Residue forms on singular hypersurfaces

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1 Introduction

The purpose of this paper is to point out a relation between the canonical sheaf and the intersection complex of a singular algebraic variety. We focus on the hypersurface case. Let M be a complex manifold, $X \subset M$ a singular hypersurface. We study residues of top-dimensional meromorphic forms with poles along X. Applying resolution of singularities sometimes we are able to construct residue classes either in L^2-cohomology of X or in the intersection cohomology. The conditions allowing to construct these classes coincide. They can be formulated in terms of the weight filtration. Finally, provided that these conditions hold, we construct in a canonical way a lift of the residue class to cohomology of X.

Let the manifold M be of dimension $n+1$. If the hypersurface X is smooth we have an exact sequence of sheaves on M:

$$0 \longrightarrow \Omega_M^{n+1} \hookrightarrow \Omega_M^{n+1}(X) \xrightarrow{Res} i_*\Omega_X^n \longrightarrow 0\,.$$

Here Ω_M^{n+1} stands for the sheaf of holomorphic differential forms of the top degree on M and $\Omega_M^{n+1}(X)$ is the sheaf of meromorphic forms with logarithmic poles along X, i.e. with the poles at most of the first order. The map $i : X \hookrightarrow M$ is the inclusion. The morphism Res is the residue map sending $\omega = \frac{ds}{s} \wedge \eta$ to $\eta_{|X}$ if s is a local equation of X. The residues of forms with logarithmic poles along a smooth hypersurface were studied by Leray ([Le]) for forms of any degree. Later such forms and their residues were applied by Deligne ([De], [GS]) to construct the mixed Hodge structure for the cohomology of open smooth algebraic varieties.

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We will allow X to have singularities. As in the smooth case the residue form is well defined differential form on the nonsingular part of X. In general it may be highly singular at the singular points of X. We will ask the following questions:

- Suppose M is equipped with a hermitian metric. Is the norm of $Res(\omega)$ square integrable? We note that this condition does not depend on the metric.
- Does the residue form $Res(\omega)$ define a class in the intersection cohomology $IH^n(X)$?

We recall that by Poincaré duality residue defines a class in homology $H_n(X)$ (precisely Borel-Moore homology, i.e. homology with closed supports), see §7. The possibility to lift the residue class to intersection cohomology means that $Res(\omega)$ has mild singularities. The intersection cohomology $IH^*(X)$, defined in [GM], is a certain cohomology group attached to a singular variety. Poincaré duality map $[X]\cap : H^*(X) \to H_{\dim_{\mathbf{R}}(X)-*}(X)$ factors through $IH^*(X)$. Conjecturally[1] intersection cohomology is isomorphic to L^2-cohomology. It was known from the very beginning of the theory, that conjecture is true if X has conical singularities ([Ch], [CGM]).

We study a resolution of singularities

$$\mu : \widetilde{M} \longrightarrow M\,, \qquad \mu^{-1}(X) = \widetilde{X} \cup E\,,$$

where $\widetilde{X}$ is the proper transform of X and E is the exceptional divisor. The pull back $\mu^*\omega$ is a meromorphic form on $\widetilde{M}$. It can happen that it has no poles along the exceptional divisors. Then we say that ω has *canonical* singularities along X. By the definition ω has canonical singularities if and only if $\omega \in adj_X \cdot \Omega^{n+1}_M(X)$, where $adj_X \subset \mathcal{O}_M$ is the adjoint ideal of [EL]. The set of forms with canonical singularities can be characterized as follows:

THEOREM 1.1 *The following conditions are equivalent*

- *ω has canonical singularities along X;*
- *the residue form $Res(\omega) \in \Omega^n_{X_{\rm reg}}$ extends to a holomorphic form on any resolution $\nu : \overline{X} \to X$;*
- *the norm of $Res(\omega)$ is square-integrable for any hermitian metric on $X_{\rm reg}$.*

Later the statement of 1.1 is divided into Proposition 3.2, Theorem 4.1, Corollary 5.2 and Theorem 6.1. Although our constructions use resolution of singularities we are primarily interested in the geometry of the singular space X itself. The resulting objects do not depend on the choice of resolution.

[1] The proof in [Oh] seems to be incomplete.

Our description of forms with canonical singularities agrees with certain results concerning intersection cohomology. We stress that on the level of forms we obtain a lift of residue to L^2-cohomology for free. On the other hand, using cohomological methods one constructs a lift of the residue class to intersection cohomology. This time the lift is obtained essentially applying the decomposition theorem of [BBD]. This lift is not unique. It is worth-while to confront these two approaches. The crucial notion in the cohomological approach is the weight filtration. We will sketch this construction below: Suppose that M is complete. Then $H^{k+1}(M{-}X)$ is equipped with the weight filtration, all terms are of the weight $\geq k+1$. The homology $H_{2n-k}(X)$ is also equipped with a mixed Hodge structure. It is of the weight $\geq k-2n$. The homological residue map preserves the weight filtration:

$$res : H^{k+1}(M{-}X) \longrightarrow H_{2n-k}(X)(-n-1).$$

Here (i) denotes the i-fold Tate twist; now $H_{2n-k}(X)(-n-1)$ is of the weight $\geq k+2$. Intersection cohomology $IH^k(X)$ maps to $H_{2n-k}(X)(-n)$. Since it is pure, the image is contained in $W_k(H_{2n-k}(X)(-n))$. We will show that:

Theorem 1.2 *If $c \in H^{k+1}(M{-}X)$ is of weight $\leq k+2$ then $res(c)$ lifts to intersection cohomology. In another words, we have a factorization of the residue map*

$$\begin{tikzcd} W_{k+2}H^{k+1}(M{-}X) \arrow[rr, "res"] \arrow[dr] & & W_{k+2}(H_{2n-k}(X)(-n-1)) \\ & IH^k(X)(-1) \arrow[ur] & \end{tikzcd} .$$

In fact for an arbitrary complete algebraic variety the image of intersection cohomology coincides with the lowest term of the weight filtration in homology, see [We4].

