lambda^3\mathbb{R}^7$ fixes one 3-form. This form defines octonionic multiplication.

15.23 $\mathfrak{g}_2$ is precisely the algebra of derivation of octonions and $Aut(\mathbb{O}) = G_2$.

15.24 For the computations see http://www.mimuw.edu.pl/%7Eaweber/triality/

15.25 The group with Lie algebra $\mathfrak{g}_2$ is simplyconnected and has the trivial center.

15.26 A another constuction of G_2 can be found in [J. F. Adams: Lectures on Exceptional Lie Groups]

16 Appendix: Triality and G_2

For real Lie algebra $\mathfrak{so}(8)$ the triality was constructed by E. Cartan, *Le principe de dualité et la théorie des groupes simples et semi–simples*, Bulletin sc. Math. (2) 49, 361–374 (1925). The approach presented here is equivalent, to the Cartan's work in the complex case. The triality automorphism given below has an advantage, that the root spaces coincide with the coordinates of the matrix and these coordinates are permuted by $\mathbb{Z}_3$.
Working with the complex coefficients we choose a basis in $\mathbb{C}^8$ (as in [Fu-Ha, §19]) in which the quadratic form is equal to

$$x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5\,.$$

For real coefficients this means that we deal with $SO(4,4)$. The maximal torus of $SO(4,4)$ consists of the diagonal matrices

$$diag(e^{t_1}, e^{t_2}, e^{t_3}, e^{t_4}, e^{-t_4}, e^{-t_2}, e^{-t_3}, e^{-t_1})\,.$$

The following weights form the root system of $\mathfrak{so}(4,4)$:

$$\pm L_i \pm L_j \quad \text{for } i \neq j\,,$$

where $L_i(t_1,t_2,t_3,t_4) = t_i$. Choosing the Borel subgroup as the upper triangular matrices we obtain the Dynkin diagram of simple roots:

$$\boxed{C = L_3 + L_4}$$

$$\boxed{A = L_1 - L_2} \;—\; \boxed{Y = L_2 - L3} \;\begin{matrix} / \\ \\ \backslash \end{matrix}$$

$$\boxed{B = L_3 - L_4}$$

The triality automorphism rotates the diagram anti-clockwise:

$$(L_1 - L_2) \mapsto (L_3 + L_4) \mapsto (L_3 - L_4) \mapsto (L_1 - L_2)$$

and fixes the root $L_2 - L_3$. In the basis consisting of the weights L_i the triality automorphism is given by the remarkable matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{1}$$

16.1 Triality in $\mathfrak{so}(4,4)$

We have given a formula for triality automorphism acting on the dual $\mathfrak{t}^*$ of the Cartan subalgebra of $\mathfrak{so}(4,4)$. It does not preserve the lattice corresponding to the group $SO(4,4)$ but it preserves the lattice spanned by L_i's and $\frac{1}{2}(L_1 + L_2 + L_3 + L_4)$, which corresponds to $Spin(4,4)$, the cover of $SO(4,4)$. We list below the set of positive roots:

$$\begin{array}{cccccc} L_1 - L_2 & L_1 - L_3 & L_1 - L_4 & L_1 + L_4 & \boxed{L_1 + L_3} & \boxed{L_1 + L_2} \\ & \boxed{L_2 - L_3} & L_2 - L_4 & L_2 + L_4 & L_2 + L_3 & \\ & & L_3 - L_4 & L_3 + L_4 & & \end{array}$$

The boxed roots are fixed by triality. It can be easily seen when we express the roots in the basis of simple roots:

$$\begin{array}{cccccc} A & AY & ABY & ACY & \boxed{ABCY} & \boxed{ABC2Y} \\ & \boxed{Y} & BY & CY & BCY & \\ & & B & C & & \end{array}$$

Here for example $ABC2Y$ denotes the root $A + B + C + 2Y$. Dividing roots into orbits of the triality automorphism we see that we have the fixed roots Y, $ABCY$ and $ABC2Y$. Three free orbits are generated by A, AY and ABY.

From general theory it follows that the triality automorphism of weights lifts to a self-map of the Lie algebra $\mathfrak{so}(4,4)$. But a priori it is not clear that one can find such a lift of order three. Not every lift satisfies $\phi \circ \phi \circ \phi = Id$. The choice of signs is not obvious and demands a careful check. The elements of $\mathfrak{so}(4,4)$ for our quadratic form defined by the matrix with 1's on the antidiagonal are the matrices $(m_{ij})_{1 \le i,j \le 8}$ which are antisymmetric with respect to the reflection in the antidiagonal. Such a matrix is transformed by the triality automorphism to the following one

$$\begin{pmatrix} \bullet & m_{34} & -m_{24} & m_{26} & m_{14} & \boxed{m_{16}} & \boxed{m_{17}} & 0 \\ m_{43} & \bullet & \boxed{m_{23}} & m_{25} & -m_{13} & m_{15} & 0 & \boxed{-m_{17}} \\ -m_{42} & \boxed{m_{32}} & \bullet & m_{35} & m_{12} & 0 & -m_{15} & \boxed{-m_{16}} \\ m_{62} & m_{52} & m_{53} & \bullet & 0 & -m_{12} & m_{13} & -m_{14} \\ m_{41} & -m_{31} & m_{21} & 0 & \bullet & -m_{35} & -m_{25} & -m_{26} \\ \boxed{m_{61}} & m_{51} & 0 & -m_{21} & -m_{53} & \bullet & \boxed{-m_{23}} & m_{24} \\ \boxed{m_{71}} & 0 & -m_{51} & m_{31} & -m_{52} & \boxed{-m_{32}} & \bullet & -m_{34} \\ 0 & \boxed{-m_{71}} & \boxed{-m_{61}} & -m_{41} & -m_{62} & m_{42} & -m_{43} & \bullet \end{pmatrix}$$

