ON TORSION IN HOMOLOGY OF SINGULAR TORIC VARIETIES

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Abstract

Let X be a toric variety. It follows from [FW] that the rational Borel-Moore homology of X is isomorphic to the homology of the Koszul complex $A^T_*(X) \otimes \Lambda^* M$. Here $A^T_*(X)$ is the equivariant Chow group and M is the character group of T. We give a simple geometric proof of that fact and we show that the same holds for coefficients which are the integers with certain primes inverted.

1 Introduction

Let G be a connected Lie group which acts on a topological space X. The equivariant cohomology of X is defined to be the cohomology of the space obtained from the Borel construction

$$H^*_G(X) = H^*(EG \times_G X)\,.$$

It often happens that this group is easily computable. This is so for example when X is a complex algebraic manifold with an action of a complex algebraic group with only finitely many orbits. Then the equivariant cohomology with rational coefficients is the direct sum

(1.1)
$$H^*_G(X;\mathbf{Q}) = \bigoplus_{\text{orbit}} H^{*-2c}(BG_x;\mathbf{Q})\,.$$

Here G_x is the stabilizer of a point from the orbit and c is the complex codimension of the orbit.

There is an other invariant which even in the singular case is easy to compute. That is the equivariant Borel-Moore homology, [EG, §2.8]. It can be interpreted as the equivariant cohomology with coefficients in the dualizing sheaf. The equivariant Borel-Moore homology usually is nontrivial in the negative degrees. Again we have

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(1.2)
$$H_*^{BM,G}(X;\mathbf{Q}) = \bigoplus_{\text{orbit}} H^{2d-*}(BG_x;\mathbf{Q})\,.$$

Here d is the complex dimension of the orbit. This formula, as well as the previous one, follows from the fact that the rational cohomology of the classifying space is concentrated in even degrees.

A passage from the equivariant cohomology to the usual one is possible due to the Eilenberg-Moore spectral sequence. The second table is of the form

$$E_2^{-p,q} = \mathrm{Tor}_p^{q,H^*(BG;\mathbf{Q})}(H_G^*(X;\mathbf{Q}),\mathbf{Q}) \Rightarrow H^{q-p}(X;\mathbf{Q})\,.$$

(The torsion functor has two gradings: p is the usual grading of the left derived functor and q is the internal grading.) A generalization of this spectral sequence for cohomology with sheaf coefficients was described in [FW]. If G is a complex algebraic group, X is an algebraic variety and the action is algebraic then all the cohomological invariants are equipped with the weight filtration. This often forces the spectral sequence to degenerate. In particular according to [FW, Th. 1.6] we have:

THEOREM 1.3 *If X is smooth and the action has only finitely many orbits then the rational cohomology of X is given additively by:*

$$H^i(X;\mathbf{Q}) = \bigoplus_{q-p=i} \mathrm{Tor}_p^{q,H^*(BG;\mathbf{Q})}(H_G^*(X;\mathbf{Q}),\mathbf{Q})\,.$$

Having the decomposition 1.2 and using the fact that the equivariant cohomology of an orbit is pure we apply [FW, Th. 1.3]. We obtain

THEOREM 1.4 *If the action has only finitely many orbits then the rational Borel-Moore homology of X is given by:*

$$H_i^{BM}(X;\mathbf{Q}) = \bigoplus_{p-q=i} \mathrm{Tor}_p^{q,H^*(BG;\mathbf{Q})}(H_*^{BM,G}(X;\mathbf{Q}),\mathbf{Q})\,.$$

In both theorems above we assume that spaces and actions are algebraic.

In this note we want to specialize our results to toric varieties. The intersection cohomology was already described in [W]. Here we study ordinary homology but, in addition, we care about integral coefficients. First we note that for a torus T the Eilenberg-Moore spectral sequence can be replaced by an easier one (which is in fact isomorphic after a renumbering of entries). This is just the spectral sequence of the fibration

$$T \subset ET \times X \longrightarrow ET \times_T X\,.$$

The second table
$$E_2^{p,q} = H_T^p(X) \otimes H^q(T)$$

with its differential is exactly the Koszul complex. Therefore

$$E_3^{p,q} = \mathrm{Tor}_q^{p+2q, H^*(BT)}(H_T^*(X), \mathbf{Z}).$$

The exact degrees are slightly surprising, but they agree with the weight filtration when X is smooth. For possibly singular varieties we will describe the homological variant of the spectral sequence in an elementary way.

