Weights in cohomology and the Eilenberg–Moore spectral sequence

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Abstract

We show that in the category of complex algebraic varieties, the Eilenberg–Moore spectral sequence can be endowed with a weight filtration. This implies that it degenerates if all spaces involved have pure cohomology. As application, we compute the rational cohomology of an algebraic G-variety X (G being a connected algebraic group) in terms of its equivariant cohomology provided that $H^*_G(X)$ is pure. This is the case, for example, if X is smooth and has only finitely many orbits. We work in the category of mixed sheaves; therefore our results apply equally to (equivariant) intersection homology.

Key words: Eilenberg–Moore spectral sequence, weight filtration, equivariant cohomology, intersection cohomology, complex algebraic G–varieties.

1 Introduction

The following spectral sequence was constructed by Eilenberg and Moore in [13]:

Theorem 1.1 *Suppose we have a pull-back square of topological spaces*

$$\begin{array}{ccccc} A\times_B C & = & D & \to & A \\ & & \downarrow & & \downarrow \\ & & C & \to & B \end{array}$$

with B simply connected and $A \to B$ a fibration. Then there exists a spectral sequence converging to $H^*(D)$ with

$$E_2^{-p,q} = \mathrm{Tor}_p^{q,H^*(B)}(H^*(A), H^*(C)).$$

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Cohomology is taken with rational coefficients. The torsion product is a functor of homological type and therefore it is denoted by $\mathrm{Tor}_p^{H^*(B)}(-,-)$. It is positively graded. Additionally it has an internal grading. Its degree–q piece is denoted by $\mathrm{Tor}_p^{q,H^*(B)}(-,-)$. According to L. Smith [25, 26], the entries of the spectral sequence are the cohomology groups of certain spaces and the differentials are induced by maps between them. From this Smith deduced that the sequence inherits all the structure of cohomology, such as an action of the Steenrod algebra. We are interested in the weight filtration of the rational cohomology of algebraic varieties. Since we want to prove some results for singular spaces, we deal with intersection cohomology. We show that Smith's construction can be carried out in the category of algebraic varieties, or rather in the derived category of sheaves on algebraic varieties. Therefore the Eilenberg–Moore spectral sequence can be endowed with a weight filtration which extends the filtration constructed by Deligne [10]. The Eilenberg-Moore spectral sequence can equally be obtained from a filtration of the bar complex $B(\Omega^*(A),\Omega^*(B),\Omega^*(C))$, which is quasi-isomorphic to $\Omega^*(A\times_B C)$ (where $\Omega^*(-)$ is a complex of differential forms on a hyperresolution computing $H^*(-)$, see [10]). According to Deligne, these complexes live in the category of mixed Hodge complexes. By [17, §3] and [18, §3], the bar complex inherits the mixed Hodge structure, hence also the entries of the Eilenberg–Moore spectral sequence. We only study the weight filtration, but we are interested in intersection cohomology as well. We prove our results in the category of mixed sheaves of Saito ([22, 23]) or Beilinson-Bernstein-Deligne ([2]). We formulate our first theorem in the following way:

Theorem 1.2 *Let B be a simply connected complex algebraic variety. Let F and G be bounded below mixed sheaves over B. Suppose that F has constant cohomology sheaves. Then there is a spectral sequence with*

$$E_2^{-p,q}=\mathrm{Tor}_p^{q,H^*(B)}\big(H^*(B;F),H^*(B;G)\big)$$

and converging to $H^*(B;F\otimes G)$. The entries of the spectral sequence are endowed with weight filtrations. The differentials preserve them.

Here $H^*(B)$ denotes the cohomology of B with coefficients in the constant sheaf $\mathbb{Q}$. The choice of the category of mixed sheaves is motivated by the following: In contrast to the topological situation, we would have to deal with simplicial varieties instead of just varieties. That is because in the construction of Smith's resolution on the geometric level a quotient of varieties appears. Such a quotient is no longer an algebraic variety. In the world of simplicial varieties we can replace the quotient by a cone construction, and every step of Smith's original proof can be imitated. But we are concerned with intersection homology and we would therefore have to introduce a simplicial intersection sheaf. We find it rather unnecessary to develop a theory of intersection cohomology for simplicial varieties. On the other hand it is much easier and more general to prove the result for sheaves. If one looks carefully, one sees that Smith's construction is carried out in the stable category of topological spaces over B (at the end of his argument he desuspends the spectral sequence). The rational stable category of topological spaces over B is nothing but the derived category of sheaves over B. The sheaves coming from algebraic geometry carry an additional structure: the weight filtration. In a purely formal way we deduce results from the fact that all maps strictly preserve the filtrations. In many situations the higher differentials of the Eilenberg-Moore spectral sequences vanish because otherwise they would mix weights. Hence, $E_2 = E_\infty$ in these cases. We note that if a variety is smooth, one could work with Hodge complexes as in [10] instead of mixed sheaves.

