Weights in the cohomology of toric varieties

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Abstract

We describe the weight filtration in the cohomology of toric varieties. We present a role of the Frobenius automorphism in an elementary way. We prove that equivariant intersection homology of an arbitrary toric variety is pure. We obtain results concerning Koszul duality: nonequivariant intersection cohomology is equal to the cohomology of the Koszul complex $IH^*_T(X)\otimes H^*(T)$. We also describe the weight filtration in $IH^*(X)$.

1 Introduction

Let X be a smooth toric variety. If X is complete then its cohomology coincides with the Chow ring $A^*(X)$. Therefore the Hodge structure is not very interesting: $H^{k,k}(X)\simeq A^k(X)\otimes\mathbb{C}$ and $H^{k,l}(X)=0$ for $k\neq l$. If X is not complete, then the cohomology of X is equipped with the weight filtration constructed by Deligne, [D2]:

$$W_0H^k(X)\subset\ldots\subset W_{2k}H^k(X)=H^k(X)\,.$$

Since X is smooth $W_{k-1}H^k(X)=0$. In the toric case the weight filtration has the property:

$$W_{2l}H^k(X)=W_{2l+1}H^k(X)$$

for each l. The pure Hodge structure on

$$Gr^W_{2l}H^k(X)=W_{2l}H^k(X)/W_{2l-1}H^k(X)$$

consists only of the Hodge type (l,l).

If X is singular we replace ordinary cohomology by intersection cohomology. Since X can be given a locally conical structure in a metric sense, the intersection cohomology may be interpreted as L^2-cohomology of the nonsingular part, see [Ch, CGM]. Nevertheless we avoid to talk about the L^2–Hodge structure. We are interested in the weight filtration, which is defined via reduction of X to a finite characteristic field $\mathbb{F}_p$, see [BBD]. The weight filtration is the filtration

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coming from the eigenvalues of the Frobenius automorphism acting on the étale intersection cohomology.

Fortunately, for toric varieties the Frobenius automorphism of intersection cohomology can be induced from an endomorphism of the complex points of X. This map preserves orbits and it is covered by a map of intersection cohomology sheaves. Finally, the group $IH^k(X)$ is decomposed into the direct sum of the eigenspaces with eigenvalues p^l, $l = \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil + 1, \dots, k$. We stress that although the concept of weights is highly nontrivial and abstract, for toric varieties whole theory reduces to easy computations involving an action of a down-to-earth map.

On the other hand we consider equivariant cohomology taken with respect to the big torus $T = (\mathbb{C}^*)^n$ acting on X, [BBFK1]. If X is singular we prefer to replace usual cohomology by the intersection cohomology. We prove that the equivariant intersection cohomology of X is pure even if X is not complete. This means that the only eigenvalue which can occur in $IH_T^{2k}(X)$ is p^k, whereas $IH_T^{2k+1}(X) = 0$.

Koszul duality allows to recover usual intersection cohomology from equivariant one [GKM, MW, Fr1]. In general one has to know not only the cohomology groups, but also whole complex in the derived category of $H^*(BT)$-modules. One has a spectral sequence with $E_2^{k,l} = IH_T^k(X) \otimes H^l(T)$ converging to $IH^{k+l}(X)$. The differential in the E_2 table is the usual Koszul differential: We identify $H^*(T)$ with the exterior algebra of the dual of the Lie algebra $\mathbf{t}^*$, and the cohomology of BT with the symmetric algebra. The differential $d_{(2)} = d_{Koszul}$ has the form:

$$d_{(2)}(x \otimes \xi) = \sum_{j=1}^{n} x\lambda^j \otimes i_{\lambda_j}\xi\,, \quad \text{for } x \in IH_T^*(X)\,,\ \ \xi \in \Lambda\mathbf{t}^*\,.$$

Here $\{\lambda_j\}$ is a basis of $\mathbf{t}$, the elements λ^j of the dual basis are generators of $H^*(BT) = S\mathbf{t}^*$, and i_{λ_j} stands for the contraction. We prove using weight argument that all the higher differentials of the spectral sequence vanish if X is toric. Therefore the nonequivariant cohomology of X are the cohomology of the Koszul complex. Moreover, the weight filtration coincides with the filtration given by the spectral sequence. The Koszul complex itself splits into a direct sum of subcomplexes, each computing the graded piece of the weight filtration $Gr_*^W IH^*(X)$. We note that the nonequivariant intersection cohomology of X is pure if and only if the equivariant intersection cohomology is free over $H^*(BT)$. The properties of the weight filtration are reflected by the behavior of Poincaré polynomials, which are the weighted Euler characteristics.