We note that if ω has canonical singularities along X then its cohomology class is of weight $\leq n{+}2$. By 1.1 $Res(\omega)$ defines a class in L^2-cohomology. Also, by 1.2 the residue of $[\omega]$ can be lifted to intersection cohomology. To completely clear up this situation we construct in §9 a canonical lift of the residue class not only to intersection cohomology, but even to cohomology of X.

An attempt to relate holomorphic differential forms to intersection cohomology was proposed by Kollár ([Ko1], II §4). It seems that his solution is not definite since he applies the (noncanonical) decomposition theorem. The construction proposed in 9.4 is elementary and geometric. As a side result of our consideration we obtain

Theorem 1.3 *Suppose an algebraic variety X is complete of dimension n. Let $\widetilde{X}$ be its resolution. Then $H^k(\widetilde{X};\Omega^n_{\widetilde{X}})$ is a direct summand both in $H^{n+k}(X)$ and $IH^{n+k}(X)$.*

One can hope that a relation between holomorphic forms of lower degrees with intersection cohomology will be explained as well.

Another approach to understand the relation between the residues and intersection cohomology was presented by Vilonen [Vi] in the language of $\mathcal{D}$-modules. His method applies to isolated complete intersection singularities.

Finally in §10-§11 we briefly describe a relation between the oscillating integrals of [Ma] or [Va] and residue theory for isolated singularities. Namely, if the order of a form at each singular point is greater than zero, then the residue class can be lifted to intersection cohomology. Again, this condition coincides with having canonical singularities.

The present paper is a continuation of [We2], where the case of isolated singularities was described. My approach here was partially motivated by a series of lectures delivered by Tomasz Szemberg on the algebraic geometry seminar IMPANGA in Polish Academy of Science.

CONTENTS:

2 Residues as differential forms

Let ω be a closed form with a first order pole on X. Then the residue form $Res(\omega)$ can be defined at the regular points of X. The case when ω is a holomorphic $(n+1,0)$–form is the most important for us:

$$\omega = \frac{g}{s} dz_0 \wedge \ldots \wedge dz_n\,,$$

where the function s describes X. The space of such forms is denoted by $\Omega^{n+1}_M(X)$. Then the residue form is a holomorphic $(n,0)$–form:

$$Res(\omega) \in \Omega^n_{X_{\rm reg}}\,.$$

The symbol $\in$ by abuse of notation means $Res(\omega)$ is a section of the sheaf $\Omega^n_{X_{\rm reg}}$. The precise formula for the residue is the following: Set $s_i = \frac{\partial s}{\partial z_i}$. We have

$$ds = \sum_{i=0}^n s_i dz_i\,.$$

At the points where $s_0 \neq 0$ we write

$$dz_0 = \frac{1}{s_0}\left(ds - \sum_{i=1}^n s_i dz_i\right)$$

and

$$\begin{aligned} \omega = \frac{g}{s\,s_0}\left(ds - \sum_{i=1}^n s_i dz_i\right) \wedge dz_1 \wedge \ldots \wedge dz_n = \\ = \frac{ds}{s} \wedge \frac{g}{s_0} dz_1 \wedge \ldots \wedge dz_n\,. \end{aligned}$$

Thus $Res(\omega) = \left(\frac{g}{s_0} dz_1 \wedge \ldots \wedge dz_n\right)_{|X_{\rm reg}} \in \Omega^n_{X_{\rm reg}}$.

To see how $Res(\omega)$ behaves in a neighbourhood of the singularities let us calculate its norm in the metric coming from the coordinate system:

$$|Res(\omega)|_X = \left|\frac{ds}{|ds|} \wedge Res(\omega)\right|_M = \left|\frac{s\,\omega}{|ds|}\right|_M = \frac{|g|}{|\mathrm{grad}(s)|}\,.$$

We conclude that $Res(\omega)$ has (in general) a pole at singular points of X.

The forms that can appear as residue forms are exactly the regular differential forms defined by Kunz for arbitrary varieties; [Ku].

3 Residues and resolution

We will analyze the residue form using resolution of singularities. Let $\mu : \widetilde{M} \to M$ be a log-resolution of (M,X), i.e. a birational map , such that $\mu^{-1}X$ is a smooth divisor with normal crossings and μ is an isomorphism when restricted to $\widetilde{M} - \mu^{-1}X_{\rm sing}$. Let $\widetilde{X}$ be the proper transform of X and let E be the exceptional divisor. The pull-back of ω to $\widetilde{M}$ is a meromorphic form with poles along $\widetilde{X}$ and E. According to the terminology of [Ko2] we define:

DEFINITION 3.1 *We say that ω has canonical singularities along X if $\mu^*\omega$ has no pole along the exceptional divisor, i.e. $\mu^*\omega \in \Omega^{n+1}_{\widetilde{M}}(\widetilde{X})$.*

We note that this notion does not depend on the resolution. Our method of studying residue forms are appropriate to tackle this class of singularities. We begin with an easy observation:

PROPOSITION 3.2 *If ω has canonical singularities along X, then for any resolution $\nu : \overline{X} \to X$ the pull-back of the residue form $\nu^* Res(\omega)$ is holomorphic on $\overline{X}$.*

Remark 3.3 We do not assume that ν extends to a resolution of the pair (M, X).

