A canonical lift of Chern-Mather classes to intersection homology

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It is well known that for an n-dimensional algebraic complex variety X, the Poincaré morphism $H^{2n-i}(X) \longrightarrow H_i(X)$, cap-product by the fundamental class $[X]$, is an isomorphism if X is a manifold but, in general, it is not. There is a factorization

$$\begin{tikzcd} H^{2n-i}(X) \arrow[rr, "\cap[X]"] \arrow[dr, "\alpha"'] & & H_i(X) \\ & IH_i(X) \arrow[ur, "\omega"'] & \end{tikzcd}$$

by intersection homology groups (we will use only middle perversity) and intersection homology is the good theory for considering intersection product of cycles.

On another hand the Chern classes for singular varieties have been defined by M.H. Schwartz [Sc] and by R. MacPherson [MP1], they are defined in homology and in general it is not possible to lift them to cohomology. A natural question arose : is it possible to lift the Chern-Schwartz-MacPherson classes to intersection homology?

Firstly J.L. Verdier [V] gave the example of a singular variety X such that the Chern-Schwartz-MacPherson class $c_1(X) \in H_2(X)$ can be lifted to $IH_2(X)$ as two distinct Chern classes of small resolutions X_j of X such that $H_2(X_1) \overset{\cong}{\rightarrow} IH_2(X) \overset{\cong}{\leftarrow} H_2(X_2)$. The computation shows that if we want to express classes of singular varieties using small resolution, we need correction terms living not only in intersection homology of the singular part.

The second counter-example, due to M. Goresky, is an example of a singular algebraic variety such that the Chern-Schwartz-MacPherson class is not in the image of intersection

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partially supported by CNRS and KBN 2 P03A 01113 grants.

homology, using integer coefficients. In [BG] was explained the fact that both examples of Verdier and Goresky are examples of Thom spaces associated to Segre and Veronese embeddings respectively : $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ and $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

The question became : is it possible to lift Chern-Schwartz-MacPherson classes to intersection homology with rational coefficients ? That time the answer was positive : firstly Yokura [Y] proved the result for isolated singularities, then [BBFGK] proved that all algebraic cycles (and in particular Chern-Schwartz-MacPherson cycles) can be lifted to intersection homology, for the middle perversity and with rational coefficients (see [We] for a simpler proof). Unfortunately, the lifting is not unique, in general. Due to the Verdier example, it is not obvious that there exists a canonical lifting.

Among ingredients of MacPherson construction are the Chern-Mather classes, in fact MacPherson classes are combination of Mather ones. Zhou [Z] proved existence of lifting of Chern-Mather classes in intersection homology for total perversity. In this paper, we show, that, for quasi-projective complex varieties, there exists a canonical lifting of Mather classes to intersection homology, as soon as the embedding is fixed. The idea of the proof is the following : The Chern–Mather classes are represented by polar varieties. Such polar variety can be considered as an element of a sequence of inclusions of polar varieties. The inclusions are of codimension one and in this case there exists an unique lifting at each step. As an application, we obtain canonical lift of Chern-Schwartz-MacPherson classes to intersection homology for isolated singularities.

1. Recollection of facts about the geometry of Grasmannians

Let $G(n,m)$ be the Grassmannian of n–dimensional spaces in $\mathbb{C}^m$ and let

$$V_\bullet = \{V_0 = \{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_m = \mathbb{C}^m\}$$

be a flag. The flag manifold, set of such flags, will be denoted by $\mathbf{F}(m)$. Define for $i \geq 0$ the Schubert variety [Ch], [Eh]

$$M^i(V_\bullet) = \{W \in G(n,m) : W + V_{m-n+i-1} \neq \mathbb{C}^m\}$$

of codimension i in $G(n,m)$. The cycle $(-1)^i M^i(V_\bullet)$ represents the Poincaré dual of the i–th Chern class $c^i \in H^{2i}(G(m,n))$. Each $M^i(V_\bullet)$ has a natural stratification whose smooth strata are:

$$M^{i,k}(V_\bullet) = \{W \in G(n,m) : codim\,(W + V_{m-n+i-1}) = k+1\}\,.$$

The regular stratum of $M^i(V_\bullet)$ is $M^{i,0}(V_\bullet)$.