Now we apply the Frobenius endomorphism of the toric variety. This allows to show that the spectral sequence degenerates not only over $\mathbf{Q}$ but also with small primes inverted. We prove two theorems.

THEOREM 1.5 *The above spectral sequence degenerates on E^3 for rational coefficients and for coefficients in $\mathbf{F}_q$ if $q > \lceil \frac{\dim X}{2} \rceil$.*

THEOREM 1.6 *If $q > \lceil \frac{\dim X+1}{2} \rceil$ then the q-torsion of the integral homology is the direct sum of the q-torsions in E^3.*

As a consequence we obtain:

THEOREM 1.7 *Let X be a toric variety and let R be the ring of integers with inverted primes which are smaller or equal to $\lceil \frac{\dim X+1}{2} \rceil$. Then*

$$H_i^{BM}(X;R) = \bigoplus_{p-q=i} \mathrm{Tor}_p^{q, H^*(BT;R)}(H_*^{BM,T}(X;R), R)\,.$$

It remains to remark that $H_{2i}^{BM}(X;R) \simeq A_i^T(X) \otimes R$ is the equivariant Chow group.

We suspect that the assumption about q is redundant. The authors of [BFMH] were studying the cohomology of toric varieties with coefficients in $\mathbf{F}_2$. They also conjecture that (a complex equivalent to) the Koszul complex computes the homology of X. It is shown that this is so if the dimension is smaller or equal to 3. Many cases are verified by a computer. Moreover the same complex (with another grading) computes the homology of the corresponding real toric variety.

2 Equivariant Borel-Moore homology

From now on we omit coefficients in the notation. Let X be an algebraic variety acted by the torus $T = (\mathbf{C}^*)^n$. Let

$$ET_d = (\mathbf{C}^{d+1} \setminus \{0\})^n$$

be an approximation of ET. The equivariant Borel-Moore homology is defined by the formula:

$$H_i^{BM,T}(X) = \lim_{\overleftarrow{d}} H_{i+2nd}^{BM}(ET_d \times_T X)\,.$$

The limit is taken with respect to the inverse system

$$\iota_d^! : H^{BM}_{i+2n(d+1)}(ET_{d+1} \times_T X) \to H^{BM}_{i+2nd}(ET_d \times_T X)\,.$$

The Gysin map $\iota_d^!$ is defined since the inclusion (of the real codimension $2n$)

$$\iota_d : ET_d \times_T X \hookrightarrow ET_{d+1} \times_T X$$

is normally nonsingular. In fact the limit stabilizes: $H_i^{BM,T}(X) = H^{BM}_{i+2d}(E_{d+1} \times_T X)$ for $i > (1-2d) + 2\dim X$.

Let X be a toric variety. By 1.2 or [BZ] the equivariant homology is the sum of the homologies of the orbits:

(2.1)
$$H_i^{BM,T}(X) = \bigoplus_{\sigma\in\Sigma} H_i^{BM,T}(O_\sigma)\,.$$

The orbits σ are labelled by the fan Σ. The equivariant Borel-Moore homology is isomorphic to the equivariant Chow group $A_*^T(X)$ considered e.g. by Brion [Br]. Each $H_*^{BM,T}(O_\sigma)$ is isomorphic to $Sym(\langle\sigma\rangle)[-2\mathrm{codim}\,\sigma]$ (the symmetric power $Sym^i(\langle\sigma\rangle)$ is placed in the degree $2(\mathrm{codim}\,\sigma - i)$). In particular it is a free abelian group. The odd part of the equivariant Borel-Moore homology vanishes. The module structure over $H^*(BT)$ is described in [Br].

3 Frobenius endomorphism

Let $p > 1$ be a natural number. Complex toric varieties are equipped with Frobenius endomorphism (power map) $\phi_p : X \to X$, see [T] . The power map of T is denoted by ψ_p. If we embedd T into X as the open orbit, then ϕ_p is the unique extension of ψ_p. Both maps induce a map at each step of the approximation of the Borel construction $ET_d \times_T X$. We denote this map by ϕ_p^T. (We note that ϕ_p^T is not the same as $1 \otimes \phi_p$ considered in [BZ].)