Cohomology and intersection cohomology of toric varieties have attracted surprisingly many authors. Jurkiewicz ([Ju]) and Danilov ([Da]) have computed cohomology of complete smooth toric varieties. Next we would have to present a long list of papers. Instead we suggest to check the references e.g. in [BP]. A complex computing intersection cohomology of a toric variety was described in

[Is]. Our paper is based on the approach of [Fi], [BBFK1-2] combined with [DL]. We will assume that the reader is familiar with basic theory of toric varieties ([Fu]), intersection cohomology ([GM]) and equivariant intersection cohomology ([Br]). Our goal is to expose the role of the weight filtration. Toric varieties serve as a toy models. The reader who is not familiar with intersection cohomology may replace it by usual cohomology and assume that X is defined by a simplicial fan, i.e. X is rational homology manifold.

Now, we would like to explain terminology concerning formality which plays an important role in our consideration.

1. A manifold is called formal if the algebra of differential forms $\Omega^*(X)$ is quasiisomorphic to its cohomology *as a dg-algebra.* In the definition of formality for an arbitrary topological space the algebra of forms is replaced by the Sullivan–de Rham complex. Except from Kähler manifolds the classifying space of a connected Lie group BG is an example of a formal space.

2. Suppose B is a formal space and $X \to B$ is a map. Then $\Omega^*(X)$ is a module over $\Omega^*(B)$. A natural notion of formality over B would be the demand that $\Omega^*(X)$ is quasiisomorphic to its cohomology *as a dg-module over* $\Omega^*(B) \simeq H^*(B)$. Formality of $ET \times_T X$ over BT implies that

$$\begin{aligned} H^*(X) &= H^*(\Omega^*(ET \times_T X) \otimes \Lambda^*, d_{Koszul}) \\ &= H^*(H^*(ET \times_T X) \otimes \Lambda^*, d_{Koszul}) \,. \end{aligned}$$

   This is exactly the content of our Theorem 5.5 for simplicial toric varieties. Recently M. Franz [Fr2] have shown that $ET \times_T X$ is formal over BT (even with $\mathbb{Z}$ coefficients) for smooth toric varieties.

3. If $B = BG$ and $\Omega^*(EG \times_G X)$ is quasiisomorphic to its cohomology *as an algebra over* $H^*(BG)$ then X is called G-formal in [Li]. Smooth toric varieties are formal in the above sense by [NR].

4. The notion of *equivariantly formal* space was introduced in [GKM]. Before it was called *totally nonhomologous to zero.* The name does not fit to the scheme of the previous definitions. It just means that $\Omega^*(EG \times_G X)$ is free (up to a quasiisomorphism) dg-module over $H^*(BG)$. It is equivalent to the statement that $H^*_G(X)$ is a free module over $H^*(BG)$.

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Contents:

2 Mixed Hodge structure

Let X be a smooth possibly noncomplete algebraic variety. According to Deligne ([D2]) one defines an additional structure on the cohomology of X. One finds a completion $X \subset \overline{X}$, such that $\overline{X} \setminus X = \sum_{i=1}^{\alpha} D_i$ is a smooth divisor with normal crossings. For $0 \leq k \leq \alpha$ let $D^{(k)}$ denote the disjoint union of the k-fold intersections of the components of the divisor D and set $D^{(0)} = \overline{X}$. Deligne has constructed a spectral sequence with

2.1 $$E_1^{k,l} = H^{2k+l}(D^{(-k)}) \Rightarrow H^{k+l}(X)$$

(the coefficient are in $\mathbb{C}$). This spectral sequence degenerates on E_2. The limit filtration is the weight filtration. The quotients are equipped with the pure Hodge structure, they are decomposed into the (p,q) summands:

$$Gr_k^W H^l(X) = \bigoplus_{p+q=k} \mathcal{H}_{(l)}^{p,q}\,.$$

The groups $Gr_k^W H^l(X)$ are quotients of subgroups of $H^{l-2k}(D^{(k)})$.