Proposition 1.1. *The Schubert varieties $M^i(V_\bullet)$ have the following properties:*
1. $M^{i+1}(V_\bullet) \subset M^i(V_\bullet)$*;*
2. $M^{i+1}(V_\bullet) \cap M^{i,0}(V_\bullet) \subset M^{i+1,0}(V_\bullet)$*;*
3. for $i < n$ the regular part of $M^{i+1}(V_\bullet)$ is not contained in the singularities of $M^i(V_\bullet)$;
4. $M^0(V_\bullet) = G(n,m)$, $M^{n+1}(V_\bullet) = \emptyset$.

Proof. The proof of 1 and 2 is clear, since $V_{m-n+i-1} \subset V_{m-n+i}$.

For proving 3, suppose $V_{m-n+i} = V_{m-n+i-1} + lin\{\alpha\}$ where $lin\{\alpha\}$ denotes the line generated by $\alpha \in \mathbb{C}^m$. This space is not equal to $\mathbb{C}^m$ and there exists $W \in M^{i,0}(V_\bullet)$ such that $\alpha \in W$. Then $dim\,(V_{m-n+i} + W) = dim\,(V_{m-n+i-1} + W) = m-1$ and $W \in M^{i+1,0}(V_\bullet)$.

The proof of 4 follows by dimension considerations. $\square$

Since $M^{i+1}(V_\bullet)$ is irreducible, then a generic point of $M^{i+1}(V_\bullet)$ is in $M^{i,0}(V_\bullet)$.

2. The Gauss map and polar varieties

Let $X^n \subset \mathbb{C}^m$ be an affine variety. Let us denote by Σ_X the singular part of X and by $X_{\mathrm{reg}} = X \setminus \Sigma_X$ the regular one. There is a natural map $s : X_{\mathrm{reg}} \to G(n,m) \times \mathbb{C}^m$ defined by $s(x) = (T_x(X_{\mathrm{reg}}), x)$. The Nash blowup is the closure of the image of s. We have the diagram:

$$\begin{array}{ccccccc} & & \hat{X} & \hookrightarrow & G(n,m) \times \mathbb{C}^m & \xrightarrow{p_1} & G(n,m)\,, \\ & {}^{s}\nearrow & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_2} & & \\ X_{\mathrm{reg}} & \hookrightarrow & X & \hookrightarrow & \mathbb{C}^m & & \end{array}$$

where $\pi = p_{2|\hat{X}}$ and $g = p_{1|\hat{X}} : \hat{X} \to G(n,m)$ is the Gauss map. For a general flag $V_\bullet$ we define a cycle

$$N^i(V_\bullet) = \pi[g^{-1}M^i(V_\bullet)] = \mathrm{closure}\{\, \pi[g^{-1}(M^i(V_\bullet)) \cap s(X_{\mathrm{reg}})]\,\}.$$

We will state the genericity conditions.

Definition 2.1. We say that the map $g : \hat{X} \to G(n,m)$ is general if it is transverse to all strata $M^{i,k}(V_\bullet)$. This means that g restricted to $s(X_{\mathrm{reg}})$ and to each stratum of the special fiber is transverse to the strata $M^{i,k}(V_\bullet)$.

Let us consider the standard flag

$$V^0_\bullet = \{\{0\} \subset lin\{e_1\} \subset lin\{e_1, e_2\} \subset \ldots \subset \mathbb{C}^m\}\,.$$

The group $Gl(m)$ acts on each $G(n,m)$ transitively. By Kleinman's theorem ([Kl] 2. Theorem) there exists an open algebraic subset U of $Gl(m)$ such that for any $a \in U$ the map $a \cdot g$ is transverse to the strata $M^{i,k}(V_\bullet)$. For an open subset $\mathcal{U} \in \mathbf{F}(m)$ define the total polar variety

$$\underline{\mathbf{N}}^i(\mathcal{U}) = \{(x, V_\bullet) \in X \times \mathbf{F}(m) : x \in N^i(V_\bullet),\, V_\bullet \in \mathcal{U}\}\,.$$

It is an algebraic set over $\mathcal{U}$. The projection on $\mathcal{U}$ is not a fibration in general. We fix a sufficiently small $\mathcal{U} \subset \mathbf{F}(m)$ such that the projections $\underline{\mathbf{N}}^i(\mathcal{U}) \to \mathcal{U}$ are fibrations for all $i \geq 0$ and $\mathcal{U} \subset U \cdot V_\bullet^0$ for U as in the Kleinman's theorem.

