Koszul Duality for modules over Lie algebras

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December 20, 2005

Math. Sub. Class.: Primary 17B55, 17B20; Secondary 55N91

Let $\mathbf{g}$ be a reductive Lie algebra over a field of characteristic zero. Suppose $\mathbf{g}$ acts on a complex of vector spaces $M^\bullet$ by i_λ and $\mathcal{L}_\lambda$, which satisfy the same identities that contraction and Lie derivative do for differential forms. Out of this data one defines the cohomology of the invariants and the equivariant cohomology of $M^\bullet$. We establish Koszul duality between each other.

1 Introduction

Let G be a compact Lie group. Set $\Lambda_\bullet = H_*(G)$ and $S^\bullet = H^*(BG)$. The coefficients are in $\mathbf{R}$ or $\mathbf{C}$. Suppose G acts on a reasonable space X. In the paper [GKM] Goresky, Kottwitz and MacPherson established a duality between the ordinary cohomology which is a module over $\Lambda_\bullet$ and equivariant cohomology which is a module over $S^\bullet$. This duality is on the level of chains, not on the level of cohomology. Koszul duality says that there is an equivalence of derived categories of $\Lambda_\bullet$–modules and $S^\bullet$–modules. One can lift the structure of an $S^\bullet$–module on $H^*_G(X)$ and the structure of a $\Lambda_\bullet$–module on $H^*(X)$ to the level of chains in such a way that the obtained complexes correspond to each other under Koszul duality. Equivariant coefficients in

*Supported by KBN 2P03A 00218 grant.

the sense of [BL] are also allowed. Later, Allday and Puppe ([AP]) gave an explanation for this duality based on the minimal Brown-Hirsh model of the Borel construction. One should remark that Koszul duality is a reflection of a more general duality: the one described by Husemoller, Moore and Stasheff in [HMS].

Our goal is to show that this duality phenomenon is a purely algebraic affair. We will construct it without appealing to topology. We consider a reductive Lie algebra $\mathbf{g}$ and a complex of vector spaces $M^\bullet$ on which $\mathbf{g}$ acts via two kinds of actions: i_λ and $\mathcal{L}_\lambda$. These actions satisfy the same identities as contraction and Lie derivative do in the case of the action on the differential forms of a G–manifold. Such differential $\mathbf{g}$–modules were already described by Cartan in [Ca]; see also [AM], [GS]. We do not assume that $M^\bullet$ is finite dimensional nor semisimple. We also wish to correct a small inaccuracy in the proof of Lemma 17.6, [GKM]. The distinguished transgression plays a crucial role in our construction. This is a canonical identification between the space of primitive elements of $(\Lambda^\bullet\mathbf{g}^*)^\mathbf{g} \simeq H^*(\mathbf{g})$ with certain generators of $(S\mathbf{g}^*)^\mathbf{g}$.

The results of the present paper were obtained with the help and assistance of Marcin Chałupnik. It is a part of joint work. Our aim is to describe Koszul duality in a much wider context. We thank the referee for careful corrections.

2 Category

Let $\mathbf{g}$ be a reductive Lie algebra over a field k of characteristic 0. We consider differential graded vector spaces $M^\bullet$ over k equipped with linear operations $i_\lambda : M^\bullet \to M^{\bullet-1}$ of degree -1 for each $\lambda \in \mathbf{g}$. We define

$$\mathcal{L}_\lambda = di_\lambda + i_\lambda d : M^\bullet \to M^\bullet .$$

We assume that i_λ is linear with respect to λ and for each $\lambda, \mu \in \mathbf{g}$ the following identities are satisfied:

$$i_\lambda i_\mu = -i_\mu i_\lambda,$$

$$[\mathcal{L}_\mu, i_\lambda] = i_{[\mu,\lambda]} .$$

Then $\mathcal{L}$ is a representation of $\mathbf{g}$ in $M^\bullet$. The category of such objects with obvious morphisms will be denoted by $K(\mathbf{g})$.

Example 2.1 Let G be a group with Lie algebra $\mathbf{g}$ and let X be a G-manifold. Then the space of differential forms $\Omega^\bullet(X)$ equipped with the contractions with fundamental vector fields is an example of an object from $K(\mathbf{g})$.