Proof. Let μ be a log-resolution of (M, X). By the assumption $\mu^*\omega \in \Omega^{n+1}_{\widetilde{M}}(\widetilde{X})$. Therefore $Res(\mu^*\omega)$ is a holomorphic form on $\widetilde{X}$. Hence $Res(\omega) \in \Omega^n_{X_{\rm reg}}$ extends to a section of $\mu_*\Omega^n_{\widetilde{X}}$. The later sheaf does not depend on the resolution of X. Indeed, let $\widehat{X}$ be a smooth variety dominating both $\widetilde{X}$ and $\overline{X}$. Then $Res(\mu^*\omega)$ can be pulled back to $\widehat{X}$ and pushed down to $\overline{X}$ (since $f_*\Omega^n_{\widehat{X}} = \Omega^n_{\overline{X}}$ if f is birational). The resulting form coincides with $\nu^* Res(\omega)$ outside the singularities. $\square$

4 Vanishing of hidden residues

We have observed that if ω has canonical singularities, then the residue form is smooth on each resolution. Let us assume the converse: suppose $Res(\omega)$ extends to a holomorphic form on $\widetilde{X}$. The extension is determined only by the nonsingular part of X. We will show, that all the other "hidden" residues along exceptional divisors vanish.

THEOREM 4.1 *If $Res(\mu^*\omega)_{|\widetilde{X}-E}$ has no pole along $E \cap \widetilde{X}$ then ω has canonical singularities along X.*

Proof. Let $E = \bigcup_{i=1}^k E_i$ be a decomposition of E into irreducible components. Assume that $Res(\mu^*\omega)_{|E_i}$ is nontrivial for $1 \le i \le l$ for some $l \le k$. Blowing up intersections $E_i \cap \widetilde{X}$ we can assume that $E_i \cap \widetilde{X} = \emptyset$ for $i \le l$. Let a_i be the order of the pole of $\mu^*\omega$ along E_i. Define a quotient sheaf $\mathcal{F}$:

4.2 $$0 \to \Omega^{n+1}_{\widetilde{M}} \hookrightarrow \Omega^{n+1}_{\widetilde{M}}(\textstyle\sum_{i=1}^l a_i E_i) \twoheadrightarrow \mathcal{F} \to 0\,.$$

LEMMA 4.3 *The direct image $\mu_*\mathcal{F}$ vanishes.*

Proof. We push forward the sequence 4.2 and we obtain again the exact sequence, since $R^1\mu_*\Omega^{n+1}_{\widetilde{M}} = 0$ e.g. by [Ko1]. But now the sections of

$$\mu_*\Omega^{n+1}_{\widetilde{M}}(\sum_{i=1}^{l} a_i E_i)$$

are forms which are holomorphic on $M - \mu(E)$. Therefore they are holomorphic and hence $\mu_*\mathcal{F} = 0$. $\square$

Proof of 4.1 cont. We tensor the sequence 4.2 with $\mathcal{O}(\widetilde{X})$. Since the support of $\mathcal{F}$ is disjoint with $\widetilde{X}$ we obtain a short exact sequence:

$$0 \to \Omega^{n+1}_{\widetilde{M}}(\widetilde{X}) \to \Omega^{n+1}_{\widetilde{M}}(\widetilde{X} + \sum_{i=1}^{l} a_i E_i) \to \mathcal{F} \to 0\,.$$

We apply μ_* and by the Lemma 4.3 we have an isomorphism

$$\mu_*\Omega^{n+1}_{\widetilde{M}}(\widetilde{X}) \simeq \mu_*\Omega^{n+1}_{\widetilde{M}}(\widetilde{X} + \sum_{i=1}^{l} a_i E_i)\,.$$

The above equality means that ω cannot have a pole along exceptional divisors. $\square$

5 Adjoint ideals

The adjoint ideals were introduced in [EL] for a hypersurface $X \subset M$. The adjoint ideal $adj_X \subset \mathcal{O}_M$ is the ideal satisfying

$$\mu_*\Omega^{n+1}_{\widetilde{M}}(\widetilde{X}) = adj_X \cdot \Omega^{n+1}_M(X)\,.$$

The ideal adj_X consists of the functions f, for which $\mu^*(\frac{f}{s}dz_1 \wedge \ldots \wedge dz_m) \in \Omega^{n+1}_{\widetilde{M}}(\mu^* D)$ has no pole along the exceptional divisors, i.e. it belongs to $\Omega^{n+1}_{\widetilde{M}}(\widetilde{X})$. Here s, as before, is a function describing X. In another words the forms $\omega \in adj_X \cdot \Omega^{n+1}_M(X)$ are exactly the forms with canonical singularities along X. Moreover the sequence of sheaves

5.1 $$0 \to \Omega^{n+1}_M \to adj_X \cdot \Omega^{n+1}_M(X) \to \mu_*\Omega^n_{\widetilde{X}} \to 0$$

is exact ([EL] 3.1). (This follows from vanishing of $R^1\mu_*\Omega^{n+1}_{\widetilde{M}}$.) The adjoint ideal does not depend on the resolution.

Corollary 5.2 *The residue form $Res(\omega) \in \Omega^n_{X_{\rm reg}}$ extends to a section of $\mu_*\Omega^n_{\widetilde{X}}$ if and only if $\omega \in adj_X \cdot \Omega^{n+1}_M(X)$.*

Proof. The implication $\Rightarrow$ follows from the Theorem 4.1. The converse follows from the exact sequence 5.1. $\square$

It turns out that every form has canonical singularities, i.e. $adj_X = \mathcal{O}_M$ if and only if X has rational singularities [Ko2], §11.