If X is a smooth toric variety then there exists a smooth toric variety compactifying X. It can be chosen so the divisors at infinity are smooth toric varieties as well as each component of $D^{(k)}$. The resulting weights can only be even. Moreover:

Proposition 2.2 *For a smooth toric variety each $Gr_{2k}^W H^l(X)$ is of the type (k,k) and $Gr_{2k+1}^W H^l(X) = 0$.*

Remark 2.3 There is an easy method of constructing a spectral sequence converging to the weight filtration. Just consider the Leray spectral sequence of the inclusion $X \subset \overline{X}$. Then $E_2^{k,l} = H^k(D^{(l)})$. The limit filtration has to be shifted in order to have $W_l H^l(X) = im(H^l(\overline{X}) \to H^l(X))$. This spectral sequence degenerates on E_3. In fact, up to a renumbering it is isomorphic to the Deligne spectral sequence.

The construction of Deligne was motivated by the previous work on Weil conjectures, [D1]. One considers varieties defined over a finite field. Instead of the Hodge structure one has an action of the Frobenius automorphism. The eigenvalues on $H^l_{et}(X)$ can have absolute values equal to $p^{\frac{k}{2}}$ with $k = l, l+1, \dots, 2l$ if X is smooth. Each complex variety is in fact defined over a finitely generated ring; toric varieties are defined over $\mathbb{Z}$. The étale cohomology of the variety reduced to the finite base field is isomorphic to the usual cohomology for almost all reductions. In the toric case every reduction $\mathbb{Z} \to \mathbb{F}_p$ is good. The weight filtration coincides with the filtration of the étale cohomology:

2.4
$$W_k H^l_{et}(X) = \bigoplus\nolimits_{|\lambda| \le p^{\frac{k}{2}}} V_\lambda \,,$$

where $V_\lambda \subset H^l_{et}(X)$ is the eigenspace of the eigenvalue λ. The eigenvalues appearing here belong to $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ in general, but in the toric case λ is a power of p.

3 Frobenius automorphism of toric varieties

As noticed by Totaro ([To]), the toric varieties are rare examples of complex varieties admitting an endomorphism which coincides with the Frobenius automorphism after reduction.

Let $p > 1$ be a natural number. Raising to the p-th power is a map of the torus $\psi_p : T \to T$. There exists an extension $\phi_p : X \to X$, such that the following diagram commutes:

3.1
$$\begin{array}{ccc} T \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \psi_p \times \phi_p}\downarrow & & \downarrow{\scriptstyle \phi_p} \\ T \times X & \xrightarrow{\mu} & X \end{array} .$$

Here μ is the action of T. The map ϕ_p is given by a subdivision of the lattice. It has been recently applied to study equivariant Todd class in homology of singular toric variety in [BZ]

The following observations (already made in [To]) illustrate the nature of the weight filtration very well. To prove them one applies elementary properties of cohomology.

3.1 X complete and smooth

Let X be smooth and complete toric variety. Then every homology class is represented by the closure of an orbit. The restriction of ϕ_p to a k-dimensional orbit is a cover of the degree p^k. Therefore the induced action of ϕ_p on $H_{2k}(X)$ is the multiplication by p^k. The conjugate action on $H^{2k}(X)$ is the multiplication by p^k. We say then that the cohomology of X is pure (see Definition 3.4 below).

3.2 X smooth

If X is smooth, but not necessarily complete, then one has Deligne spectral sequence 2.1. The induced action of ϕ_p on $Gr^W_{2k}H^l$ is the multiplication by p^k. It can be verified by comparison with the étale cohomology or in a elementary way: if Y is the closure an orbit of the codimension one, then the following diagram commutes

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{i_!} & H^{*+2}(\overline{X}) \\ \phi_p^*\downarrow & & \downarrow\phi_p^* \\ H^*(Y) & \xrightarrow{pi_!} & H^{*+2}(\overline{X}) \end{array} \quad .$$

Here $i_!$ is Gysin the map induced by the inclusion via Poincaré Duality. (It is a general rule that Gysin map shifts the weight by the codimension of the subvariety.) The differential in the Deligne spectral sequence 2.1 are induced by the inclusions. Therefore, to have an equivariant spectral sequence with respect to ϕ_p one encodes the action in the usual way:

$$E_1^{-k,l} = H^{2k+l}(D^{(k)})(k)\,,$$

where the symbol (k) denotes action of ϕ_p via the multiplication by p^k. The spectral sequence degenerates on E_2. This is because the higher differentials

$$d_{(i)} : E_i^{k,l} \to E_i^{k+i,l-i+1}$$

do not preserve the eigenspaces of ϕ_p^*. The domain of $d_{(i)}$ has the eigenvalue $p^{\frac{l}{2}}$ and the target has $p^{\frac{l-i+1}{2}}$ (the relevant values of l and $l-i+1$ are even). For the limit filtration of $H^*(X)$ we have:

Proposition 3.2 *The action of ϕ_p^* on $Gr^W_{2k}H^l(X)$ is the multiplication by p^k.*

Again this is just a simple characterization of the weight filtration for toric varieties. On the other hand, not appealing to the general theory, we can define weights in the following way. To be consistent with the usual terminology we will say that:

Definition 3.3 *A vector space V is of weight k if*

- *ϕ_p acts on V as multiplication by $p^{k/2}$ for k even,*
- *$V = 0$ for k odd.*

Definition 3.4 *The cohomology group $H^k(X)$ is pure if it is of weight k.*

It turns out (see Corollary 4.12) that our definition of purity agrees with the one in [BBFK2], §2. Namely the condition on the eigenvalues in even cohomology redundant. It is implied by vanishing of odd cohomology.

In fact the decomposition of $H^*(X)$ into the eigenspaces of ϕ_p^* splits the weight filtration into a gradation, [To].

Example 3.5 Let $X=\mathbb{P}^1\times\mathbb{P}^1\setminus\{(0,\infty),(\infty,0)\}$. Let $\overline{X}$ be the compactification of X obtained by blowing up two removed points in $\mathbb{P}^1\times\mathbb{P}^1$. The Deligne spectral sequence has the following E_1 table

$$\begin{array}{ccc|c} 0 & \mathbb{C}^2(2) & \mathbb{C}(2) & 4 \\ 0 & 0 & 0 & 3 \\ 0 & \mathbb{C}^2(1) & \mathbb{C}^4(1) & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{C} & 0 \\ \hline \text{-2} & \text{-1} & 0 & \end{array} \qquad d_{(1)}: E_1^{k,l}\to E_1^{k+1,l}.$$

The second table is equal to E_∞:

$$\begin{array}{ccc|c} 0 & \mathbb{C}(2) & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & \mathbb{C}^2(1) & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{C} & 0 \\ \hline \text{-2} & \text{-1} & 0 & \end{array}$$

The cohomology of X is following:

$$\mathbb{C},\ 0,\ \mathbb{C}^2(1),\ \mathbb{C}(2),\ 0$$

in the dimensions 0,1,2,3,4. The group $H^3(X)$ is not pure, since it is of the weight 4.

3.3 X arbitrary

For singular X we study intersection cohomology ([GM]) instead of usual cohomology. The map ϕ_p preserves orbits, which form a stratification of X. Therefore ϕ_p induces a map of intersection cohomology groups $IH^*(X)$. Moreover this map exists on the level of sheaves. Thus it is natural with respect to inclusions. Avoiding the formalism of [BBD] we can define a weight filtration by the formula 2.4. Later it will be clear, that the action of ϕ_p^* decomposes $IH^*(X)$ into a sum of eigenspaces with the eigenvalues p^k as in the smooth case, see 5.5.

4 Equivariant cohomology

Although the equivariant cohomology and equivariant intersection cohomology are complicated objects, they have surprisingly nice properties for toric varieties. Equivariant cohomology is defined by means of Borel construction:

$$H^*_T(X)=H^*(ET\times_T X)\,. \qquad IH^*_T(X)=H^*(ET\times_T X;IC^\bullet_T)\,.$$

For the basic features we refer to [Br] and [BBFK1].

4.1 Smooth case

Equivariant cohomology of a smooth (or just simplicial) toric variety can be easily recovered from the fan. If Δ is a simplicial fan, then the associated toric variety X_Δ is a rational homology manifold. The equivariant cohomology can be identified with the algebra of the continuous functions on the support of the fan, which are polynomial on each cone of Δ. The odd part of the equivariant cohomology vanishes also for noncomplete fans. From the commutativity of 3.1 it follows that ϕ_p acts on the equivariant cohomology.