Using the same notation as before, our main result reads as follows:

THEOREM 1.3 *If $H^*(B)$, $H^*(B;F)$ and $H^*(B;G)$ are pure (i.e., the degree-n cohomology group is entirely of weight n), then $E_r^{-p,*}$ is pure for all r and p, and the Eilenberg–Moore spectral sequence degenerates on the E_2 level. The resulting filtration is the weight filtration*

$$W_\nu H^n(F \otimes G) = \bigoplus_{q-p=n,\ q\leq\nu} \mathrm{Tor}_p^{q,H^*(B)}(H^*(B;F), H^*(B;G))\,.$$

Suppose we have a pull-back square

1.4
$$\begin{array}{ccccc} A\times_B C & = & D & \xrightarrow{\tilde g} & A \\ & & \tilde f\big\downarrow & & \big\downarrow f \\ & & C & \xrightarrow[g]{} & B \end{array}$$

of algebraic varieties, where B is smooth and simply connected and the map $A \to B$ is a fibration. (It is enough to assume that the map f is a topological locally trivial fibration.) When we apply Theorem 1.3 to the push forwards $F = Rf_*IC_A$ and $G = Rg_*IC_C$ of the intersection sheaves, we obtain:

THEOREM 1.5 *If $H^*(B)$, $IH^*(A)$ and $IH^*(C)$ are pure, then*

$$IH^n(D) = \bigoplus_{q-p=n} \mathrm{Tor}_p^{q,H^*(B)}(IH^*(A), IH^*(C))\,.$$

The sum of terms with $q \leq \nu$ coincides with $W_\nu IH^*(D)$.

Let G be a linear algebraic group. We assume that G is connected. Our goal is to study the weight structure in the cohomology and equivariant cohomology of algebraic G-varieties. If the equivariant cohomology is pure, it determines the non-equivariant cohomology additively:

THEOREM 1.6 *If the rational equivariant cohomology $H^*_G(X) = H^*(EG\times_G X)$ is pure, then the rational cohomology of X is given additively by:*

$$H^n(X) = \bigoplus_{q-p=n} \mathrm{Tor}_p^{q,H^*(BG)}(H^*_G(X), \mathbb{Q})\,.$$

The sum of terms with $q \leq \nu$ coincides with $W_\nu H^*(X)$.

Theorem 1.6 follows from Theorem 1.5 by approximating BG in the pull-back square

$$\begin{array}{ccc} X & \to & EG \times_G X \\ \downarrow & & \downarrow \\ point & \to & BG \end{array} .$$

In the special case where $X = G/H$ is a homogeneous spaces we recover a result of Borel [5, Th. 25.1 or 25.2].

An analogous theorem can be formulated for intersection cohomology or Borel–Moore homology. It remains to say when equivariant cohomology is pure. Without difficulty, we find (see Propositions 4.3 and 4.5):

Theorem 1.7 *If a G-variety X is smooth and has only finitely many orbits, then $H^*_G(X)$ is pure. The rational cohomology of X is given additively by:*

$$H^*_G(X) = \bigoplus H^{*-2c}(BH)\,,$$

where the sum is taken over all orbits $\mathcal{O} = G/H \subset X$, and $c = \mathrm{codim}\ \mathcal{O}$.

For singular varieties the result holds for equivariant Borel–Moore homology as defined in [12, Section 2.8]. The multiplicative structure of $H^*_G(X)$ is harder. It involves Chern classes of normal bundles of orbits. Except for special cases like toric varieties or some other spherical varieties, we do not have a satisfactory description.

In order to apply Theorem 1.6 to intersection cohomology of singular varieties, we have to show that $IH^*_G(X)$ is pure. An important class of varieties having this property is that of spherical varieties. One can prove it by using a local description of singularities as in [8].

Standing assumptions: Unless stated otherwise, all cohomology groups are taken with rational coefficients. All algebraic varieties are defined over the complex numbers. We consider only algebraic actions of linear algebraic groups.

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2 Preliminaries

2.1 Weight filtration

The weight filtration in the cohomology of algebraic varieties can be constructed in various ways: either using arithmetic methods, [9, §6] or through analytic methods, [10]. To tackle not necessarily constant coefficient systems one should work with the mixed Hodge modules of M. Saito, [22, 23]. Our results will follow formally from the existence of a weight filtration. Therefore, instead of going into various constructions, we will list its properties. Let X be a smooth variety. We consider cohomology with coefficients in $\mathbb{Q}$. The weight filtration

$$0 = W_{k-1}H^kX \subset W_kH^kX \subset W_{k+1}H^kX \subset \ldots \subset W_{2k}H^kX = H^kX$$

satisfies the following conditions:

1. $W_kH^kX = H^kX$ if X is complete. (We say that the cohomology is *pure*.)