Example 2.2 Another example of an object of $K(\mathbf{g})$ is $\Lambda^\bullet\mathbf{g}^*$, the exterior power of the dual of $\mathbf{g}$. The generators of $\Lambda^\bullet\mathbf{g}^*$ are given the gradation 1. This is a differential graded algebra with a differential d_Λ induced by the Lie bracket. The operations i_λ are the contractions with $\mathbf{g}$.

Example 2.3 For a representation V of $\mathbf{g}$ define the invariant subspace

$$V^{\mathbf{g}} = \{v \in V :\ \forall\lambda \in \mathbf{g}\ \ \mathcal{L}_\lambda v = 0\}.$$

Then $V^{\mathbf{g}}$ with trivial differential and i_λ's is an object of $K(\mathbf{g})$. In particular we take $V = S^\bullet\mathbf{g}^*$, the symmetric power of the dual of $\mathbf{g}$. The generators of $S^\bullet\mathbf{g}^*$ are given the gradation 2.

Example 2.4 Suppose there are given two objects $M^\bullet$ and $N^\bullet$ of $K(\mathbf{g})$. Then $M^\bullet \otimes N^\bullet$ with operations i_λ defined by the Leibniz formula is again in $K(\mathbf{g})$.

Note that the objects of $K(\mathbf{g})$ are the same as differential graded modules over a dg–Lie algebra $C\mathbf{g}$ (the cone over $\mathbf{g}$) with

$$C\mathbf{g}^0 = C\mathbf{g}^{-1} = \mathbf{g} \qquad C\mathbf{g}^{\neq -1,0} = 0\,.$$

The elements of $C\mathbf{g}^0$ are denoted by $\mathcal{L}_\lambda$ and the elements of $C\mathbf{g}^{-1}$ are denoted by i_λ. They satisfy the following identities:

$$di_\lambda = \mathcal{L}_\lambda\,, \qquad [\mathcal{L}_\lambda, \mathcal{L}_\mu] = \mathcal{L}_{[\lambda,\mu]}\,, \qquad [\mathcal{L}_\lambda, i_\mu] = i_{[\lambda,\mu]}\,, \qquad [i_\lambda, i_\mu] = 0\,.$$

The enveloping dg–algebra of $C\mathbf{g}$ is the Chevaley–Eilenberg complex $V(\mathbf{g})$ ([Wei] p. 238) which is a free $U(\mathbf{g})$–resolution of k. Thus the objects of $K(\mathbf{g})$ are just the dg–modules over $V(\mathbf{g})$.

Example 2.5 Suppose that a Lie algebra $\mathbf{g}$ acts on a graded commutative k-algebra A by derivations. Let $\Omega_k A$ be the algebra of forms; it is generated by symbols a and da with $a \in A$. The cone of $\mathbf{g}$ acts by derivations. The action on generators is given by

$$i_\lambda a = 0, \quad \mathcal{L}_\lambda a = \lambda a, \quad i_\lambda da = \lambda a, \quad \mathcal{L}_\lambda da = d\lambda a\,.$$

Then $\Omega_k A$ is in $K(\mathbf{g})$.

Another point of view (as in [GS]) is that the objects of $K(\mathbf{g})$ are the representations of the super Lie algebra $\widehat{\mathbf{g}} = C\mathbf{g} \oplus k[-1]$ (where $k[-1]$ is generated by d) with relations:

$$[d,d]=0, \qquad [d,x]=dx \qquad \text{for } x\in C\mathbf{g}\,.$$

3 The twist

We will describe a transformation, which plays the role of the canonical map

$$\begin{array}{rccc}\Phi: & G\times X & \to & G\times X\\ & (g,x) & \mapsto & (g,gx)\end{array}$$

for topological G-spaces.