We will prove a key theorem:

THEOREM 4.1 *The equivariant cohomology of a smooth toric variety is pure.*

Remark 4.2 Since the space ET is a limit of algebraic varieties (as defined in [Br]) or a simplicial variety (see [D3] 6.1) it is equipped with a mixed Hodge structure. This structure is not only pure, but also of Hodge type (i.e. (k,k) in $2k$-th cohomology). Every class is represented by an algebraic cycle.

Remark 4.3 The same statement holds for arbitrary smooth G-varieties consisting of finitely many orbits. In fact $H^*_G(X)$ is generated by algebraic cycles. We will develop this remark in a subsequent paper [FW]. We want to keep the present paper as elementary as possible. Therefore we do not treat the case of general G-varieties here. This would demand at least relying on the theory of mixed Hodge structures (in the smooth case). We prefer to exploit the map ϕ_p.

Proof of 4.1. Our proof is the induction on the number of cones.
1) If Δ consists of one cone σ, then

$$H^*_T(X_\Delta) = H^*(B(T_\sigma))\,,$$

where T_σ is the subtorus with Lie algebra spanned by σ. The isomorphism is induced by the inclusion of the minimal orbit $T/T_\sigma \hookrightarrow X_\Delta$. The action of ϕ_p on $H^*(B(T_\sigma))$ is as desired.
2) Now suppose that X is decomposed into a sum of toric varieties $X = X_1 \cup X_2$. The Mayer-Vietoris sequence splits into short exact sequences:

$$0 \to H^{2k}_T(X) \hookrightarrow H^{2k}_T(X_1) \oplus H^{2k}_T(X_2) \twoheadrightarrow H^{2k}_T(X_1 \cap X_2) \to 0\,.$$

The groups $H^{2k}_T(X_i)$ are pure, therefore the subgroup of their direct sum is pure too. □

Remark 4.4 From the intuitive point of view Theorem 4.1 is clear: If one identifies the equivariant cohomology with the piecewise polynomial functions (Stanley-Reisner ring in the complete case), then the gradation is given by the homogeneity degree. Purity means that homogeneous functions are homogeneous.

4.2 Singular case

The construction of the equivariant intersection cohomology in terms of the fan is more complicated, $IH^*_T(-)$ can be described axiomatically ([BBFK1], Def. 3.1). As in the smooth case the odd part vanishes. The even part is equipped with an action of ϕ_p. The map $IH^*_T(X) \to IH^*(X)$ is ϕ_p-invariant. We generalize Theorem 4.1:

THEOREM 4.5 *The equivariant intersection cohomology of a toric variety is pure.*

Remark 4.6 Proceeding as in [BJ] we can prove the same statement for arbitrary G-variety, provided, that it consists of finitely many orbits and singularities are not to bad. This is the case for spherical varieties. See [FW].

Proof. The proof of the theorem is analogous except the first step. We will introduce an induction on $\dim X$. If the dimension is one, then X_Δ is smooth ($X_\Delta = \mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$) and the equivariant intersection cohomology is pure.

We recall the definition of equivariant formality:

DEFINITION 4.7 *We say that X is equivariantly formal if $IH^*_T(X)$ is a free module over $H^*(BT)$.*

Equivariant formality is equivalent to the degeneration on E_2 of the spectral sequence:

4.8 $$E_2^{k,l} = H^k(BT) \otimes IH^l(X) \Rightarrow IH^{k+l}_T(X)\,.$$

In the toric case it is also equivalent to vanishing of $IH^*(X)$ in odd degrees. Other equivalent conditions are stated in [BBFK1], Lemma 4.1 and [BBFK2], Theorem 3.8.

For the inductive step we will need the following lemma:

LEMMA 4.9 *Assume that X is equivariantly formal. Then $IH^*_T(X)$ is pure if and only if $IH^*(X)$ is pure.*

Proof. "$\Rightarrow$" If $IH^*_T(X)$ is equivariantly formal then the map $IH^*_T(X) \to IH^*(X)$ is surjective. Purity of $IH^*(X)$ follows. "$\Leftarrow$" If $IH^*(X)$ is equivariantly formal, then the terms of the spectral sequence 4.8 are pure. Therefore $IH^*_T(X)$ is pure. □