2. Let $f\colon X \to Y$ be an algebraic map. The induced map in cohomology strictly preserves weight, i.e., $f^*W_\nu H^kY = W_\nu H^kX \cap f^*H^kY$.

3. The weight filtration is strictly preserved by the maps in the Gysin (localization) sequence

$$\cdots \longrightarrow H^kX \xrightarrow{j^*} H^kU \xrightarrow{\delta} H^{k+1-2c}Y \xrightarrow{i_!} H^{k+1}X \longrightarrow \cdots.$$

Here $i\colon Y \hookrightarrow X$ is a smooth subvariety of codimension $2c$, and j is the inclusion $U = X \setminus Y \hookrightarrow X$. The boundary map δ lowers weight by $2c$ and $i_!$ raises it by $2c$.

4. The weight filtration is defined for relative cohomology. The long exact sequence of a pair strictly preserves it.

5. The weight filtration is also defined for simplicial varieties. This time weights smaller than k can appear in $H^k(X_\bullet)$. By hyperresolution of singularities, the cohomology of singular varieties can be endowed with a weight structure.

We deal with singular varieties, therefore we have to consider not only constant sheaves, but also the "mixed sheaves" of [2, §5, p. 126]. In particular, we compute cohomology with coefficients in the intersection sheaf IC_X, that is, intersection cohomology [15]. In the main part of [2] the varieties are actually defined over a field of finite characteristic. For complex varieties we are allowed to work with "sheaves of geometric origin" ([2, §6.2.4]). For such sheaves it is possible to find a good reduction and apply the results which are valid in finite characteristic. In particular we have $IH^*(X_{\mathbb{C}}; \mathbb{Q}_\ell) \simeq IH^*(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. (We recall that for varieties over a finite field we have to use coefficients in $\mathbb{Q}_\ell$. The formalism of [9] which allows to compute the cohomology with coefficients in a complex of sheaves generalizes the construction of étale cohomology.) This way we transport the weight filtration to the intersection cohomology of complex varieties. The theory of M. Saito also applies: an intersection sheaf is an object in the category of mixed Hodge modules. The weight filtration in $IH^k(X)$ also starts with W_kIH^kX as in the smooth case. It turns out that the weight filtration is defined over rational numbers. We will use again just formal properties of mixed sheaves. A suitable category to work with is the category of mixed Hodge modules over a base B, i.e. $\mathbf{D}_{\text{mix}}(B)$. The properties 3 and 4 above can be extended to the following:

6. For a distinguished triangle of mixed sheaves

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

the maps in the long exact sequence

$$\cdots \to H^k(X;A) \to H^k(X;B) \to H^k(X;C) \to H^{k+1}(X;A) \to \cdots$$

strictly preserve weights.

2.2 Varieties associated with a group action

Let G be a connected linear algebraic group. The classifying space of G as a simplicial variety has already been considered by Deligne [10]:

$$BG_\bullet = \left\{ pt \;\substack{\longleftarrow\\ \longleftarrow}\; G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G \times G \;\cdots \right\}$$

(the arrows are multiplications or forgetting the edge factors). In an appropriate category, $BG_\bullet$ represents the functor which associates to X the isomorphism classes of Zariski-locally trivial G-bundles over X.

Another algebraic model $BG_{ét}$ was described by Totaro [28], see also [21, §4.2]. It classifies G-bundles which are étale-locally trivial. Totaro's $BG_{ét}$ has the same cohomology as the simplicial $BG_\bullet$. The spaces $EG_{ét}$ and $BG_{ét}$ are infinite-dimensional; one therefore has to approximate them by U and U/G where $U \subset V$ runs over all open G-invariant subsets in representations $G \to GL(V)$, such that the geometric quotient U/G exists and $U \to U/G$ is a G-bundle (étale locally trivial). We prefer to work with this model. Both models have isomorphic cohomology.

The Borel construction of X is the simplicial variety

$$[X/G]_\bullet = (EG \times_G X)_\bullet = \left\{ X \;\substack{\longleftarrow\\ \longleftarrow}\; X \times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X \times G \times G \;\cdots \right\}$$

(the arrows are the action, multiplications or forgetting the last factor). Of course, $[pt/G]_\bullet = BG_\bullet$.

Note that in Totaro's model the space $U \times_G X$ might not be an algebraic variety. But it is so if all orbits admit G-invariant quasi-projective neighbourhoods. See the discussion in [29].

Equivariant cohomology $H^*_G(X)$ is defined as the cohomology of $EG \times_G X$. It does not matter which model of $EG \to BG$ we use. For a fixed degree i we have $H^i_G(X) = H^i(U \times_G X)$ if the codimension of $V \setminus U$ is sufficiently large.

2.3 The weight filtration in $H^*(BG)$ and in $H^*(G)$

The following observations were made by Deligne [10], §9.

