FORMALITY OF EQUIVARIANT INTERSECTION COHOMOLOGY OF ALGEBRAIC VARIETIES

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Abstract. We present a proof that the equivariant intersection cohomology of any complete algebraic variety acted by a connected algebraic group G is a free module over $H^*(BG)$.

Introduction

Let X be a complex algebraic variety and G be a linear algebraic group acting algebraically on X. Denote by $IH^*_G(X) = H^*(EG \times_G X; \mathcal{IC}^\bullet_G)$ the equivariant intersection cohomology of X with rational coefficients, first introduced in [Br]. It is a module over $H^*(BG)$. We say that X is formal if $IH^*_G(X)$ is free. We have a spectral sequence

$$(*) \qquad E_2^{pq} = H^p(BG, IH^q(X)) \Rightarrow IH^{p+q}_G(X)\,.$$

Suppose that G is connected. Then BG is 1-connected and the coefficients are not twisted. Then the spectral sequence (*) degenerates if and only if X is formal.

The aim of this note is to show that complete G-varieties are formal. We will prove:

Theorem 1. *Let X be a complete algebraic variety acted by a connected algebraic linear group. Then X is formal.*

In this way we add to the list of [GKM, 14.1] an important class of examples. In fact Theorem 1 is known for specialists. Each of the proofs comes back to the fundamental paper of Beilinson-Bernstein-Deligne-Gabber, [BBD]. After all, in the most general case, Theorem 1 follows from a simple weight argument, as presented in Remark 7. Nevertheless, we think it is worth to compare several approaches to equivariant intersection cohomology and ways of possible proofs.

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1. Proof for projective varieties

Our proof of Theorem 1 uses the decomposition theorem or reduction to a finite characteristic, which are strong tools. Thus we think that it is worth to note that for projective X the situation is much simpler. We proceed as follows:

Step 1: We choose an equivariant embedding of X into a projective space. This is possible by a result of Sumihiro [Sum, Theorem 1].

Step 2: Let $h \in H^2(X)$ be the class of the hyperplane section. We show that h can be lifted to $H^2_G(X)$. This is so for $X = \mathbb{P}^N$ and $G = GL(N+1)$ acting naturally. Indeed, take a infinite Grassmannian as a model of BG, then

$$EG \times_G \mathbb{P}^N = \{(V,l) \, : \, l \subset V \subset \mathbb{C}^\infty \, , \, \dim l = 1 \, , \, \dim V = N+1 \} \, .$$

One has a tautological bundle, its Chern class is a lift of $-h$. If $X \subset \mathbb{P}^N$ and G acts linearly, we construct a lift for $\mathbb{P}^N$ by naturality with respect to G and we restrict it to X.

Step 3: We have a map of spectral sequences

$$L : E_r^{p,q} \to E_r^{p,q+2} \, ,$$

which is induced by the multiplication by a cycle (or a form) representing h. Let $d = \dim X$. By the hard Lefschetz theorem

$$L^k : E_2^{p,d-k} = H^p(BG) \otimes IH^{d-k}(X) \to E_2^{p,d+k} = H^p(BG) \otimes IH^{d+k}(X)$$

is an isomorphism for every p and k. From the Lefschetz criterion, [D1], it follows that the spectral sequence degenerates, as noticed [Br], 4.2.2.

This argument works for varieties over any field of any characteristic. One should replace the ordinary cohomology and intersection cohomology by their étale version. In the Step 3 an isomorphism $IH^*_G(X) \simeq H^*(BG) \otimes IH^*(X)$ can be made canonical by [D3]. It would depend only on the embedding $X \subset \mathbb{P}^N$ and the representation $G \to GL(k)$.

For smooth X the Step 2 can also be carried out by methods of symplectic geometry. Let K be a maximal compact subgroup of G. Then $H^*_G(\mathbb{P}^N) \simeq H^*_K(\mathbb{P}^N)$. The class h is represented by a K-equivariant symplectic form ω. One can define a moment map by an explicit formula [K, 2.5]. The choice of the moment map is precisely a lift of ω to $H^2_K(\mathbb{P}^N)$ in the Cartan model of $\Omega^*(EK \times_K \mathbb{P}^N)$, see [AB, 6.18]. Another proof for smooth projective varieties, can be obtained by means of a Morse function coming from a moment map, see [K, 5.8].

