Positivity

of Schur function expansions

# of Thom polynomials

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joint with

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J^k – space of jets $(C^m, 0) \to (C^n, 0)$

$\Sigma \subset J^k$ – singularity type
= any closed algebraic subset, invariant w/r to automorphisms

$f : M^m \to N^n$ – holomorphic map of complex manifolds

Let $\Sigma(f)$ – the set of points in M of the type Σ

Suppose Σ is closed under suspension.

Thom: There exists a polynomial $P_\Sigma \in Z[c_1, c_2, c_3, \ldots]$ such that

$$[\Sigma(f)] = P_\Sigma(f^* c_* N / c_* M) \in H^*(M),$$

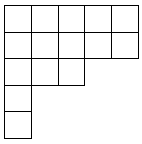
i.e. $[\Sigma(f)] \in H^*(M)$ depends only on Chern classes of M, N and the homotopy class of f.

Schur polynomials

$$Z[c_1, c_2, c_3, \ldots] = \lim_{i,j \to \infty} H^*(\mathrm{Grass}_i(C^j))$$

Schubert cells $\longrightarrow$ basis of $H_*(\mathrm{Grass}_i(C^j))$

Dual basis = Schur polynomials S_I
(for I - Young tableau)

E.g.  $= S_{55311}$

$$= \det \begin{pmatrix} c_5 & c_6 & c_7 & c_8 & c_9 \\ c_4 & c_5 & c_6 & c_7 & c_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 0 & 0 & 1 & c_1 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \end{pmatrix}$$

Theorem: (P. Pragacz, AW) For any singularity type P_Σ is a nonnegative combination of Schur polynomials.

Method of the proof:
– reduction to the universal quotient bundle Q over $\mathrm{Grass}_m(C^j)$ (instead of $T^*M - f^*T^*N$)

$$\widetilde{\Sigma} \quad \subset \quad \tilde{J}^k \quad = \quad \left(\oplus_{\ell=1}^k \mathrm{Sym}^\ell Q\right)^n \\ \downarrow \\ \mathrm{Grass}_m(C^j)$$

$\widetilde{\Sigma}$ – universal singularity locus

Q globally generated

$\Downarrow$

$\tilde{J}^k$ globally generated

– (Fulton-Lazarsfeld) "positivity of cone classes"

$\Downarrow$

intersection $[\widetilde{\Sigma}] \cdot [\text{Schubert cell}] \geq 0$

Fulton–Lazarsfeld

E – ample vector bundle
i.e. $\mathrm{Sym}^i(E)$ for $i >> 0$ has lots of sections

Theorem: If $E \to X$ is an ample bundle $C \subset E$ a cone, $\operatorname{codim} C = \dim X$ then

$$[C] \cdot [X] > 0$$

By limit process Theorem applies to globally generated bundles with $\geq$.

Fulton–Lazarsfeld $\Leftarrow$ Bloch–Gieseker

Theorem: If $E \to X$ is an ample bundle $\dim E \geq \dim X = n$ then

$$\int_X c_n(E) > 0$$

Maps $C^n \to C^n$

A_1: $\ydiagram{1}$

A_2: $2\ydiagram{2} + \ydiagram{1,1}$

A_3: $6\ydiagram{3} + 5\ydiagram{2,1} + \ydiagram{1,1,1}$

A_4: $24\ydiagram{4} + 26\ydiagram{3,1} + 10\ydiagram{2,2} + 9\ydiagram{2,1,1} + \ydiagram{1,1,1,1}$

$I_{2,2}$: $\ydiagram{2,2}$

A_5: $120\ydiagram{5} + 154\ydiagram{4,1} + 92\ydiagram{3,2} + 71\ydiagram{3,1,1}$
$+14\ydiagram{2,1,1,1} + 25\ydiagram{2,2,1} + \ydiagram{1,1,1,1,1}$

$I_{2,3}$: $4\ydiagram{3,2} + 2\ydiagram{2,2,1}$

A_6: $720\, S_6 + 1044\, S_{51} + 770\, S_{42} + 266\, S_{33}$

$+580\, S_{411} + 455\, S_{321} + 70\, S_{222} + 155\, S_{3111}$

$+84\, S_{2211} + 20\, S_{21111} + S_{111111}$

Important in the proof:

Functor

$$Q \mapsto \left(\bigoplus_{\ell=1}^{k} \mathrm{Sym}^\ell Q \right)^n$$

preserves globally generated bundles.

(Any polynomial functor is OK /C)

Another situation: Lagrangian singularities.