THEOREM 2.1 *The cohomology of BG is pure, i.e.,*

$$W_{k-1}H^k(BG) = 0\,, \qquad W_k H^k(BG) = H^k(BG)\,.$$

For example, $H^*(B\mathbb{C}^*) \simeq H^*(\mathbb{P}^\infty)$ is a polynomial algebra on one pure generator of degree 2.

Now let $P^\bullet = P^1 \oplus P^3 \oplus P^5 \oplus \ldots$ be the space of primitive elements of the Hopf algebra H^*G, so that $H^*G = \bigwedge P^\bullet$.

THEOREM 2.2 *Let $k > 0$. Then $W_k H^k G = 0$ and $W_{k+1}H^k G = P^k$ (which is 0 if k is even).*

Hence one can filter the cohomology of G by "complexity": $C_a H^*(G) = \bigwedge^{\leq a} P^\bullet$. Then $C_a H^*(G) \cap H^k(G) = W_{a+k}H^k(G)$.

3 Filtration in the Eilenberg–Moore spectral sequence

Let us come back to the pull-back diagram 1.4 and the associated Eilenberg-Moore spectral sequence. We want to endow its entries with a weight filtration. The goal of this section is to repeat Smith's construction (as presented in [26]). We find it convenient to consider the category of sheaves over B instead of the category of spaces over B as in Smith's papers. More precisely, we work in the category $\mathbf{D}_{\mathrm{mix}}(B)$ of mixed sheaves of [22, 23] or [2]. Instead of a map $f\colon A \to B$ we consider the sheaf Rf_*IC_A since $IH^*(A) = H^*(B; Rf_*IC_A)$. The condition: *$f\colon A \to B$ is a fibration with the fibre X* is replaced by: *Rf_*IC_A has constant cohomology sheaves with stalks $IH^*(X)$*. Since we assume B to be simply connected there is no need to consider locally constant cohomology sheaves. It is clear that if $f\colon A \to B$ is a fibration which is locally trivial with respect to the classical topology, then Rf_*IC_A has constant cohomology sheaves. Note that in this case the weight filtration is constant, [2, Proposition 6.2.3], although the Hodge structure might vary. Suppose that B is smooth. We claim that the intersection cohomology of the pull-back is the cohomology of B with coefficients in the tensor product $Rf_*IC_A \otimes Rg_*IC_C$:

$$IH^*(D) \simeq H^*(B; Rf_*IC_A \otimes Rg_*IC_C)$$

or, more precisely, that $\bar{f}^*IC_C \otimes \bar{g}^*IC_A$ is quasi-isomorphic to IC_D. Indeed,

- this sheaf is constant when restricted to the regular part of D, which is equal to $C_{\mathrm{reg}} \times_B A_{\mathrm{reg}}$;
- locally, for an open set $U \subset C$ (in the classical topology) over which the fibration is trivial, i.e., $\bar{f}^{-1}(U) \simeq U \times X$, we have $(\bar{f}^*IC_C \otimes \bar{g}^*IC_A) \simeq IC_U \otimes IC_X$.

Therefore our goal is to construct a spectral sequence converging to cohomology with coefficients in a tensor product of mixed sheaves. The same applies to constant sheaves instead of intersection sheaves. For $B = BG$ one can generalize our construction to equivariant sheaves in the sense of Bernstein-Lunts, [3].

We set $\mathcal{A}^* = H^*(B)$, and we write H^*F instead of $H^*(B;F)$ for a mixed sheaf F.

THEOREM 3.1 *Let B be a simply connected complex algebraic variety. Let F and G be bounded below mixed sheaves over B. Suppose that F has constant cohomology sheaves. Then there is a spectral sequence with*

$$E_2^{-p,q} = \mathrm{Tor}_p^{q,\mathcal{A}^*}(H^*F, H^*G)$$

and converging to $H^*(F \otimes G)$. The entries and differentials of the spectral sequence lie in the category $\mathbf{D}_{\mathrm{mix}}(point)$.

Proof. Let $\pi\colon B \times B \to B$ be the projection onto the first factor and let $\Delta\colon B \to B \times B$ be the diagonal. For a mixed sheaf F over B we define $QF = R\pi_*\pi^*F$. It comes with a map $QF \to F$ constructed in the following way:

1. $F \to F = R\pi_* R\Delta_* F$ is the identity map over B,
2. $\pi^*F \to R\Delta_*F$ is the adjoint map,
3. applying $R\pi_*$, we obtain $R\pi_*\pi^*F \to R\pi_*R\Delta_*F = F$.