We define a linear map $\mathbf{i}:\Lambda^{\bullet}\mathbf{g}^*\otimes M^{\bullet}\to\Lambda^{\bullet}\mathbf{g}^*\otimes M^{\bullet}$ by the formula:

$$\mathbf{i}(\xi\otimes m)=\sum_k \xi\wedge\lambda^k\otimes i_{\lambda_k}m\,,$$

where $\{\lambda_k\}$ is a basis of $\mathbf{g}$ and $\{\lambda^k\}$ is the dual basis. It commutes with $\mathcal{L}_\lambda$. The operation $\mathbf{i}$ is nilpotent. Define an automorphism $\mathbf{T}:\Lambda^{\bullet}\mathbf{g}^*\otimes M^{\bullet}\to\Lambda^{\bullet}\mathbf{g}^*\otimes M^{\bullet}$ (which is not in $K(\mathbf{g})$):

$$\mathbf{T}=\exp(-\mathbf{i})=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\mathbf{i}^n\,,$$

$$\mathbf{T}(\xi\otimes m)=\sum_{I=\{i_1<\ldots<i_n\}}(-1)^{\frac{n(n+1)}{2}}\xi\wedge\lambda^I\otimes i_{\lambda_I}m\,.$$

It satisfies ([GHV], Prop. V, p.286, see also [AM])

$$i_\mu(\mathbf{T}(\xi\otimes m))=\mathbf{T}((i_\mu\xi)\otimes m)\,,$$

$$d\,\mathbf{T}(\xi\otimes m)=\mathbf{T}\left(d(\xi\otimes m)+\sum_k\lambda^k\wedge\xi\otimes\mathcal{L}_{\lambda_k}m\right)\,.$$

Remark 3.1 Note that for the self-map Φ of $G\times X$ we have

$$d\Phi^*\omega(1,x)=\left(d\omega+\sum_k p^*\lambda^k\wedge\mathcal{L}_{\lambda_k}\omega\right)(1,x)\,,$$

where $p:G\times X\to X$ is the projection and $x\in X$.

The twist on the level of the Weil algebra has already been used by Cartan [Ca] and later by Mathai and Quillen [MQ]. From another point of view, for a d.g.vector space to be an object of $K(\mathbf{g})$ is equivalent to having such a twist which satisfies certain axioms. We will not state them here. We just remark that understanding of this twist allows to develop a theory of actions of $\mathcal{L}_\infty$-algebras.

4 Weil algebra

Following [GHV], Chapter VI, p.223 we define the Weil algebra

$$W(\mathbf{g}) = \Lambda^\bullet (C\mathbf{g})^* = S^\bullet \mathbf{g}^* \otimes \Lambda^\bullet \mathbf{g}^* .$$

The generators of $S^\bullet \mathbf{g}^*$ are given the gradation 2 whereas the generators of $\Lambda^\bullet \mathbf{g}^*$ are given the gradation 1. The differential in $W(\mathbf{g})$ is the sum of three operations:

$$d_W(a \otimes b) = a \otimes d_\Lambda b + \sum_k \lambda^k a \otimes i_{\lambda_k} b + \sum_k ad^*_{\lambda_k} a \otimes \lambda^k \wedge b \, ,$$

where $\{\lambda_k\}$ is a basis of $\mathbf{g}$.

The differential d_W satisfies Maurer–Cartan formula

$$d_W(1 \otimes \xi) - 1 \otimes d_\Lambda \xi = \xi \otimes 1 \, ,$$

for $\xi \in \mathbf{g}^*$.

The operations i_λ are contractions with the second term. The resulting action $\mathcal{L}_\lambda = d_W i_\lambda + i_\lambda d_W$ is induced by the co-adjoint action on $S^\bullet \mathbf{g}^* \otimes \Lambda^\bullet \mathbf{g}^*$, [GHV], rel. (6.5), p.226. With this structure $W(\mathbf{g})$ becomes an object of $K(\mathbf{g})$.

The cohomology of $W(\mathbf{g})$ is trivial except in dimension 0, where it is k, [GHV], Prop. I, p.228. There are given canonical maps in $K(\mathbf{g})$:

- inclusion $(S^\bullet \mathbf{g}^*)^{\mathbf{g}} \simeq (S^\bullet \mathbf{g}^*)^{\mathbf{g}} \otimes 1 \subset W(\mathbf{g})$,
- restriction $W(\mathbf{g}) \twoheadrightarrow \Lambda^\bullet \mathbf{g}^*$, which sends all the positive symmetric powers to 0.