6 L^2-cohomology

Let us assume that the tangent space of M is equipped with a hermitian metric. For example if M is a projective variety, then one has the restriction of the Fubini-Study metric from projective space. The nonsingular part of the hypersurface X also inherits this metric. One considers the complex of differential forms which have square-integrable pointwise norm (and the same holds for differential). Its cohomology is an important invariant of the singular variety called the L^2-cohomology, [CGM]. This is why we are led to the question: when the norm of the residue form is square-integrable? Moreover, for the forms of the type $(n,0)$ on the n-dimensional manifold the condition of integrability does not depend on the metric. This is because $\int_X |\eta|^2 dvol(X)$ is equal up to a constant to $\int_X \eta \wedge \overline{\eta}$.

THEOREM 6.1 *The residue form $Res(\omega)$ has the square-integrable norm if and only if ω has canonical singularities.*

Proof. Instead of asking about integrability on X_{reg} we ask about integrability on $\widetilde{X}$. Now, local computation shows that if ω has a pole, then its norm is not square-integrable. $\square$

Remark 6.2 Note that the class of the residue form does not vanish provided that ω has a pole along X. This is because $Res(\omega)$ can be paired with its conjugate $\overline{Res(\omega)}$ in cohomology.

Remark 6.3 The connection between integrability condition and multiplicities were studied by Demailly, see e.g. [Dm].

Remark 6.4 For homogeneous singularities (which are conical) integrals of the residue forms along conical cycles converge provided that the cycle is allowable in the sense of intersection homology and $|Res(\omega)| \in L^2(X)$.

7 Residues and homology

Suppose for a moment that $X \subset M$ is smooth. Let Tub_X be a tubular neighbourhood of X in M. We have a commutative diagram:

$$\begin{array}{ccc}
H^*(M{-}X) & \stackrel{[M]\cap}{\longrightarrow} & H^{BM}_{2n+2-*}(M,X)(-n-1) \\
\downarrow d & & \downarrow \partial \\
H^{*+1}(M, M{-}X) & \stackrel{[M]\cap}{\longrightarrow} & H^{BM}_{2n+1-*}(X)(-n-1) \\
\| & & \uparrow [X]\cap \\
H^{*+1}(Tub_X, Tub_X{-}X) & \stackrel{\tau}{\longleftarrow} & H^{*-1}(X)(-1)
\end{array}$$

In the diagram H^{BM}_* denotes Borel–Moore homology, i.e. homology with closed supports. All coefficients are in $\mathbb{C}$. The entries of the diagram are equipped with the Hodge structure. The map τ is the Thom isomorphism, the remaining maps in the bottom square are also isomorphisms by Poincaré duality for X and M. The residue map

$$res = \tau^{-1} \circ d : H^*(M{-}X) \longrightarrow H^{*-1}(X)$$

is defined to be the composition of the differential with the inverse of the Thom isomorphism. By [Le] we have:

$$res([\omega]) = \frac{1}{2\pi i}[Res(\omega)]$$

for a closed form with the first order pole along X. (We use small letter for the homology class $res(c) \in H^{BM}_{2n+1-*}(X)$ to distinguish it from the differential form $Res(\omega) \in \Omega^*_X$.)

When X is singular then there is no tubular neighbourhood of X nor Thom isomorphism, but we can still define a homological residue

$$res : H^*(M{-}X) \longrightarrow H^{BM}_{2n+1-*}(X)(-n-1)$$

$$res(c) = [M] \cap dc = \partial([M] \cap c)$$

If X was nonsingular, then this definition would be equivalent to the previous one since $\xi \mapsto [X] \cap \xi$ is Poincaré duality isomorphism and the diagram above commutes.

Remark 7.1 One should mention here the work of M. Herrera [He1] and [HeL] who defined a residue current for a meromorphic $k+1$–form. This current is supported by the divisor of poles. For a closed form it defines a homology class in $H^{BM}_{2n-k}(X)$.

In general there is no hope to lift the residue morphism to cohomology. For $M = \mathbb{C}^{n+1}$ the morphism res is the Alexander duality isomorphism and $[X]\cap$ may be not onto. Instead we ask if the residue of an element lifts to the intersection homology of X. The intersection homology groups, defined by Goresky and MacPherson in [GM], are the groups that 'lie between' homology and cohomology; i.e. there is a factorization:

$$\begin{array}{ccc} H^*(X) & \xrightarrow{[X]\cap} & H^{BM}_{2n-*}(X)(-n)\,. \\ \searrow & & \nearrow \\ & IH^*(X) & \end{array}$$

In fact for complete X the map $[X]\cap$ factors through

$$H^k(X)/W_{k-1}H^k(X) \xhookrightarrow{\alpha} IH^k(X) \xrightarrow{\beta} W_k(H_{2n-k}(X)(-n))\,.$$

The injectivity of α and surjectivity of β is proved in [We4]. The composition $\beta\alpha$ does not have to be an isomorphism. For example, if X admits an algebraic cellular decomposition then its cohomology is pure (i.e. $W_{k-1}H^k(X) = 0$ and $W_k(H_{2n-k}(X)(-n)) = H_{2n-k}(X)(-n)$) but the Poincaré duality map $[X]\cap$ does not have to be an isomorphism. We will analyze the arguments of [We4] for the particular situation of a hypersurface.