The sheaf QF is isomorphic to $R\epsilon_*\mathbb{Q}_B \boxtimes F$, where $\epsilon\colon B \to point$. The cohomology of the stalk $\mathcal{H}_x^*(QF)$ is equal to $\mathcal{A}^* \otimes \mathcal{H}_x^*(F)$. We note that H^*QF is a free $\mathcal{A}^*$-module and that the induced map $H^*QF \to H^*F$ is surjective. Indeed, $H^*QF = \mathcal{A}^* \otimes H^*F$ and the map to H^*F is the action of $\mathcal{A}^*$ on H^*F. Let us summarize the ingredients which L. Smith has used to construct the spectral sequence (most of them are listed in [25]):

1. For all $F \in Ob(\mathbf{D}_{\mathrm{mix}}(B))$ the map $QF \to F$ satisfies the properties
   a) $H^*QF \to H^*F$ is surjective,
   b) H^*QF is a free $\mathcal{A}^*$-module,
   c) if $H^qF = 0$ for $q < a$, then $H^qF \to H^qQF$ is an isomorphism for $q < a + 2$. (This condition is necessary for convergence, see [26], p. 42.)
2. If F is a fibration (i.e., has constant cohomology sheaves), then so is QF.
3. If H^*F is of finite type, then H^*QF is of finite type as well.

4. If F is a fibration and H^*F a free $\mathcal{A}^*$-module, then the natural map
$$H^*F \otimes_{\mathcal{A}^*} H^*G \to H^*(F \otimes G)$$
is an isomorphism for all $G \in Ob(\mathbf{D}_{\mathrm{mix}}(B))$.

5. Let F, $G \in Ob(\mathbf{D}_{\mathrm{mix}}(B))$ with F being a fibration. If $H^qF = 0$ for $q < a$ and $H^qG = 0$ for $q < b$, then $H^q(F \otimes G) = 0$ for $q < a + b$. (This is again necessary for convergence, compare [26], p. 43 and Proposition A.5.2.)

Properties (4) and (5) follow from the Grothendieck spectral sequence
$$E_2^{p,q} = H^p(\mathcal{H}^qF \otimes G) \Rightarrow H^{p+q}(F \otimes G)\,.$$

Now we mimic Smith's construction of a free resolution of H^*F. We inductively define

- $F_0 = F$;
- $Q_p = QF_p$;
- F_{p+1} is the fibre of $Q_p = QF_p \to F_p$, i.e., it fits into the distinguished triangle
$$\begin{tikzcd} Q_p \arrow[rr] & & F_p. \arrow[dl, "{[+1]}"] \\ & F_{p+1} \arrow[ul] & \end{tikzcd}$$

We thus obtain a free resolution of H^*F coming on the level of sheaves from
$$\begin{tikzcd} F = F_0 \arrow[rr, "{[+1]}"] & & F_1 \arrow[rr, "{[+1]}"] \arrow[dl] & & F_2 \ \ldots \arrow[rr, "{[+1]}"] & & F_p \arrow[rr, "{[+1]}"] \arrow[dl] & & F_{p+1} \arrow[dl] & \\ & Q_0 \arrow[ul] \arrow[u, phantom, "\star"] & & Q_1 \arrow[ll, "\triangle"'] \arrow[ul] \arrow[u, phantom, "\star"] & & \ldots\ Q_p \arrow[ul] \arrow[ll, phantom] & & Q_{p+1} \arrow[ll, "\triangle"'] \arrow[ul] \arrow[u, phantom, "\star"] & & \ldots \end{tikzcd}$$

Here $\star$ denotes a distinguished triangle and $\triangle$ a commutative triangle. The sheaves Q_p have constant cohomology. We tensor the entire diagram by G. The triangles remain distinguished. Now we apply $H^*(-)$. The Eilenberg–Moore spectral sequence is obtained from the exact couple
$$\begin{tikzcd} \bigoplus_p H^*(F_p \otimes G) \arrow[rr, "{[+1]}"] & & \bigoplus_p H^*(F_p \otimes G). \arrow[dl] \\ & \bigoplus_p H^*(Q_p \otimes G) \arrow[ul] & \end{tikzcd}$$

This is an exact couple in the abelian category of vector spaces with an action of the Frobenius automorphism. The maps strictly preserve the weight filtration. If we worked with Saito's category of mixed Hodge sheaves, the cohomology groups would lie in the abelian category of mixed Hodge structures; here the maps also strictly preserve the weight filtration. The elements of the couple are bigraded by p and by the internal degree q. The map $\to$ is of bidegree $(1,1)$, the map $\swarrow$ of bidegree $(0,0)$ and the map $\nwarrow$ of bidegree $(1,0)$. The resulting spectral sequence has $E_1^{-p,q} = H^q(Q_p \otimes G)$. By property (4) we have

$$E_1^{-p,*} = H^*(Q_p \otimes G) = H^*Q_p \otimes_{\mathcal{A}^*} H^*G\,.$$

The E_2 term is

$$E_2^{-p,*} = H_p(H^*(Q_\bullet \otimes G)) = \mathrm{Tor}_p^{\mathcal{A}^*}(H^*F, H^*G)\,.$$

Remark 3.2 The properties of this spectral sequence were studied in detail by L. Smith in [26]. His arguments fit perfectly into the formalism of the stable homotopy category over B. In the final step he desuspends his spectral sequence [26, p. 38]. The formalism of the derived category of sheaves over B is even better for this purpose.