In the rest of the paper we will show how to deduce formality for an arbitrary complete variety, not necessarily projective.

2. Algebraic models of $EG \times_G X$

There are several constructions of BG that can be performed in the category of algebraic varieties. Of course BG itself cannot be an algebraic variety, since it cannot be of a finite dimension. There are two well known approaches of giving a meaning to BG: an approximation or the simplicial construction.

The first approach is to approximate BG by algebraic varieties B_nG. For a detailed exposition see [T]. If $G \subset GL(k)$, then for example we take as E_nG the Stiefel variety of k independent vectors in $\mathbb{C}^{n+k}$. The quotient $B_nG = E_nG/G$ exists and it is an algebraic variety. It can be presented as the quotient of groups $GL(n+k)/H$, where $H = \begin{pmatrix} G & * \\ 0 & GL(n) \end{pmatrix}$. Another candidate for an approximation is the quotient $B'_nG = GL(n+k)/H'$, where $H' = \begin{pmatrix} G & 0 \\ 0 & GL(n) \end{pmatrix} \simeq G \times GL(n)$. In this case $E'_nG = GL(n+k)/H'_0$, where $H'_0 = \begin{pmatrix} I & 0 \\ 0 & GL(n) \end{pmatrix}$.

Now we construct the associated bundle. Using the approximation B'_nG we have $E'_nG \times_G X = GL(n+k) \times_{H'} X$ (where $GL(n)$ acts trivially on X). But even here, with such a simple construction the resulting space does not have to be a variety. The spaces $E'_nG \times_G X$ (approximations of the Borel construction) are quotients of $E'_nG \times X$ by G and it can be shown, that they are algebraic spaces. Let us quote [B-B]:

Theorem 2. *Let G be an algebraic group. Let E and X be algebraic spaces and let E be a principal locally isotrivial G-fibration. Assume that E is normal. Then $E \times_G X$ exists in the category of algebraic spaces. If, moreover E is an algebraic variety, X normal and can be covered by G-invariant open quasi-projective subsets, then $E \times_G X$ is an algebraic variety.*

Note that, by [B-B, Theorem 3], if H is a connected affine algebraic group, $G \subset H$ and X normal, then $H \times_G X$ is an algebraic variety if and only if the assumption about covers is fulfilled. For the proof of formality one would like to apply the decomposition theorem (or other methods based on Weil conjectures). It is clear that either we have to make some assumptions about X, or to generalize the results of [BBD] to the case algebraic spaces.

In order to avoid algebraic spaces it is possible to apply a simplicial construction of BG, see [D2, 6.1]. Then $(BG)_\bullet$ is a simplicial variety: $(BG)_n = G^n$. The associated Borel construction is also a simplicial variety: $(EG \times_G X)_n = G^n \times X$. The topological realization of the Borel construction coincides with the realization of the quotient stack X/G as defined in [Si, §8].

Remark 3. Applying any of the described construction we obtain a Hodge structure on $H^*(BG)$ which is pure, see [D2, 9.1]. If we perform the same constructions for groups over a finite field, we also obtain a pure structure with respect to the action of the Frobenius authomorphism.

Remark 4. Suppose the base field is $\mathbb{F}_q$. Then the action of the Frobenius authomorphism $Fr^q : x \mapsto x^q$ induces an operation on $H^*_{et}(BG; \mathbb{Q}_\ell)$, which is just the Adams operation ψ^q; compare [AM; Sul, Ch.5 §4].

3. Degeneration of the spectral sequence without the assumption of projectivity

Suppose X is a complete algebraic variety. We replace X by its normalization. It does not change the intersection cohomology. Assume that X can be covered by G-invariant open quasi-projective subsets, then the space $E_nG \times_G X$ (or $E'_nG \times_G X$)

is an algebraic variety by Theorem 2. The projection

$$E_nG \times_G X \to B_nG$$

is a proper algebraic morphism.

Corollary 5. *If the normalization of X can be covered by G-invariant quasiprojective open subsets, then the spectral sequence (*) converging to $IH^*(E_nG \times_G X)$ degenerates at the E_2 term.*

Proof. The Corollary follows from the decomposition theorem of [BBD]. □

Remark 6. We would be able to remove the assumption about covers, provided that we proved the decomposition theorem for separable algebraic spaces. The statement of [BBD, 6.2.5] seems to remain valid. To prove it one would have to develop the formalism of mixed sheaves for algebraic spaces over finite fields.