The Weil algebra is a model of differential forms on EG and the sequence of morphisms in $K(\mathbf{g})$

$$(S^\bullet \mathbf{g}^*)^{\mathbf{g}} \hookrightarrow W(g) \twoheadrightarrow \Lambda^\bullet \mathbf{g}^*$$

is a model of

$$\Omega^\bullet(BG) \hookrightarrow \Omega^\bullet(EG) \twoheadrightarrow \Omega^\bullet(G)\,.$$

Remark 4.1 It is easy to see that $W(\mathbf{g}) = \Omega_k \Lambda^\bullet \mathbf{g}^*$. Thus for any commutative d.g-algebra A

$$\mathrm{Hom}_{g-comm}(\Lambda^\bullet \mathbf{g}^*, A) = \mathrm{Hom}_{d.g-comm}(W(\mathbf{g}), A)\,.$$

5 The distinguished transgression

The invariant algebra $(\Lambda^\bullet \mathbf{g}^*)^{\mathbf{g}}$ is the exterior algebra spanned by the space of primitive elements $P^\bullet$, whereas $(S^\bullet \mathbf{g}^*)^{\mathbf{g}}$ is the symmetric algebra spanned by some space $\widetilde{P}^\bullet$. The point is that $\widetilde{P}^\bullet$ can be canonically chosen.

Proposition 5.1 [GHV] Prop. VI, p.239. *Suppose $\xi \in P^\bullet$ is a primitive element. Then there exist an element $\omega \in W(\mathbf{g})^{\mathbf{g}}$ such that*

$$\omega_{|\Lambda^\bullet \mathbf{g}^*} = \xi\,,$$

$$i_\lambda \omega = i_\lambda(1 \otimes \xi) \quad \mathit{for\ all}\ \lambda \in (\Lambda^\bullet \mathbf{g})^{\mathbf{g}}$$

$$d_W(\omega) = \widetilde{\xi} \otimes 1\,.$$

The element ω is not unique, but $\widetilde{\xi}$ is. The set of $\widetilde{\xi}$ for $\xi \in P^*$ is the distinguished space of generators of $(S^\bullet \mathbf{g}^*)^{\mathbf{g}}$.

6 Example – $\mathbf{su}_2$

The algebra $\mathbf{su}_2$ is spanned by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with relation $[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}$ and its cyclic transposition. In $\Lambda^\bullet \mathbf{g}^*$ we have

$$d_\Lambda \mathbf{i}^* = 2\mathbf{j}^* \wedge \mathbf{k}^* \quad \text{and cycl.}$$

In the Weil algebra we have

$$d_W(1 \otimes \mathbf{i}^*) = 1 \otimes 2\mathbf{j}^* \wedge \mathbf{k}^* + \mathbf{i}^* \otimes 1 \quad \text{and cycl.}\,,$$

$$d_W(\mathbf{i}^* \otimes 1) = 2(\mathbf{k}^* \otimes \mathbf{j}^* - \mathbf{j}^* \otimes \mathbf{k}^*) \quad \text{and cycl.}$$

The primitive elements in $\Lambda^{\bullet}\mathbf{g}^*$ are spanned by $\xi = \mathbf{i}^* \wedge \mathbf{j}^* \wedge \mathbf{k}^*$. As ω of Proposition 5.1 we take

$$\omega = 1 \otimes \mathbf{i}^* \wedge \mathbf{j}^* \wedge \mathbf{k}^* + \frac{1}{2}(\mathbf{i}^* \otimes \mathbf{i}^* + cycl.)$$

Then

$$\widetilde{\xi} \otimes 1 = d_W(\omega) = \frac{1}{2}(\mathbf{i}^{*2} + \mathbf{j}^{*2} + \mathbf{k}^{*2}) \otimes 1\,,$$

whereas

$$d_W(1 \otimes \xi) = \mathbf{i}^* \otimes \mathbf{j}^* \wedge \mathbf{k}^* + cycl.$$

7 Invariant cohomology and equivariant cohomology

Denote $(\Lambda^{\bullet}\mathbf{g})^{\mathbf{g}}$ by $\Lambda_{\bullet}$. Let $D(\Lambda_{\bullet})$ be the derived category of graded differential $\Lambda_{\bullet}$–modules. For $M^{\bullet} \in K(\mathbf{g})$ the invariant submodule $(M^{\bullet})^{\mathbf{g}}$ is a differential module over $\Lambda_{\bullet}$. We obtain an object in $D(\Lambda_{\bullet})$. We call it the invariant cohomology of $M^{\bullet}$.