The convergence of the Eilenberg–Moore spectral sequence is guaranteed by the following. Suppose that the mixed sheaves F and G have cohomology concentrated in non-negative degrees. Then the resulting spectral sequence lies in the second quadrant. Inductively one checks that $H^qF_p = 0$ for $q < 2p$. Indeed, by property (1c) the map $H^qQ_p \to H^qF_p$ is an isomorphism for $q < 2p+2$ and a surjection for $q = 2p+2$. Therefore $H^qF_{p+1} = 0$ for $q < 2p+2$. Now by property (5) $H^q(Q_p \otimes G) = 0$ for $q < 2p$ (since $H^q(Q_p) = H^q(F_p) = 0$). We find that the terms $E_2^{-p,q}$ may be nonzero only for $0 \geq q \geq 2p$. □

The entries of the spectral sequence are cohomologies of objects in $\mathbf{D}_{\mathrm{mix}}(point)$, therefore:

COROLLARY 3.3 *The Eilenberg–Moore spectral sequence inherits a weight filtration.*

Let us assume that $\mathcal{A}^*$, H^*F and H^*G are pure. Then $H^*Q_0 = \mathcal{A}^* \otimes H^*F$ is pure. The cohomology H^*F_1 is the kernel of $H^*Q_0 \twoheadrightarrow H^*F$, hence also pure. Reasoning inductively, we find that H^*Q_p is pure for all p. Hence $E_1^{-p,q} = (H^*Q_p \otimes_{\mathcal{A}^*} H^*G)^q$ is pure of weight q. Therefore the differentials vanish from d_2 on. We obtain:

THEOREM 3.4 *If $\mathcal{A}^*$, H^*F and H^*G are pure, then $E_r^{-p,*}$ is pure. Consequently, the Eilenberg–Moore spectral sequence degenerates and*

$$W_\nu H^n(F \otimes G) = \bigoplus_{q-p=n,\ q\leq\nu} \mathrm{Tor}_p^{q,\mathcal{A}^*}(H^*F, H^*G)\,.$$

Remark 3.5 Our construction is explicit. The spectral sequence can be deduced from the simplicial sheaf $F \otimes Q_\bullet G$. The term $E_1^{-p,*} = H^*(Q_p \otimes G)$ can be identified with the bar complex

$$\mathrm{Bar}_p(H^*F, \mathcal{A}^*, H^*G) = H^*F \otimes (\overline{\mathcal{A}}^*)^{\otimes p} \otimes H^*G\,.$$

Here $\overline{\mathcal{A}}^* = \overline{H}^*(B)$ is the reduced cohomology of the base.

Remark 3.6 We will apply the Eilenberg–Moore spectral sequence to sheaves over BG. We can extend Theorem 3.4 to this case by approximating BG the way Totaro does.

4 Applications

4.1 Rational cohomology of principal bundles

Let G be a connected linear algebraic group and let X be a complete complex algebraic variety. Let $P \to X$ be a principal algebraic G-bundle which is étale locally trivial. It does not have to be Zariski-locally trivial, but the map $P \to X$ is affine. In general a classifying map $X \to BG$ might not exist in the category of algebraic varieties. Instead, we use a construction of [28, Proof of Theorem 1.3]: Let V be a representation of G and $X_V = P \times_G V \to X$ the associated vector bundle. The algebraic variety X_V is homotopy equivalent to X. Let U be an open G-invariant subset of V with a quotient U/G which approximates $BG_{\acute{e}t}$. The intersection cohomology of the open set $X_U = P \times_G U \subset X_V$ approximates $IH^*(X)$ as $\operatorname{codim}(V \setminus U) \to \infty$. Similarly $IH^*(P \times U)$ approximates $IH^*(P)$. Now there is a pull-back

$$\begin{array}{ccc} P \times U & \rightarrow & BE_{\acute{e}t} \\ \downarrow & & \downarrow \\ X_U & \rightarrow & BG_{\acute{e}t} \end{array}$$

Set $B = BG_{\acute{e}t}$, $A = EG_{\acute{e}t}$ and $C = \lim X''$. The Eilenberg–Moore spectral sequence (1.1) allows to compute the intersection cohomology of P. Since $IH^*(X)$ is pure (as is $H^*(EG)$, of course), by (1.6) we obtain:

Corollary 4.1

$$W_\nu IH^n(P) = \bigoplus_{q-p=n,\ q\le\nu} \operatorname{Tor}_p^{q,H^*(BG)}(IH^*(X), \mathbb{Q})\,.$$

The approximation is justified by the following proposition, which follows simply from the fact that Tor can be computed from the bar-resolution.