Proof of 4.5 cont. 1) If Δ consists of a single cone, then (we divide X_Δ by a finite group if necessary, as in the proof Theorem 4.4 in [BBFK1]) there is a decomposition
$$X_\Delta \simeq T/T_\sigma \times C(X_{L\sigma})\,,$$

where L_σ is a fan in $\mathbf{t}_\sigma/\mathrm{lin}(\alpha)$ with $\alpha \in int(\sigma)$. The toric variety $X_{L\sigma}$ comes with an ample bundle and $C(X_{L\sigma})$ is the affine cone over $X_{L\sigma}$. It is the toric variety associated to the cone σ considered in $\mathbf{t}_\sigma$. The variety $X_{L\sigma}$ is complete, thus it is equivariantly formal ([GKM]). The equivariant intersection cohomology is pure by the inductive assumption on the dimension of X, so $IH^*(X_{L\sigma})$ is pure by 4.9. The intersection cohomology of the cone is the primitive part of $IH^*(X_{L\sigma})$:

$$IH^{n-i}(C(X_{L\sigma})) = \begin{cases} ker(IH^{n-i}(X_{L\sigma}) \xrightarrow{h^{i+1}} IH^{n+i+2}(X_{L\sigma})) & \text{for } i \geq 0 \\ 0 & \text{for } i < 0 \end{cases} .$$

Here h is the class of the hyperplane section. Again we see that $IH^*(C(X_{L\sigma}))$ is pure. The cone $C(X_{L\sigma})$ is equivariantly formal (since the odd part vanishes). Therefore $IH^*_T(C(X_{L\sigma}))$ is pure by 4.9.

2) The induction with respect to the number of cones is the same as in the proof of 4.1. One should remember that at this point decomposition theorem ([BBD]) is used to justify that $IH^{odd}_T(X) = 0$. □

Remark 4.10 Proof that the odd equivariant cohomology vanish and that even equivariant cohomology is pure may be done in one shot.

COROLLARY 4.11 *Intersection cohomology of X is pure if and only if X is equivariantly formal.*

Proof. The vanishing of $IH^{odd}(X)$ implies degeneration of the spectral sequence 4.8. To prove the converse suppose X is equivariantly formal. Equivariant intersection cohomology is always pure by 4.5. Purity of $IH^*(X)$ follows from 4.9. □

We can restate 4.11:

COROLLARY 4.12 *Intersection cohomology of X is pure if and only if it vanishes in odd degrees.*

Warning: Our corollaries do not hold for an arbitrary algebraic variety with a torus action.

5 Koszul duality

Let X be a topological space acted by a torus T. Borel construction produces a space and a map:

$$X_T = ET \times_T X \longrightarrow BT \,.$$

The homotopy type of X can be recovered from X_T by a pull-back diagram:

$$\begin{array}{ccccccc} X & \sim & ET \times X & = & ET \times_{BT} X_T & \longrightarrow & X_T \\ & & & & \downarrow & & \downarrow \\ & & & & ET & \longrightarrow & BT \end{array} ,$$

The map $ET \times X \to X_T$ is a fibration with the fiber T. In the stack language it is the quotient map $X \to X/T$, [Si].

Remark 5.1 The T-homotopy type of X is not preserved by the procedure

$$X \mapsto X_T \mapsto ET \times_{BT} X_T = ET \times X$$

since the action of T on $ET \times X$ is free and the action on X does not have to be.

On the level of cohomology one has a spectral sequence

5.2
$$E_2^{k,l} = IH_T^k(X) \otimes H^l(T) \Rightarrow IH^{k+l}(X)\,.$$

(Note that X_T is simply-connected.) To recover $IH^*(X)$ one would have to know a complex defining equivariant intersection cohomology: precisely an element of the derived category of $H^*(BT)$-modules. This is a form of Koszul duality as studied in [GKM], [AP], [MW], [Fr1].

Remark 5.3 After certain renumbering of the entries the E_r table of the spectral sequence 5.2 are isomorphic to E_{r-1} of the Eilenberg–Moore spectral sequence ([EM, Sm]). The analog of the Eilenberg–Moore spectral sequence for intersection cohomology is studied in [FW]. It is shown there that the weight structure is inherited.

In the toric case the spectral sequence is acted by ϕ_p. The cohomology of T is not pure: $H^l(T) = \Lambda^l \mathbf{t}^*$ is of weight $2l$. Because (by 4.5) the equivariant intersection cohomology is pure, the weight of $E_2^{k,l}$ is $k+2l$. The differentials

$$d_{(2)} : E_2^{k,l} \to E_2^{k+2,l-1}$$

do preserve weights.