Remark 7.1 Let X be a manifold on which a compact group G acts. Let $\mathbf{g}$ be the Lie algebra of G. Then $\Omega^{\bullet}(X)$ is a $\mathbf{g}$-module. The invariants of $\Lambda^{\bullet}\mathbf{g}$ act on $\Omega^{\bullet}(X)$, but this action does not commute with the differential in general. We have $[d, i_\lambda]\omega = \mathcal{L}_\lambda\omega$. To obtain an action which commutes with the differential one restricts it to $(\Omega^{\bullet}(X))^{\mathbf{g}}$. Fortunately the resulting cohomology does not change. We obtain a complex with an action of $\Lambda_{\bullet} = H_*(G)$, which is quasi-isomorphic to $\Omega^{\bullet}(X)$. The cohomology is equal to $H^*(X)$.

Denote $(S^{\bullet}\mathbf{g}^*)^{\mathbf{g}}$ by $S^{\bullet}$. Let $D(S^{\bullet})$ be the derived category of graded differential $S^{\bullet}$–modules. Following [Ca] we define:

$$(M^{\bullet})_{\mathbf{g}} := (S^{\bullet}\mathbf{g}^* \otimes M^{\bullet})^{\mathbf{g}}$$

with differential

$$d_{M,\mathbf{g}}(a \otimes m) = a \otimes d_M m - \sum_k \lambda^k a \otimes i_{\lambda_k} m\,.$$

It is a differential $S^\bullet$–module. We obtain an object $(M^\bullet)_{\mathbf{g}}$ in $D(S^\bullet)$. We call it the equivariant cohomology of $M^\bullet$.

For an object $N^\bullet$ of $K(\mathbf{g})$ we define horizontal elements

$$(N^\bullet)_{hor} = \{n \in N^\bullet \,:\, \forall\lambda \in \mathbf{g}\;\; i_\lambda n = 0\}\,.$$

Then define basic elements

$$(N^\bullet)_{basic} = (N^\bullet)^{\mathbf{g}}_{hor} = \{n \in N^\bullet \,:\, \forall\lambda \in \mathbf{g}\;\; i_\lambda n = 0,\, i_\lambda dn = 0\}\,,$$

which form a complex.

The following Lemma can be found in [Ca], but we need to have an explicit form of the isomorphism, as in [AM] §4.1.

Lemma 7.2 [Ca] *The map ψ_0 is an isomorphism of differential graded $S^\bullet$–modules:*

$$\psi_0 = 1 \otimes \mathbf{T}_{|1\otimes M^\bullet} : (M^\bullet)_{\mathbf{g}} \to (W(\mathbf{g}) \otimes M^\bullet)_{basic}\,.$$

Proof. The elements of $(\Lambda^\bullet \mathbf{g}^* \otimes M^\bullet)_{hor}$ are of the form

$$\mathbf{T}(1 \otimes m) = 1 \otimes m - \sum_k \lambda^k \otimes i_{\lambda_k} m - \sum_{k<l} \lambda^k \wedge \lambda^l \otimes i_{\lambda_k} i_{\lambda_l} m \pm \dots\,,$$

thus they are determined by m. The conclusion follows since

$$(W(\mathbf{g}) \otimes M^\bullet)_{hor} = S^\bullet \mathbf{g}^* \otimes (\Lambda^\bullet \mathbf{g}^* \otimes M^\bullet)_{hor}\,.$$

$\square$

Remark 7.3 From the above description we see that $(M^\bullet)_{\mathbf{g}}$ is an analog of $\Omega^\bullet(EG \times_G X)$. Let X and $\mathbf{g}$ be as in 7.1. The construction of the equivariant cohomology presented here is the so-called Cartan model of $\Omega^\bullet(EG \times_G X)$. We obtain a complex with an action of $S^\bullet = H^*(BG)$. The cohomology is equal to $H^*_G(X)$.