The upper half of the diagonal $(t_1, t_2, t_3, t_4) = (m_{11}, m_{22}, m_{33}, m_{44})$ is transformed by the matrix (1) to

$$\begin{array}{l} \frac{1}{2}(m_{11} + m_{22} + m_{33} - m_{44}) \\ \qquad \frac{1}{2}(m_{11} + m_{22} - m_{33} + m_{44}) \\ \qquad\qquad \frac{1}{2}(m_{11} - m_{22} + m_{33} + m_{44}) \\ \qquad\qquad\qquad \frac{1}{2}(m_{11} - m_{22} - m_{33} - m_{44}) \end{array}$$

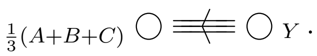
We remark that this is an unique automorphism with real coefficients extending the self-map of the maximal torus. We would like to stress, that both: the Cartan construction of triality for $\mathfrak{so}(8)$ and the triality for $\mathfrak{so}(4,4)$ presented here works for any ring in which 2 is invertible. On the other hand we note that the remaining real forms of the orthogonal algebra $\mathfrak{so}(k, k-8)$ for $k = 1, 2, 3, 5, 6, 7$ do not admit any triality automorphism with real coefficients. Already on the level of $\mathfrak{t}^*$ we obtain the rotation matrices with imaginary coefficients.

16.2 Noncompact version of G_2

The algebra fixed by ϕ consist of matrices of the form

$$\begin{pmatrix} t_2+t_3 & a_1 & -a_2 & a_3 & a_3 & a_4 & a_5 & 0 \\ a_6 & t_2 & a_7 & a_2 & a_2 & a_3 & 0 & -a_5 \\ -a_8 & a_9 & t_3 & a_1 & a_1 & 0 & -a_3 & -a_4 \\ a_{10} & a_8 & a_6 & 0 & 0 & -a_1 & -a_2 & -a_3 \\ a_{10} & a_8 & a_6 & 0 & 0 & -a_1 & -a_2 & -a_3 \\ a_{11} & a_{10} & 0 & -a_6 & -a_6 & -t_3 & -a_7 & a_2 \\ a_{12} & 0 & -a_{10} & -a_8 & -a_8 & -a_9 & -t_2 & -a_1 \\ 0 & -a_{12} & -a_{11} & -a_{10} & -a_{10} & a_8 & -a_6 & -t_2-t_3 \end{pmatrix}$$

16.1 Theorem. The fixed points of the triality automorphism is a Lie algebra of the type $\mathfrak{g}_2$. Indeed, the fixed elements of $\mathfrak{t}^*$ is spanned by the simple roots $Y = L_2 - L_3$ (the longer root) and $\frac{1}{3}(A + B + C) = \frac{1}{3}(L_1 - L_2 + 2L_3)$ (the shorter root)



The shorter root as the functional on $\mathfrak{t}^\phi$ is equal to A or B or C, but we represent it as an invariant element $\frac{1}{3}(A + B + C) \in (\mathfrak{t}^*)^\phi$. The positive root spaces of $\mathfrak{so}(4,4)^\phi$ are the following:

- the eigenspaces associated to the longer roots

$$\mathfrak{so}(4,4)_Y, \quad \mathfrak{so}(4,4)_{ABCY}, \quad \mathfrak{so}(4,4)_{ABC2Y}$$

- the diagonal subspaces associated to the shorter roots $\frac{1}{3}(A+B+C)$, $\frac{1}{3}(AY+BY+CY)$, $\frac{1}{3}(ABY+ACY+BCY)$

$$\big(\mathfrak{so}(4,4)_A \oplus \mathfrak{so}(4,4)_B \oplus \mathfrak{so}(4,4)_C\big)^\phi$$

$$\big(\mathfrak{so}(4,4)_{AY} \oplus \mathfrak{so}(4,4)_{BY} \oplus \mathfrak{so}(4,4)_{CY}\big)^\phi,$$

$$\big(\mathfrak{so}(4,4)_{ABY} \oplus \mathfrak{so}(4,4)_{ACY} \oplus \mathfrak{so}(4,4)_{BCY}\big)^\phi.$$

16.3 The twin-brother of octonion algebra

The Lie group corresponding to $\mathfrak{so}(4,4)^\phi$ is the noncompact form of G_2 since its maximal torus is $(\mathbb{R}^*)^2$. The Lie algebra $\mathfrak{so}(4,4)^\phi$ annihilates the vector $v_0 = e_4 - e_5$ and its action restricted to $v_0^\perp$ preserves the form

$$\widetilde{\omega} = 2e^*_{167} + 2e^*_{238} + e^*_{2(4+5)7} - e^*_{1(4+5)8} + e^*_{3(4+5)6}\,,$$

where $e^*_{i(4+5)j}$ means $e^*_i \wedge (e^*_4 + e^*_5) \wedge e^*_j$. The associated algebra is the algebra of pseudo-Cayley numbers (also called split octonions).