Λ_∞ – Lagrangian Grassmannian

$\widetilde{Q}_I$ basis of $H^*(\Lambda_\infty)$ dual to the cell decomposition (Pragacz–Ratajski)

$\widetilde{Q}$-polynomials defined by Pfaffians

$$I = (i_1 > i_2 > \ldots > i_{2k} \geq 0)$$

$$\mathrm{Pf}(\widetilde{Q}_{i_k, j_\ell})$$

where

$$\widetilde{Q}_{ij} = c_i c_j + 2 \sum_{p>0} c_{i+p} c_{j-p} \quad \text{for } i > j$$

Lagrangian singularities - char. classes

Arnold, Fuks, Vassilyev, ... , Kazarian

Theorem:(M. Mikosz, P. Pragacz ,AW) Thom polynomial of any Lagrangian singularity is nonnegative combination of $\widetilde{Q}_I$'s.

Method of the proof:
$\mathcal{L}_n$ – the space parametryzing k-jets of Lagrangian submanifolds

Λ_n – Lagrangian Grassmannian

$\mathcal{L}_n \to \Lambda_n$ – retraction, contractible fibers

Normal bundle of Λ_n in $\mathcal{L}_n$ isomorphic to

$$\bigoplus_{\ell=3}^{k+1} \mathrm{Sym}^\ell(Q)$$

$\mathrm{A}_\bullet$**-series:**

$\mathrm{A}_2: \ \widetilde{\mathrm{Q}}_1$

$\mathrm{A}_3: \ 3\,\widetilde{\mathrm{Q}}_2$

$\mathrm{A}_4: \ 3\,\widetilde{\mathrm{Q}}_{21} + 12\,\widetilde{\mathrm{Q}}_3$

$\mathrm{A}_5: \ 27\,\widetilde{\mathrm{Q}}_{31} + 60\,\widetilde{\mathrm{Q}}_4$

$\mathrm{A}_6: \ 87\,\widetilde{\mathrm{Q}}_{32} + 228\,\widetilde{\mathrm{Q}}_{41} + 360\,\widetilde{\mathrm{Q}}_5$

$\mathrm{A}_7: \ 135\,\widetilde{\mathrm{Q}}_{321} + 1275\,\widetilde{\mathrm{Q}}_{42} + 2004\,\widetilde{\mathrm{Q}}_{51} + 2520\,\widetilde{\mathrm{Q}}_6$

$\mathrm{D}_\bullet$**-series:**

$\mathrm{D}_4: \ \widetilde{\mathrm{Q}}_{21}$

$\mathrm{D}_5: \ 6\,\widetilde{\mathrm{Q}}_{31}$

$\mathrm{D}_6: \ 12(\widetilde{\mathrm{Q}}_{32} + 2\,\widetilde{\mathrm{Q}}_{41})$

$\mathrm{D}_7: \ 24(\widetilde{\mathrm{Q}}_{321} + 5\,\widetilde{\mathrm{Q}}_{42} + 6\,\widetilde{\mathrm{Q}}_{51})$

$\mathrm{E}_\bullet$**-series:**

$\mathrm{E}_6: \ 9\,\widetilde{\mathrm{Q}}_{32} + 6\,\widetilde{\mathrm{Q}}_{41}$

$\mathrm{E}_7: \ 9\,\widetilde{\mathrm{Q}}_{321} + 60\,\widetilde{\mathrm{Q}}_{42} + 24\,\widetilde{\mathrm{Q}}_{51}$

Singularity P_8 **(modality** 1**):**

$\mathrm{P}_8: \ \widetilde{\mathrm{Q}}_{321}$

$\mathrm{D}_\bullet$**-series:**

$\mathrm{D}_4: \ \ydiagram{2,1}$

$\mathrm{D}_5: \ 6\,\ydiagram{3,1}$

$\mathrm{D}_6: 12(\ydiagram{3,2} + 2\,\ydiagram{4,1})$

$\mathrm{D}_7: 24(\ydiagram{3,2,1} + 5\,\ydiagram{4,2} + 6\,\ydiagram{5,1})$

$\mathrm{E}_\bullet$**-series:**

$\mathrm{E}_6: \ 9\,\ydiagram{3,2} + 6\,\ydiagram{4,1}$

$\mathrm{E}_7: \ 9\,\ydiagram{3,2,1} + 60\,\ydiagram{4,2} + 24\,\ydiagram{5,1}$

Singularity P_8 **(modality** 1**):**

$\mathrm{P}_8: \ \ydiagram{3,2,1}$