8 Hodge theory

According to Deligne ([De], see also [GS]) any algebraic variety carries a mixed Hodge structure. Suppose the ambient variety M is complete. To construct the mixed Hodge structure on $M-X$ one finds a log-resolution of (M, X), denoted by $\mu : \widetilde{M} \to M$, (see §3). Then one defines $A^*_{\log} = A^*_{\widetilde{M}}(\log\langle\mu^{-1}X\rangle)$, the complex of C^∞ forms with logarithmic poles along $\mu^{-1}X$. Its cohomology computes $H^*(\widetilde{M}-\mu^{-1}X) = H^*(M-X)$. The complex $A_{\log}$ is filtered by the weight filtration

$$0 = W_{k-1}A^k_{\log} \subset W_kA^k_{\log} \subset \ldots \subset W_{2k}A^k_{\log} = A^k_{\log}\,,$$

which we describe below. Let $z_0, z_1, \ldots z_n$ be local coordinates in which the components of $\mu^{-1}X$ are given by the equations $z_i = 0$ for $i \le m$. The space $W_{k+\ell}A^k_{\log}$ is spanned by the forms

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_\ell}}{z_{i_\ell}} \wedge \eta$$

where $i_j \le m$ and $\eta \in A^{k-\ell}_{\widetilde{M}}$ is a smooth form on $\widetilde{M}$. The weight filtration in $A^*_{\log}$ induces a filtration in cohomology. The quotients of subsequent terms $W_{k+\ell}H^k(M-X)/W_{k+\ell-1}H^k(M-X)$ are equipped with pure Hodge structure of weight $k+\ell$.

Our goal is to tell whether the residue of a differential form or the residue of a cohomology class can be lifted to intersection cohomology. The Hodge structure on intersection cohomology hasn't been constructed yet in the setup of differential forms. On the other hand, there are alternative constructions in which intersection homology has weight filtration. If X is a complete variety, then $IH^*(X)$ is pure. This property is fundamental either in [BBD] or in Saito's theory, [Sa].

The homology of X is also equipped with the mixed Hodge structure. Since X is complete

$$W_{k-1}(H_{2n-k}(X)(-n)) = 0\,, \qquad W_{2k}(H_{2n-k}(X)(-n)) = H_{2n-k}(X)(-n)\,.$$

Due to purity of intersection cohomology

$$im(IH^k(X) \to H_{2n-k}(X)) \subset W_k(H_{2n-k}(X)(-n))\,.$$

The residue map

$$res : H^{k+1}(M{-}X) \to H_{2n-k}(X)(-n-1)$$

preserves the weights. In particular it vanishes on

$$W_{k+1}H^{k+1}(M{-}X) = im(H^{k+1}(\widetilde{M}) \to H^{k+1}(\widetilde{M}{-}\mu^{-1}X))\,.$$

Suppose we have a class $c \in W_{k+2}H^{k+1}(M{-}X)$. Then $res(c)$ is of weight $k+2$ in $H_{2n-k}(X)(-n-1)$. It is reasonable to ask if it comes from intersection cohomology.

THEOREM 8.1 *Suppose that M is complete. Then the residue of each class $c \in W_{k+2}H^{k+1}(M{-}X)$ can be lifted to intersection cohomology.*

Proof. Let $\mu : \widetilde{M} \to M$ be a log-resolution of (M, X). We consider the residue $res(\mu^* c) \in H_{2n-k}(\mu^{-1}X)(-n-1)$.

LEMMA 8.2 *The homology class $res(\mu^* c)$ is a lift of $res(c)$ to $H_{2n-k}(\mu^{-1}X)(-n-1)$, i.e.*

$$\mu_*(res(\mu^* c)) = res(c)\,.$$

Proof.

$$\mu_*(res(\mu^* c)) = \mu_*([\widetilde{M}] \cap d\mu^* c) = (\mu_*[\widetilde{M}]) \cap dc = res(c)\,.$$

□

Proof of 8.1 cont. Now assume that c has weight $k+2$. Then $\mu^* c$ is represented by a form ω with logarithmic poles of weight $k+2$. The residue of ω consists of forms $Res_i(\omega)$ on each component $E_i \subset \mu^{-1}X$ (we set $E_0 = \widetilde{X}$). These forms have no poles along the intersections of components. This means that $res(\mu^*c)$ comes from $\sum_\alpha[Res_i(\omega)] \in \bigoplus_i H^k(E_i) = IH^k(\mu^{-1}X)$. By [BBFGK] (see [We3] for a short proof) we can close the following diagram with a map θ of intersection cohomology groups:

$$\begin{array}{ccccccc}
\sum_i[Res_i(\omega)] & \in & IH^k(\mu^{-1}X) & \xrightarrow{\iota} & H_{2n-k}(\mu^{-1}X) & \ni & res(\mu^*c) \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\mu_*} & & \\
 & & IH^k(X) & \xrightarrow{\iota} & H_{2n-k}(X) & \ni & res(c)\,.
\end{array}$$

Here ι is the natural transformation from intersection cohomology to homology. The class $\theta(\sum_i[Res_i(\omega)])$ is the desired lift of $res(c)$. □

Remark 8.3 The completeness assumption can be removed in 8.1 and it is clear that the orders of poles at infinity do not matter.