4. The end of the proof

If X can be covered by G-invariant quasiprojective open subsets, then $EG \times_G X$ can be approximated by algebraic varieties. The formality follows from Corollary 5. For arbitrary complete X there is a way of going around the problem with $EG \times_G X$ not being a variety. First we consider the maximal torus $T \subset G$. The classifying space for the torus can be realized as a product of infinite projective spaces. Then the approximation of $ET \to BT$ is locally Zariski trivial. Thus $ET \times_T X$ is a limit of algebraic varieties. We can apply directly the techniques of [BBD] to deduce that the spectral sequence (*) degenerates for T. The restriction to the maximal torus induces an injection $H^*(BG) \hookrightarrow H^*(BT)$ and an injection of the E_2 terms of the spectral sequences (*). Therefore, since the differentials vanish in the E_2 term for T, they vanish in the E_2 term for G. Then $E_2 = E_3$ and we continue our reasoning. We find that all the differentials in the spectral sequence for G vanish. The reduction to the maximal torus was described in [Gi, §4, prof of 3.3].

Remark 7. There is another solution of the problem with Borel construction. We apply the simplicial model of EG. We reduce the base ring to a finite field. Now we follow Deligne's suggestion: the Galois group acts on the spectral sequence (*). Since $H^*(BG)$ and $IH^*(X)$ are pure, E_2^{pq} is pure of weight $p+q$. All the differentials in the spectral sequence vanish because they mix weights. Note that this proof works for noncomplete algebraic varieties, provided that $IH^*(X)$ is pure. One can generalize our proof further and consider cohomology with coefficients in an arbitrary 'sheaf' $\mathcal{F}$, an object of the derived category. In this case a suitable extension of $\mathcal{F}$ on the simplicial variety $(EG \times_G X)_\bullet$ has to be chosen to make sense of the equivariant cohomology. The extension $\mathcal{F}_G$ should be an equivariant sheaf in the sense of [BL] and it should be defined over a finitely generated subring $R \subset \mathbb{C}$. Moreover $\mathcal{F}_G$ should have enough good reductions to finite fields, so that $H^*_{G_\mathbb{C}}(X_R \otimes \mathbb{C}; \mathcal{F}_{G_\mathbb{C}}) \simeq H^*_{G_{\overline{\mathbb{F}}_q}}(X_R \otimes \overline{\mathbb{F}}_q; \mathcal{F}_{G_{\overline{\mathbb{F}}_q}})$ for some residue field $\mathbb{F}_q = R/\mathfrak{m}$. The conclusion is the same: purity of $H^*(X;\mathcal{F})$ implies formality.

5. Algebra structure

Theorem 1 shows that the $H^*(BG)$-module structure of $H^*_G(X)$ caries no information about the action. The algebra structure of $H^*_G(X)$ is more interesting, but hard to recover. It involves some characteristic classes of the action.

Example 8. Let $X = \mathbb{P}^N$ and let $G = GL(N+1)$ with the obvious action on X (as in the Step 2 of §1). The effect of the Borel construction is the projectivization of the tautological bundle over BG. The cohomology algebra is well known:

$$H^*_G(X) = \mathbb{Q}[c_1, c_2, \ldots, c_{N+1}, h]/(c_{N+1} - c_N h + \cdots + c_1(-h)^N + (-h)^{N+1}).$$

6. Dual structure

Due to Koszul duality [GKM; MW] the nonequivariant cohomology of X is equipped with the structure of a module over $H_*(G)$: for $\lambda \in H_k(G)$ and $x \in H^l(X)$ we set $\lambda \cdot x = \lambda \setminus \mu^*(x) \in H^{l-k}(X)$, where $\mu : G \times X \to X$ is the action and $\setminus$ is the slant product $\setminus : H_k(G) \otimes H^l(G \times X) \to H^{l-k}(X)$. According to [GKM, 9.3] if the spectral sequence (*) degenerates, then the cohomology of X is trivial module over $H_*(G)$, even on the level of the derived category of $H_*(G)$-modules. Therefore for any λ with $\deg \lambda > 0$ we have $\lambda \cdot x = 0$, i.e.:

Corollary 9. *The structure of $H_{>0}(G)$-module on $IH^*(X)$ is trivial for a complete variety X acted by a connected group G.*

This generalizes a result of Deligne [D2, 9.1.7].

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