8 Koszul duality

By [GKM], §8.5 the following functor $h : D^+(S^\bullet) \to D^+(\Lambda_\bullet)$ is an equivalence of categories:

$$h(A^\bullet) = \mathrm{Hom}_k(\Lambda_\bullet, A^\bullet) = (\Lambda^\bullet \mathbf{g}^*)^{\mathbf{g}} \otimes A^\bullet$$

with differential

$$d_h((\xi_1 \wedge \ldots \wedge \xi_n) \otimes a) =$$

$$= \sum_j (-1)^{j+1} (\xi_1 \wedge \ldots \vee^j \ldots \wedge \xi_n) \otimes \widetilde{\xi}_j a + (-1)^n (\xi_1 \wedge \ldots \wedge \xi_n) \otimes da\,,$$

where ξ_j's are primitive.

THEOREM 8.1 (KOSZUL DUALITY) *Let $\mathbf{g}$ be a reductive Lie algebra. Suppose $M^\bullet$ is an object of $K^+(\mathbf{g})$ then in $D^+(\Lambda_\bullet)$*

$$h((M^\bullet)_{\mathbf{g}}) \simeq (M^\bullet)^{\mathbf{g}}\,.$$

PROOF. The action of $\mathbf{g}$ on $H^*(W(\mathbf{g}))$ is trivial and, since $\mathbf{g}$ is reductive, $W(\mathbf{g})$ is semisimple. Thus by [GHV], Th. V, p.172 the inclusion

$$k \otimes (M^\bullet)^{\mathbf{g}} = W(\mathbf{g})^{\mathbf{g}} \otimes (M^\bullet)^{\mathbf{g}} \subset (W(\mathbf{g}) \otimes M^\bullet)^{\mathbf{g}}$$

is a quasi-isomorphism. We want to construct a quasi-isomorphism ψ from

$$h((M^\bullet)_{\mathbf{g}}) = (\Lambda^\bullet \mathbf{g}^*)^{\mathbf{g}} \otimes (S^\bullet \mathbf{g}^* \otimes M^\bullet)^{\mathbf{g}}$$

to

$$(W(\mathbf{g}) \otimes M^\bullet)^{\mathbf{g}} = (S^\bullet \mathbf{g}^* \otimes \Lambda^\bullet \mathbf{g}^* \otimes M^\bullet)^{\mathbf{g}}\,.$$

First we choose a linear map $\omega : P^* \to W(\mathbf{g})$ satisfying the conditions of Proposition 5.1. We construct ψ by the formula extending ψ_0 of Lemma 7.2 with help of the distinguished transgression of §5.

$$\psi((\xi_1 \wedge \ldots \wedge \xi_n) \otimes m) = \omega(\xi_1) \ldots \omega(\xi_n) \psi_0(m)\,.$$

It is well defined since $(W(\mathbf{g}) \otimes M^\bullet)^{\mathbf{g}}$ is $W(\mathbf{g})^{\mathbf{g}}$–module and

$$\psi_0(m) \in (W(\mathbf{g}) \otimes M^\bullet)_{basic}$$

$$\omega(\xi_1) \in W(\mathbf{g})^{\mathbf{g}}\,.$$

The map ψ commutes with i_λ since the image of ψ_0 is horizontal. It commutes with the differential because $d_W(\omega(\xi)) = \widetilde{\xi} \otimes 1$. This corrects an error in the proof of Lemma 17.6, [GKM], where ψ does not commute with the differential unless $\mathbf{g}$ is abelian.