Note that if a meromorphic $n+1$–form ω has canonical singularities along X then $\mu^*\omega$ has no pole along the exceptional divisors. Therefore it belongs to the logarithmic complex, it is closed and

$$\mu^*\omega \in W_{n+2}A^{n+1}_{\log}\,.$$

Conversely, a closed $n+1$–form which belongs to the top piece of the Hodge filtration $F^{n+1}A^{n+1}_{\log}$ has to be meromorphic. We obtain a surjection

$$\Omega^{n+1}_{\widetilde{M}}(\mu^{-1}X) \twoheadrightarrow F^{n+1}H^{n+1}(M-X)\,.$$

A meromorphic form has canonical singularities if and only if it belongs to $W_{n+2}A^{n+1}_{\log}$ since by 4.1 it has no poles along the exceptional divisors. Therefore we have a surjective map

$$\Omega^{n+1}_{\widetilde{M}}(\widetilde{X}) \twoheadrightarrow F^{n+1}W_{n+2}H^{n+1}(M-X)\,.$$

This way we have solved positively the problem of lifting to $IH^n(X)$ the residue classes of forms which have canonical singularities. Nevertheless it is possible to do much more. We will find a lift to cohomology $H^n(X)$ in a canonical way.

9 Residues in cohomology

In this section we ignore the Tate twist.

Suppose a meromorphic $(n+1)$-form ω has canonical singularities along X. We will show how to construct a lift of the residue class $res(\omega) \in H_n(X)$ to $H^n(X)$. It is enough to define an integral

$$\widehat{res}(\omega) : H_n(X) \to \mathbb{C}\,.$$

For the construction we need the following (probably well known) fact.

Proposition 9.1 *Let X be a variety of pure dimension. Let $TC_*^{alg}(X) \subset C_*(X)$ be the subcomplex of geometric chains which are semialgebraic and satisfy the conditions*

$$\dim(\xi \cap X_{\mathrm{sing}}) < \dim \xi \,,$$

$$\dim(\partial\xi \cap X_{\mathrm{sing}}) < \dim \partial\xi \,.$$

The inclusion of complexes induces an isomorphism of homology.

Remark 9.2 To show that the support condition does not spoil the homology one can proceed as in [Ha] computing inductively local cohomology.

For a cycle $\xi \in TC_n^{alg}(X)$ let us define

$$\langle \widehat{res}(\omega), \xi \rangle = \frac{1}{2\pi i} \int_{\mu^*\xi} Res(\mu^*\omega) \,,$$

where

$$\mu^*\xi = \mathrm{closure}(\mu^{-1}(\xi - X_{\mathrm{sing}}))$$

is the strict transform of the cycle ξ. Note that $\mu^*\xi$ is a semialgebraic chain, which does not have to be a cycle. Alternatively, we may define $\langle \widehat{res}(\omega), \xi \rangle = \frac{1}{2\pi i} \int_\xi Res(\omega)$ and say that the integral always converges for $\xi \in TC_n^{alg}(X)$. We have to prove, that our definition does not depend on the choice of a cycle. Suppose that ξ' is another cycle, such that $\xi - \xi' = \partial\eta$. Again we assume that both ξ' and η belong to $TC_*^{alg}(X)$. Set

$$\Delta = \mu^*\xi - \mu^*\xi' - \partial\mu^*\eta \,.$$

The residue form $Res(\mu^*\omega)$ is closed, therefore by Stokes theorem

$$\begin{aligned} \langle \widehat{res}(\omega), \xi \rangle - \langle \widehat{res}(\omega), \xi' \rangle &= \tfrac{1}{2\pi i} \left(\int_{\mu^*\xi} Res(\mu^*\omega) - \int_{\mu^*\xi'} Res(\mu^*\omega) \right) \\ &= \tfrac{1}{2\pi i} \int_\Delta Res(\mu^*\omega) \,. \end{aligned}$$

The chain Δ is contained in the exceptional locus of $\mu_{|\widetilde{X}}$, which is of dimension $n-1$. The form $Res(\mu^*\omega)$ is of type $(n,0)$, therefore it vanishes on Δ. This way we have defined $\widehat{res}(\omega) \in (H_n(X))^* = H^n(X)$.

We have to show that $\widehat{res}(\omega)$ is a lift of $res(\omega) \in H_n(X)$. In fact we will argue that it is a lift of $res(\mu^*\omega) \in H_n(\mu^{-1}X)$. By our assumption $res(\mu^*\omega)$ comes from $\bigoplus_i H^n(E_i)$. By 4.1 the residues $Res_i(\mu^*\omega)$ vanish along the exceptional divisors. It is enough to show that

$$\langle \widehat{res}(\omega), [\mu_*(\xi)] \rangle = \frac{1}{2\pi i} \langle Res_0(\mu^*\omega), \xi \rangle = \frac{1}{2\pi i} \int_\xi Res_0(\mu^*\omega)$$

for a cycle $\xi \in C_n(\widetilde{X})$. We may assume that ξ is semialgebraic and $\dim(\xi \cap \mu^{-1}(X_{\mathrm{sing}}) \leq n-1$. Then $\mu^*\mu_*\xi = \xi$ and the formula follows from the definition of $\widehat{res}(\omega)$.