We would like to encode the action of ϕ_p^T in equivariant homology. The map $\iota_d^!$ does not commute with $\phi^T_{p,*}$, but $\phi^T_{p,*}\iota_d^! = p\,\iota_d^!\phi^T_{p,*}$. Then we set $H_i^{BM,T}(X) = H^{BM}_{i+2d}(ET_d \times_T X)(d)$ for d sufficiently large. If we study homology with coefficients in the field $\mathbf{F} = \mathbf{Q}$ or $\mathbf{F}_q$, provided that $(p,q) = 1$, the symbol (d) denotes tensoring with $\mathbf{F}$ acted by ϕ_p via the multiplication by p^{-d}. If we want to study integral homology we just analyze homology of a sufficiently large approximation of the Borel construction. Now we can state

Proposition 3.1 *$\phi^T_{p,*}$ induces the multiplication by p^i on $H_{2i}^{BM,T}(X)$.*

Proof. The homology of an orbit is generated over $H^*(BT)$ by its fundamental class. The map $\phi^T_{p,*}$ restricted to the orbit O_σ is a covering of the degree p^i, where $i = \dim O_\sigma = \operatorname{codim} \sigma$. Thus $\phi^T_{p,*}[\sigma] = p^i[\sigma] \in H^{BM,T}_{2i}(O_\sigma)$. Now we apply the additivity 2.1. $\square$

We consider the system of T-fibrations

$$ET_d \times X \to ET_d \times_T X\,.$$

We obtain a system of spectral sequences with

$${}_dE^2_{k,l} = H^{BM}_k(ET_d \times_T X) \otimes H^{BM}_l(T)\,.$$

The map $\iota^!_d$ passes to a map of spectral sequences

$${}_{d+1}E^j_{k+2,l}(1) \to {}_dE^j_{k,l}\,.$$

We set ${}_\infty E^j_{k,l} = {}_dE^j_{k+2nd,l}(d)$ for d large enough. The map ϕ^*_p of $H^{BM}_q(T)$ is the multiplication by p^{q-n} (for $q < n$ this group is trivial). Therefore the resulting map of the spectral sequences is the multiplication by $p^{\frac{k}{2}+l-n}$ on ${}_\infty E^2_{k,l}$.

THEOREM 3.2 *The spectral sequence* ${}_\infty E^j_{k,l}$ *converges to* $H^{BM}_{k+l-2n}(X)$. *For rational coefficients it degenerates on* E^3. *The resulting filtration coincides after renumbering with the weight filtration of homology.*

Remark 3.3 There is a shift $-2n$ in the degree which is repaired when we move the generators of $H^{BM}_*(T)$ to the negative degrees. They should be placed there since we compare E_2 with the Koszul complex, see below.

Remark 3.4 The homology $H^{BM,T}_*(X)$ is a module over $H^*(BT) = Sym\, M$, where $M = Hom(T, \mathbf{C}^*)$. The differential $d_2 : {}_\infty E^2_{k,l} \to {}_\infty E^2_{k-2,l+1}$ after the identification $H^{BM}_*(T) = H^{2n-*}(T)$ with $\Lambda^* M$ becomes the Koszul differential

$$H^{BM,T}_k(X) \otimes \Lambda^{2n-l} M \to H^{BM,T}_{k-2}(X) \otimes \Lambda^{2n-l-1} M\,.$$

Remark 3.5 The Koszul complex contains a complex constructed in [T]. Totaro considered rational coefficients, but he remarked that some information about the torsion can be obtained, see also [J, Remark 2.4.8].

Remark 3.6 From another point of view the Koszul duality and Frobenius endomorphism appears in [B] and [BL] for dual pairs of affine toric varieties.

Proof of Theorem 3.2. At each step ${}_dE^*_{k,l}$ converges to $H^{BM}_{k+l}(X \times (\mathbf{C}^{d+1} \setminus \{0\})^n)$, which is equal to $H^{BM}_{k+l}(X \times (\mathbf{C}^{(d+1)n}) = H^{BM}_{k+l-2n(d+1)}(X)$ for d sufficiently large. Therefore ${}_\infty E^*_{k,l}$, which is equal to ${}_dE^*_{k+2nd,l}$ (for d sufficiently large) converges to $H^{BM}_{k+l-2n}(X)$. The eigenvalue of ϕ_p acting on ${}_\infty E^*_{k-r,l+r-1}$ is equal to $p^{\frac{k}{2}+l-n+\frac{r}{2}-1}$. There is no obstruction for the differential d_2, but the higher differentials have to vanish. $\square$