Proposition 4.2 *Let A^* be an inverse limit of graded rings A^*_i ($i \in \mathbb{Z}$) and let M^* and N^* be graded A^*-modules which are inverse limits of A^*_i-modules M^*_i and N^*_i. We assume that the grading is nonnegative and that for each degree n, the limits stabilize (i.e., $A^n = A^n_i$ for large i and analogously for M^* and N^*). Then for a fixed q and sufficiently large i*

$$\operatorname{Tor}_*^{q,A^*}(M^*, N^*) = \operatorname{Tor}_*^{q,A^*_i}(M^*_i, N^*_i).$$

4.2 From equivariant to non-equivariant cohomology

Let G be as before and X a G-variety. Set $B = BG$, $A = EG \times_G X$ and $C =$ *point*. We list examples to which Theorem 1.6 applies, that is, for which $H^*(EG \times_G X) = H^*_G(X)$ or $IH^*(EG \times_G X) = IH^*_G(X)$ are pure. Let us start with some simple observations.

PROPOSITION 4.3 *Let $X \to Y$ be a fibration (in the classical topology) with fibre, a homogeneous G-space. If Y is complete, then $IH^*_G(X)$ is pure.*

Proof. Consider the fibration $EG \times_G X \to Y$. The fibres are homotopy equivalent to BH, where H is the stabilizer of a point in X. Then by the Leray spectral sequence we find that $IH^*(EG \times_G X)$ is pure. (Note that this spectral sequence degenerates.) □

PROPOSITION 4.4 *Suppose X is equivariantly fibred over a homogeneous G-space. If the fibres are complete, then $IH^*_G(X)$ is pure.*

Proof. The fibration is of the form $X \to G/H$. Let F be a fibre, which is complete. Consider the fibration $EG \times_G X \to EG \times_G G/H \simeq BH$. The fibres are again isomorphic to F. The Leray spectral sequence has $E_2^{p,q} = H^p(BH; IH^q(F))$. (The coefficients might be twisted.) We have

$$H^p(BH; IH^q(F)) = (H^p(\widetilde{BH}) \otimes IH^q(F))^{\pi_1(BH)} = (H^p(BH^0) \otimes (IH^q(F)))^{H/H^0},$$

where H^0 is the identity component of H. We see that $E_2^{p,q}$ is pure of weight $p+q$. Therefore $IH^*_G(X)$ is pure. Moreover, the spectral sequence degenerates. □

Now suppose that X is smooth. We show that purity is additive in this case.

PROPOSITION 4.5 *Suppose that X is smooth and has a G-equivariant stratification $\{S_\alpha\}$ such that each $H^*_G(S_\alpha)$ is pure. Then*

$$H^*_G(X) = \bigoplus_p \bigoplus_{\operatorname{codim} S_\alpha = p} H_G^{*-2p}(S_\alpha)(p) \tag{1}$$

is pure.

Here (p) denotes the shift of weights by $2p$.

Proof. Consider the filtration of X by codimension of strata:

$$U_p = \bigcup_{\operatorname{codim} S_\alpha \le p} S_\alpha \,.$$

The resulting spectral sequence for equivariant cohomology has

$$E_1^{p,q} = H_G^{p+q}(U_p, U_{p-1}) \,.$$

By the Thom isomorphism,

$$E_1^{p,q} = \bigoplus_{\operatorname{codim} S_\alpha = p} H_G^{q-p}(S_\alpha)\,.$$

The Thom isomorphism lowers weight by $2p$. Hence, $E_1^{p,q}$ is pure of weight $p+q$ because $H_G^{q-p}(S_\alpha)$ is pure of weight $q-p$. We conclude that the spectral sequence degenerates, which was to be shown. □

By Theorem 2.1, Proposition 4.5 applies in particular to varieties with finitely many orbits. Since the right hand side of (1) is the E_1 term of a cohomology spectral sequence, it carries a canonical product. For a toric variety X defined by a fan Σ, this "is" the product in $H_G^*(X)$ (which equals the Stanley–Reisner ring of Σ). The same holds for complete symmetric varieties [4, Thm. 36] and, more generally, for toroidal varieties [6, §2.4].

Now we switch to the singular case.

Remark 4.6 First we note that our results can be generalized straightforwardly if we replace the constant sheaf by the dualizing sheaf. This way one compares Borel–Moore homology with equivariant Borel–Moore homology, defined in [12]. We will not develop this remark here.