We have proved

Theorem 9.3 *If ω is a holomorphic form of the top degree, then there exists a canonical lift of $res(\omega)$ to cohomology $H^n(X)$.*

Remark 9.4 By the same procedure one can define a map

$$\iota : H^k(\widetilde{X}, \Omega^n_{\widetilde{X}}) \to H^{n+k}(X)\,,$$

such that $\mu^* \circ \iota$ is the canonical map $H^k(\widetilde{X}, \Omega^n_{\widetilde{X}}) \to H^{n+k}(\widetilde{X})$. By [BBFGK] the map $\mu^* : H^*(X) \to H^*(\widetilde{X})$ factors through $IH^*(X)$. On the level of derived category $D(X)$ we have a chain of maps

$$R\mu_*\Omega^n_{\widetilde{X}}[-n] \simeq \mu_* A^{n,*}[-n] \to \mathbb{C}_X \to IC_X \to R\mu_*\mathbb{C}_{\widetilde{X}}$$

factorizing the natural $R\mu_*\Omega^n_{\widetilde{X}}[-n] \to R\mu_*\mathbb{C}_{\widetilde{X}}$. This proves Theorem 1.3. Note that a map to intersection cohomology or rather a dual one

$$IC_X \to DR\mu_*\Omega^n_{\widetilde{X}}[-n] \simeq R\mu_*\mathcal{O}_{\widetilde{X}}$$

was described by Kollár in [Ko1], II 4.8. The decomposition theorem of [BBD] is applied. Our map is constructed surprisingly easily and in a canonical way.

For complete X we obtain a side result:

Theorem 9.5 *Suppose an algebraic variety X is complete of dimension n. Let $\widetilde{X}$ be its resolution. Then $H^k(\widetilde{X}; \Omega^n_{\widetilde{X}})$ is a direct summand both in $H^{n+k}(X)$ and $IH^{n+k}(X)$. The inclusion is adjoint to the strict transform of cycles.*

The statement for intersection cohomology also follows from [Ko1] II 4.9.

Remark 9.6 In [He2] there are studied residues of the meromorphic forms which can be written as $\omega = \frac{ds}{s} \wedge \eta + \theta$. For the forms of top degree this condition is more restrictive then having canonical singularities. For example if $n \geq 2$ and X has isolated simple singularities then all forms $\omega \in \Omega^{n+1}(X)$ have canonical singularities (see §11) but not necessarily can be written as above. For the forms considered by Herrera the residue $res(\omega) = \eta_{|X}$ is well defined as an element of a suitable complex of forms on the singular variety X. The space M is allowed to be singular. For nonsingular M this result is rather tautological.

10 Isolated singularities

Residue forms for hypersurfaces with isolated singularities are strongly related to oscillating integrals. The first references for this theory are [Ma] or [Va]. In [AGV]§10-15 the reader can find a review, samples of proofs and other precise references to original papers. A relation of oscillating integrals with the theory of singularities of pairs is explained in [Ko2], §9.

Suppose $0 \in \mathbb{C}^{n+1}$ is an isolated singular point of s. Let $X_t = s^{-1}(t) \cap B_\epsilon$ for $0 < |t| < \delta$ be the Milnor fiber with the usual choice of $0 < \delta \ll \epsilon \ll 1$. For a given germ at 0 of a holomorphic $(n+1)$-form $\eta \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ define a quotient of forms by:

$$(\eta/ds)_{|X_t} = Res\left(\frac{\eta}{s-t}\right) \in \Omega^n_{X_t}\,.$$

Let $\zeta_t \subset X_t$ be a continuous multivalued family of n-cycles in the Milnor fibers. The function

$$I^\eta_\zeta(t) = \int_{\zeta_t} \eta/ds$$

is a holomorphic (multi–valued) function. By [Ma] or [AGV] §13.1 the function $I^\eta_\zeta(t)$ can be expanded in a series

$$I^\eta_\zeta(t) = \sum_{\alpha,k} a_{\alpha,k} t^\alpha (\log t)^k\,,$$

where the numbers α are rationals greater then -1 and k are natural numbers or 0. When we consider all the possible families of cycles we obtain so-called *geometric section* $S(\eta)$ of the cohomology Milnor fiber. We recall that cohomology Milnor fiber is a flat vector bundle equipped with Gauss-Manin connection. Its fiber over t is $H^n(X_t)$. If we fix $t_0 \neq 0$ we can write

$$S(\eta) = \sum_{\alpha,k} A_{\alpha,k} t^\alpha (\log t)^k,$$

with $A_{\alpha,k} \in H^n(X_{t_0})$. The smallest exponent α occurring in the expansion of $S(\eta)$ is called the *order* of η. The smallest possible order among all the forms η is the order of $dz_0 \wedge \ldots \wedge dz_n$.

PROPOSITION 10.1 *Suppose that X has isolated singularities. Let $\omega \in \Omega^{n+1}_M(X)$ be a meromorphic form with a first order pole along X. If the order of $s\omega$ is greater than zero at each singular point, then the residue class of ω lifts to intersection cohomology of X.*

Remark 10.2 For simple singularities with $n \geq 2$ the order of any form is greater than zero.

Proof is based on the following easy local homological computation ([We2], 2.1):

PROPOSITION 10.3 *If X has isolated singularities then a differential n-form on $X_{\rm reg}$ defines an element in intersection cohomology if and only if it vanishes in cohomology when restricted to the links of the singular points.*

Each cycle ζ_0 in the link can be extended to a family of cycles in the neighbouring fibers. We can approximate the value of the integral $\int_{\zeta_0} Res(\omega)$ by the oscillating integral of $\eta = s\omega$. If all the exponents in $I^\eta_\zeta(t)$ are greater than zero, then the limit integral for $t = 0$ vanishes. Therefore $[Res(\omega)] = 0$ in the cohomology of each link. $\square$

Remark 10.4 Proposition 10.1 is a special case of the Theorem 8.1, although formulation of 10.1 is in terms of oscillating integrals. By [AGV], §13.1 Th.1, the order of $s\omega$ is greater than zero if and only if ω has canonical singularities. Then $[\omega] \in W_{n+2}H^{n+1}(M{-}X)$ and 8.1 applies.