4 Torsion.

It is not possible to detect the q-torsion of X for small prime q, but we prove Theorems 1.5 and 1.6 announced in the introduction:

Proof of Theorem 1.5. To show that the higher differentials vanish we consider the eigenvalues of ϕ_p acting on ${}_dE^2_{k-r,l+r-1}$ for $r = 2, \dots n+1$ and $k-r$ even. These are at most $\lceil \frac{n}{2} \rceil$ subsequent powers of p, as the reader may easily check (see the picture below). We will choose p such that all these powers are different modulo q. It is enough to take p which generate the group $\mathbf{F}_q^* \simeq \mathbf{Z}/(q-1)$. $\square$

$$
\begin{array}{lccccccccccccc}
2n & \dots & \circ & \cdot & \bullet & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \bullet & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \star & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \star & \cdot & \circ & \dots \\
n & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots
\end{array}
$$

A picture of the spectral sequence for $n = 6$.
$\star$ denotes the source and the target of d_2 (which always have the same weight),
$\bullet$ denotes the remaining possibly nonzero entries of the spectral sequence which are hit by the higher differentials.

If we want to determine the q-torsion for integral homology we may meet problems with extensions.

Proof of Theorem 1.6. The part of the integral spectral sequence computing the q-torsion degenerates as in the previous proof. To avoid problems with extensions we have to know that the eigenvalues of ϕ_p on $E^3_{k,l}$ are different along the lines $k + l = const$. There are exactly $\lceil \frac{n+1}{2} \rceil$ subsequent powers of p. We proceed as before, that is we find p with different powers modulo q. $\square$

$$
\begin{array}{lccccccccccccc}
2n & \dots & \circ & \cdot & \bullet & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \bullet & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \bullet & \cdot & \circ & \cdot & \circ & \dots \\
 & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \dots \\
n & \dots & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \bullet & \cdot & \circ & \dots
\end{array}
$$

A picture of the spectral sequence for $n = 6$.
The entries $\bullet$ should have different weights in $\mathbf{F}_q$.

We conjecture that the theorems above are true without assumptions on q. The conjecture holds if X is smooth by the work of M. Franz [F].

5 Weight filtration and gradation.

Let V_j be the eigensubspace of ϕ_p acting on $H_*^{BM}(X; \mathbf{Q})$ for the eigenvalue p^j. It does not depend on p. The weight filtration in homology usually is denoted by $W^i = W_{-i}$ and it is decreasing. Our gradation is related to the weight filtration:

$$W^i = \bigoplus_{j \geq \frac{i}{2}} V_j \,.$$

In particular $W^{2j} = W^{2j-1}$.

Conjugation. Toric varieties are defined over real numbers. The complex conjugation acts on the complex points of X. We can also determine the action on the homology: it acts by $(-1)^{\frac{i}{2}}$ on the i-th term of the weight gradation.

References

[BFMH] F. Bihan, M. Franz, C. McCrory, J. van Hamel *Is every toric variety an M-variety?* arXiv.org:math.AG/0510228

[B] T. Braden *Koszul duality for toric varieties* , arXiv.org:math/0308216

[BL] T. Braden, V. A. Lunts *Equivariant-constructible Koszul duality for dual toric varieties*, arXiv.org:math.AG/0409495

[Br] M. Brion, *Equivariant Chow groups for torus actions* Transformation groups, Vol 2, No 3 (1997), pp. 1-43

[BZ] J-L. Brylinski, B. Zhang: *Equivariant Todd classes for toric varieties*, arXiv.org:math.AG/0311318

[EG] D. Edidin, W. Graham *Equivariant intersection theory* Invent. Math. vol 131 (1998) no. 3, pp. 595–634

[F] M. Franz *On the integral cohomology of smooth toric varieties*, arXiv.org:math.AT/0308253

[FW] M. Franz, A. Weber *Weights in cohomology and the Eilenberg-Moore spectral sequence*, Ann. Inst Fourier (Grenoble) 55 (2005), no. 2, 673–691

[J] A. Jordan *Homology and cohomology of toric varieties,* thesis, Konstanz 1998.

[T] B. Totaro *Chow groups, Chow cohomology, and linear varieties.* Journal of Algebraic Geometry, to appear.
http://www.dpmms.cam.ac.uk/~bt219/papers.html

[W] A. Weber *Weights in the cohomology of toric varieties*. Cent. Eur. J. Math. 2 (2004), no. 3, 478–492, arXiv.org:math.AG/0301314

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