Let us make some comments about intersection sheaves, which compute intersection cohomology [15]. If the local intersection sheaf is pointwise pure, we can argue as in the proof of Proposition 4.5. This condition is satisfied if the singularities are quasi-homogeneous (as in [11], for instance). Other examples are obtained from the decomposition theorem ([2, Théorème 6.2.5], [24]):

PROPOSITION 4.7 *If X admits an equivariant resolution $\widetilde{X}$ with pure equivariant cohomology, then the equivariant intersection cohomology of X is pure.*

Proof. Indeed, $IH_G^*(X)$ injects into $H_G^*(\widetilde{X})$ by the decomposition theorem, and the inclusion preserves weights. □

There are lots of examples of G-varieties admitting a resolution which can be stratified by S_α as in 4.5. If X is a spherical variety, one can prove purity without appealing to the decomposition theorem. Here a precise analysis of spherical singularities ([8] §3.2) shows that the intersection sheaf is pointwise pure. The stalk $\mathcal{H}_x^* IC_X$ coincides with the primitive part of intersection cohomology of some projective spherical variety of lower dimension. We refer the reader to [7], Theorem 2 and [8], Theorem 3.2. Filtering X by codimension of orbits, we find that

$$IH_G^*(X) = \bigoplus_\alpha H^*(BG_\alpha; \mathcal{H}_{x_\alpha}^* IC_X)\,.$$

The sum is taken over all orbits $G\cdot x_\alpha = G/G_\alpha$. The decomposition follows from the purity of the summands. The coefficients may be twisted. The intersection cohomology localized at the orbit $G\cdot x_\alpha$ is equal to

$$H^*(BG_\alpha;\mathcal{H}^*_{x_\alpha}IC_X) = H^*(BG^0_\alpha;\mathcal{H}^*_{x_\alpha}IC_X)^{G_\alpha/G^0_\alpha}\,.$$

Corollary 4.8 *If X is a spherical variety, then $IH^*_G(X)$ is pure and*

$$IH^n(X) = \bigoplus_{q-p=n} \mathrm{Tor}_p^{q,H^*(BG)}(IH^*_G(X),\mathbb{Q})\,.$$

The sum of terms with $q\le\nu$ coincides with $W_\nu IH^*(X)$.

This theorem generalizes a result for toric varieties, [30].

5 Koszul duality

Our paper is motivated by the work of Goresky, Kottwitz and MacPherson, [16]. One of their main results states that the nonequivariant "cohomology" of a G-space X can be recovered from the "equivariant cohomology" (and vice versa) through Koszul duality. But one has to be careful, since here by "equivariant cohomology" we mean a complex $C^*_G(X)$ (an object in a derived category) computing $H^*_G(X)$. Precisely, there is an equivalence of derived categories

$$D\,(H^*(BG)\text{-modules}) \simeq D\,(H_*(G)\text{-modules})$$

such that $C^*_G(X)$ corresponds to $C^*(X)$. The argument of [16] contains a gap. A proof that the cohomology of the Koszul complex $\Omega^*_G(X)\otimes H^*(G)$ is equal to $H^*(X)$ appeared in [20][1], while the correct action of $H_*(G)$ has been constructed in [19] and [1]. The question arises: can one recover $H^*(X)$ knowing only the cohomology groups $H^*_G(X)$, not the whole complex? In general one cannot. The higher differentials can be expressed in terms of Massey products in $H^*_G(X)$.

Remark 5.1 A better question seems to be: Can one find interesting spaces for which knowledge of $H^*_G(X)$ suffices to recover $H^*(X)$? This property deserves to be called "formality" (or maybe BG-formality), but unfortunately the word "formality" has been reserved for something else (namely freeness over $H^*(BG)$). See the remarks in the introduction to [30].

In general there is a spectral sequence converging to $H^*(X)$ with

$$E_1 = H^*_G(X)\otimes H^*(G)\,,$$

$$d_1 = \text{Koszul differential}\,.$$

[1] The action of $H^*(G)$ in [20] is not correct.

This is just the Eilenberg-Moore spectral sequence. The cohomology of the complex (E_1, d_1) is the torsion product:

$$E_2 = \mathrm{Tor}^{H^*(BG)}(H^*_G(X), \mathbb{C}) .$$

Our Theorem 1.6 says that under suitable purity assumptions the higher differentials vanish.

COROLLARY 5.2 *Suppose that $H^*_G(X)$ is pure then the cohomology of X is the cohomology of the Koszul complex $(H^*_G(X) \otimes H^*(G), d_1)$.*

There are still open questions: While Koszul duality allows one to recover the $H_*(G)$-module structure of $H^*(X)$, the Eilenberg-Moore spectral sequence gives $H^*(X)$ only additively.

QUESTION 5.3 *Suppose that a smooth complex algebraic variety X consists of finitely many orbits. Is the cohomology of the Koszul complex*

$$(H^*_G(X) \otimes H^*(G); d_1).$$

isomorphic to $H^*(X)$ as a module over $H_*(G)$*?*

By [14] this is the case for toric varieties.

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