PROPOSITION 5.4 *The higher differentials*

$$d_{(i)} : E_i^{k,l} \to E_2^{k+i,l-i+1}$$

for $i > 2$ vanish.

Proof. The weight of the target is $k+i+2(l-i+1) = k+2l-i+2 < k+2l$ for $i>2$. Hence $d_{(i)}$ vanishes. □

The differential $d_{(2)}$ has the usual Koszul form:

$$d_{(2)}(x \otimes \xi) = \sum_{j=1}^{n} x\lambda^j \otimes i_{\lambda_j}\xi\,, \quad \text{for } x \in IH_T^*(X)\,,\ \ \xi \in \Lambda\mathbf{t}^*\,.$$

Here $\{\lambda_j\}$ is the basis of $\mathbf{t}$ and the elements λ^j of the dual basis are identified with generators of $H^*(BT) = S\mathbf{t}^*$. We obtain a description of the nonequivariant intersection cohomology:

THEOREM 5.5 *Intersection cohomology of a toric variety is isomorphic to the cohomology of the Koszul complex $IH^*_T(X) \otimes \Lambda\mathbf{t}^*$. The filtration induced from the gradation of $\Lambda\mathbf{t}^*$ is the weight filtration up to a shift.*

Remark 5.6 The analogous statement holds for arbitrary G varieties with $H^*(G)$ instead of $\Lambda\mathbf{t}^*$, provided that $IH^*_G(X)$ is pure. The proof will appear in [FW].

Remark 5.7 In the simplicial case the complex $H^*_T(X) \otimes \Lambda\mathbf{t}^*$ has appeared in [BP], 4.2.2 and [Fr1]. Ishida complex ([Od]) is related to it. If X is simplicial toric variety, then our complex contains and is quasiisomorphic to Ishida complex.

Let us describe the weight filtration precisely. The Koszul complex splits into a direct sum of subcomplexes

5.8 $$IH^*_T(X) \otimes \Lambda\mathbf{t}^* = \bigoplus_{l=0}^{2n} C^*_{[l]}\,, \qquad C^i_{[l]} = IH_T^{2(k-l)} \otimes \Lambda^{2l-k}\mathbf{t}^*\,.$$

The weight filtration of intersection cohomology also splits into the sum of eigenspaces of the Frobenius action

$$Gr^W_{2l}\, IH^k(X) \simeq (IH^k(X))_{p^l} = H^k(C^*_{[l]})\,.$$

Although the complexes $C^*_{[l]}$ might be nonzero in high degrees, their cohomology vanish:

$$H^k(C^*_{[l]}) = 0 \quad \text{for } k > l \text{ or } k > 2n\,.$$

In particular we have:

COROLLARY 5.9 *The pure component of $IH^k(X)$ is equal to the cokernel of $d_{(2)} : IH^{k-2}_T(X) \otimes \mathbf{t}^* \to IH^k_T(X)$. Equivalent descriptions of $W_k IH^k(X)$ are*

$$IH^*_T(X)/\mathbf{m}IH^*_T(X) = IH^*_T(X) \otimes_{S\mathbf{t}^*} \mathbb{C}\,,$$

where $\mathbf{m} = S^{>0}\mathbf{t}^*$ is the maximal ideal.

Example 5.10 Let us come back to the Example 3.5. It is the toric variety associated to the fan consisting of two quarters in $\mathbb{R}^2$ having only the origin in common. According to §4.1 the equivariant cohomology can be identified with the pairs of polynomials in two variables having the same value at the origin. The spectral sequence 5.2 has the E_2 table

2	$\mathbb{C}(2)$	0	$\mathbb{C}^4(3)$	0	$\mathbb{C}^6(4)$	0	$\mathbb{C}^8(5)$	0	$\mathbb{C}^{10}(6)$	$\dots$
1	$\mathbb{C}^2(1)$	0	$\mathbb{C}^8(2)$	0	$\mathbb{C}^{12}(3)$	0	$\mathbb{C}^{16}(4)$	0	$\mathbb{C}^{20}(5)$	$\dots$
0	$\mathbb{C}$	0	$\mathbb{C}^4(1)$	0	$\mathbb{C}^6(2)$	0	$\mathbb{C}^8(3)$	0	$\mathbb{C}^{10}(4)$	$\dots$
	0	1	2	3	4	5	6	7	8	