We will check that ψ is a quasi-isomorphism. Let's filter both sides by $S^{\geq i}(\mathbf{g}^*)$. Then the corresponding quotient complexes are

$$Gr_i^S \psi : (\Lambda^{\bullet}\mathbf{g}^*)^{\mathbf{g}} \otimes (S^i\mathbf{g}^* \otimes M^\bullet)^{\mathbf{g}} \longrightarrow (S^i\mathbf{g}^* \otimes \Lambda^{\bullet}\mathbf{g}^* \otimes M^\bullet)^{\mathbf{g}} .$$

The differential on the LHS is just $\epsilon \otimes 1 \otimes d_M$ (where $\epsilon = (-1)^{\deg \xi}$) and the differential on the RHS is

$$1 \otimes d_\Lambda \otimes 1 + \sum_k ad^*_{\lambda_k} \otimes \lambda^k \wedge \cdot \otimes 1 + 1 \otimes \epsilon \otimes d_M .$$

The map $Gr_i^S \psi$ equals $1 \cdot (1 \otimes \mathbf{T}_{|1\otimes M^\bullet})$. When we untwist it (i.e. we apply $\exp(\mathbf{i}) = \mathbf{T}^{-1}$ to the RHS) the differential takes the form

$$1 \otimes d_\Lambda \otimes 1 + \sum_k ad^*_{\lambda_k} \otimes \lambda^k \wedge \cdot \otimes 1 + 1 \otimes \epsilon \otimes d_M + \sum_k 1 \otimes \lambda^k \wedge \cdot \otimes \mathcal{L}_{\lambda_k} .$$

Since we stay in the invariant subcomplex the differential equals

$$1 \otimes d_\Lambda \otimes 1 - \sum_k 1 \otimes \lambda^k \wedge ad^*_{\lambda_k} \otimes 1 + 1 \otimes \epsilon \otimes d_M .$$

Moreover $\sum_k 1 \otimes \lambda^k \wedge ad^*_{\lambda_k} = 2d_\Lambda$, thus the differential on the RHS is

$$-1 \otimes d_\Lambda \otimes 1 + 1 \otimes \epsilon \otimes d_M .$$

The cohomology of the LHS is

$$H^*\left((\Lambda^{\bullet}\mathbf{g}^*)^{\mathbf{g}} \otimes \left(S^i\mathbf{g}^* \otimes M^\bullet\right)^{\mathbf{g}}\right) = (\Lambda^{\bullet}\mathbf{g}^*)^{\mathbf{g}} \otimes H^*\left(\left(S^i\mathbf{g}^* \otimes M^\bullet\right)^{\mathbf{g}}\right)$$

and the cohomology of the RHS is

$$H^*\left(\left(S^i\mathbf{g}^* \otimes \Lambda^{\bullet}\mathbf{g}^* \otimes M^\bullet\right)^{\mathbf{g}}\right) = H^*\left((\Lambda^{\bullet}\mathbf{g}^*)^{\mathbf{g}}\right) \otimes H^*\left(\left(S^i\mathbf{g}^* \otimes M^\bullet\right)^{\mathbf{g}}\right)$$

again by [GHV], Th. V, p.172, since the action on $H^*(\Lambda^{\bullet}\mathbf{g}^*)$ is trivial and $\Lambda^{\bullet}\mathbf{g}^*$ is semisimple. Thus cohomology of the graded complexes are the same. The conclusion of 8.1 follows. □

Remark 8.2 Let X and $\mathbf{g}$ be as in 7.1. Following [GKM] let us explain the meaning of Koszul duality. The invariant and the equivariant cohomology of X are defined on the level of derived categories. Theorem 8.1 gives a procedure to reconstruct the invariant cohomology from the equivariant cohomology. Since this duality is an isomorphism of categories the invariant cohomology determines the equivariant cohomology as well. The corresponding statement on the level of graded modules over $\Lambda_\bullet$ and $S^\bullet$ is not true as an easy example in [GKM] shows. One cannot recover $H^*_G(X)$ from $H^*(X)$ with an action of $H_*(G)$ even in the case $X = S^3$, $G = S^1$.

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Correction: The paper contains a gap. The action of $H^*(G)$ in the proof of Theorem 8.1 on the page 13 is not correct. The correct action of $H_*(G)$ has been constructed in [A. Alekseev, E. Meinrenken, *Equivariant cohomology and the Maurer-Cartan equation* Preprint math.DG/0406350].