11 Quasihomogeneous isolated hypersurface singularities

More precise information about the exponents occurring in the oscillating integrals can be obtained for isolated quasihomogeneous singularities. All the simple singularities are of this form. The resulting statement for the residue forms is expressed in terms of weights. The weights of polynomials considered here should not be confused with the weights in the mixed Hodge theory. It is rather related to the Hodge filtration. The relation is subtle and it will not be discussed here. Let $a_0, a_1, \ldots, a_n \in \mathbb{N}$ be the weights attached to coordinates in which the function s is quasihomogeneous. For a meromorphic form of the top degree we compute the weight in the following way:

$$v\left(\frac{g}{s}dz_0 \wedge \ldots \wedge dz_n\right) = v(g) - v(s) + \sum_{i=0}^{n} v_i \,.$$

Theorem 11.1 *Suppose that X has isolated singularities given by quasihomogeneous equations in some coordinates. Let $\omega \in \Omega^{n+1}_M(X)$ be a meromorphic form with a first order pole along X. Suppose ω has no component of the weight 0 at each singular point. Then the residue class of ω lifts to intersection cohomology of X.*

Proof. To apply Proposition 10.3 we will show that $res(\omega)_{|L} = 0$. It suffices to check that ω is exact in a neighbourhood of the singular points. The calculation is local, we may assume that $M = \mathbb{C}^{n+1}$ and $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1}}(X)$ is rational. Suppose that ω is quasihomogeneous:

$$\omega = \frac{g}{s}dz_0 \wedge \ldots \wedge dz_n$$

with g quasihomogeneous of degree $v(g)$. Then g/s is quasihomogeneous of degree $v(g) - v(s)$. This means that

$$\sum_{i=0}^{n} a_i \frac{\partial(g/s)}{\partial z_i} z_i = (v(g) - v(s))\frac{g}{s} \,.$$

Let us define a form

$$\eta = \frac{g}{s}\sum_{i=0}^{n}(-1)^i a_i\, z_i\, dz_0 \wedge \dots \overset{i}{\vee} \dots \wedge dz_n\,.$$

Then

$$d\eta = \sum_{i=0}^{n} a_i \left(\frac{\partial(g/s)}{\partial z_i} z_i + \frac{g}{s}\right) dz_0 \wedge \dots \wedge dz_n = \left(v(g) - v(s) + \sum_{i=0}^{n} a_i\right) \frac{g}{s} dz_0 \wedge \dots \wedge dz_n\,.$$

Therefore if $v(\omega) = v(g) - v(s) + \sum_{i=0}^{n} a_i \neq 0$ then $\omega = \frac{1}{v(g)-v(s)+\sum_{i=0}^{n} a_i} d\eta$.
□

Remark 11.2 Conversely, if $\omega \neq 0$ is quasihomogeneous of degree 0 then the residue form restricted to the link L of the singular point is nonzero, $res(\omega)_{|L} \neq 0$. To see this consider the quotient

$$L/S^1 \subset \mathbb{P}(a_0, \dots a_n)$$

in the weighted projective space. Here L is the link of the singular point; it is homeomorphic to the intersection of X with the unit sphere. The circle acts on $\mathbb{C}^{n+1}$ diagonally with weights a_i. Integrating along the fibers of the quotient map one obtains a holomorphic form called by us *the second residue*

$$Res^{(2)}(\omega) = \int_{S^1} Res(\omega) \neq 0 \in \Omega^{n-1}_{L/S^1}\,.$$

Although L/S^1 is not smooth, it may have only quotient singularities and the Hodge theory applies. Therefore $[\int_{S^1} Res(\omega)] \neq 0 \in H^{n-1}(L/S^1)$. We will illustrate this construction by an example.

Remark 11.3 Fix a real number $p > 1$. If $v(\omega) > 0$ one can construct on X_{reg} a conelike metric adapted to the quasihomogeneous coordinates such that $|\omega|$ is integrable in the p-th power. By [We1] L^p-cohomology is isomorphic to intersection cohomology for a perversity $\underline{q}$ with $\frac{2n}{p} - 1 \leq \underline{q}(2n) < \frac{2n}{p}$. For large p it is isomorphic to cohomology of the normalization of X. This way (again) we obtain an explicit lift to cohomology.

Example 11.4 Elliptic singularity: Consider a singularity of the type P_8 (elliptic singularity) in a form

$$s(z_0, z_1, z_2) = z_1^3 + p z_0^2 z_1 + q z_0^3 - z_0 z_2^2$$

where p and q are real numbers such that the polynomial $z^3 + pz + q$ does not have double roots. Let

$$\omega = \frac{1}{s} dz_0 \wedge dz_1 \wedge dz_2\,.$$

Then

$$Res(\omega) = -\frac{1}{2z_0z_2}dz_0 \wedge dz_1$$

for $z_0z_2 \neq 0$. The second residue is equal to

$$Res^{(2)}(\omega) = \frac{dz_1}{z_2} = \frac{dz_1}{\sqrt{z_1^3 + pz_1 + q}}\,.$$

If we integrate $Res^{(2)}(\omega)$ along the real part of the elliptic curve $L/S^1 \subset \mathbb{P}^2$ we obtain the classical elliptic integral.

Remark 11.5 It would be enough to show that $Res^{(2)}(\omega)$ is nonzero as a form, since it is holomorphic it cannot vanish in cohomology. Counting the homogenity degree it is immediate to check that the second residue of the of the form $\frac{1}{s}dz_0 \wedge dz_1 \wedge dz_2$ is nontrivial for any homogeneous polynomial of the degree 3. The coefficients of s do not have to be real. Nevertheless we find it interesting to see exactly what kind of numbers can appear as values of the second residue.

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