The E_3 table is

2	0	0	0	0	0	0	0	0	0	...
1	0	0	$\mathbb{C}(2)$	0	0	0	0	0	0	...
0	$\mathbb{C}$	0	$\mathbb{C}^2(1)$	0	0	0	0	0	0	...
	0	1	2	3	4	5	6	7	8	

Remark 5.11 The dual spectral sequence 4.8 does not have to degenerate on E_3. The counterexample is $\mathbb{C}^2 \setminus \{0\} \sim S^3$ with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on the coordinates. The tables E_2, E_3 and E_4 are equal. The differential $d_{(4)}$ is nontrivial: $d_{(4)}(1 \otimes x) = \lambda^1\lambda^2 \otimes 1$, were $x \in H^3(\mathbb{C}^2 \setminus \{0\})$ is the generator and λ^1, λ^2 is the basis of $\mathbf{t}^*$.

6 Poincaré polynomials

For a toric variety we define the virtual Poincaré polynomial by the formula:

$$IP_{cld}(X) = \sum_{k,l} (-1)^k \dim Gr^W_{2l} IH^k(X) q^l \,.$$

The subscript *cld* indicates that the supports of cohomology are closed, whereas the usual Poincaré polynomials are taken with respect to compact supports. One can find the same formula for cohomology with compact supports in [Fu] §4.5 and for intersection cohomology in [DL] §5.5. Our approach does not refer to the abstract theory. The polynomial $IP_{cld}(X)$ can be treated as Euler characteristic with weights:

$$IP_{cld}(X) = \chi^W(IH^*(X)) := \sum_l \chi(Gr^W_{2l} IH^*(X)) q^l \,.$$

Due to purity of equivariant intersection cohomology the equivariant Poincaré polynomial is simpler to define:

$$IP_{cld}(X_T) = \chi^W(IH^*_T(X)) = \sum_l \dim IH^{2l}_T(X) q^l \,.$$

In general it would be the alternating sum as before. Poincaré polynomials were studied in [BBFK2], §4. The recursive definition of $IP_{cld}(X)$ applies in our case, as it will be clear from the following propositions. We note that for complete toric varieties our Poincaré polynomial differs from the one of [BBFK2] by a substitution $q = t^2$, whereas for noncomplete X the polynomial of [BBFK2] can be defined as the weighted Euler characteristic of intersection cohomology with compact supports.

Proposition 6.1 *The Poincaré polynomials are related by the formula*

$$IP_{cld}(X_T)(1-q)^n = IP_{cld}(X) \,.$$

Proof. The weighted Euler characteristic is multiplicative, therefore

$$IP_{cld}(X) = \chi^W(IH^*(X)) = \chi^W(IH^*_T(X) \otimes H^*(T)) = \chi^W(IH^*_T(X))\chi^W(H^*(T)).$$

The Poincaré polynomial of $H^*(T)$ is $(1-q)^n$, thus

$$IP_{cld}(X) = IP_{cld}(X_T)(1-q)^n.$$

□

Our Poincaré polynomials $IP_{cld}(X)$ are not additive with respect to union of strata, but:

Proposition 6.2 *The polynomials $IP_{cld}(X)$ and $IP_{cld}(X_T)$ are additive in the following sense: if $X = X_1 \cup X_2$, then*

$$IP_{cld}(X) = IP_{cld}(X_1) + IP_{cld}(X_2) - IP_{cld}(X_1 \cap X_2)$$

and the same for $IP_{cld}(X_T)$.

Proof. The additivity of $IP_{cld}(X_T)$ follows from the fact that Mayer-Vietoris sequence splits into the short exact sequences. The additivity of $IP_{cld}(X)$ follows from 6.1. □

The recursive formula for $IP_{cld}(X)$ is the following:

$$IP_{cld}(X_\Delta) = \sum_{\sigma \in \Delta} \partial IP_{cld}(X_{L\sigma})(1-q)^{n-\dim \sigma},$$

where

$$\partial IP_{cld}(X) = \tau_{> \frac{\dim X}{2}}(1-q) IP_{cld}(X)$$

if $\dim X \geq 0$ and $\partial IP_{cld}(\emptyset) = 1$. The symbol $\tau_{> \frac{\dim X}{2}}$ denotes truncating the coefficients of the monomials of degrees which are not $> \frac{\dim X}{2}$. The above formula holds for equivariantly formal X by [BBFK2] and Poincaré duality. Hence it holds for an affine X. By additivity it holds for any X.

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