Definition 2.2. The flag $V_\bullet$ is called *good* if $V_\bullet \in \mathcal{U}$.

For any good flag $V_\bullet \in \mathcal{U}$ the cycle $(-1)^i N^i(V_\bullet)$ is called the polar variety [LT], [Pi], it represents the Chern–Mather class $c^M_{n-i}(X)$ [MP1]. It is the closure of critical points of the projection $X_{\mathrm{reg}} \to V^\perp_{m-n+i-1}$.

3. Construction of a lift of Chern classes to intersection homology

Let us denote by $\mathbf{D}_X$ the dual sheaf (Borel-Moore homology sheaf) of an algebraic complex variety, and IC_X its intersection homology sheaf. All coefficients of homology and intersection homology are rationals.

The dual proof of [BBFGK] (§3.5) shows that for closed embedding of codimension one $W \hookrightarrow X$, there exists a lift ν of the natural morphism $\iota : \mathbf{D}_W \longrightarrow \mathbf{D}_X$ providing a commutative diagram

$$\begin{array}{ccc} IC_W & \xrightarrow{\nu} & IC_X \\ \downarrow & & \downarrow \\ \mathbf{D}_W & \xrightarrow{\iota} & \mathbf{D}_X \end{array} \qquad \text{and then} \qquad \begin{array}{ccc} IH_*(W) & \xrightarrow{\nu_*} & IH_*(X) \\ \downarrow & & \downarrow \\ H_*(W) & \xrightarrow{\iota_*} & H_*(X)\,. \end{array}$$

It is unique as soon as there exists an unique lift on the smooth part of W (*loc. cit.* p. 167). Here IC_W is the intersection homology sheaf of W considered as a sheaf on X supported by W and $\mathbf{D}_W$ is the Borel–Moore homology sheaf also considered as a sheaf on X.

Let $N^\bullet(V_\bullet)$ be the sequence of polar varieties associated to a good flag:

$$N^n(V_\bullet) \subset \ldots \subset N^1(V_\bullet) \subset N^0(V_\bullet) = X^n\,.$$

Then, for each j, $N^j(V_\bullet)$ has codimension one in $N^{j-1}(V_\bullet)$ and no component of $N^j(V_\bullet)$ is contained in the singularities of $N^{j-1}(V_\bullet)$. In this situation, by the previous result, there exist unique sheaf morphisms

$$IC_{N^n(V_\bullet)} \longrightarrow \ \ldots \ \longrightarrow IC_{N^1(V_\bullet)} \longrightarrow IC_{N^0(V_\bullet)} = IC_X$$

which are lifts of the natural morphisms

$$\mathbf{D}_{N^n(V_\bullet)} \longrightarrow \ \ldots \ \longrightarrow \mathbf{D}_{N^1(V_\bullet)} \longrightarrow \mathbf{D}_{N^0(V_\bullet)} = \mathbf{D}_X\,.$$

We obtain the induced diagram of morphisms of intersection homology and homology groups:

$$\begin{array}{ccccccccc}
IH_*(N^n(V_\bullet)) & \longrightarrow & \dots & \longrightarrow & IH_*(N^1(V_\bullet)) & \longrightarrow & IH_*(N^0(V_\bullet)) & = & IH_*(X) \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
H_*(N^n(V_\bullet)) & \longrightarrow & \dots & \longrightarrow & H_*(N^1(V_\bullet)) & \longrightarrow & H_*(N^0(V_\bullet)) & = & H_*(X)\,.
\end{array}$$

The fundamental class of the polar variety $[N^i(V_\bullet)]$ belongs to $IH_{2(n-i)}(N^i(V_\bullet))$.

Definition 3.1. We define an element $\tilde{c}^i(X) = \tilde{c}_{n-i}(X) \in IH_{2(n-i)}(X)$ as the image (under the considered sequence of morphisms) of the fundamental class of the polar variety $(-1)^i[N^i(V_\bullet)] \in IH_{2(n-i)}(N^i(V_\bullet))$.

Proposition 3.2. *The class $\tilde{c}^i(X)$ does not depend on the choice of the good flag.*

Proof. We have two sequences of morphisms:

$$\begin{array}{ccccccccc}
IH_*(\underline{\mathbf{N}}^n(\mathcal{U})) & \to & \dots & \to & IH_*(\underline{\mathbf{N}}^1(\mathcal{U})) & \to & IH_*(\underline{\mathbf{N}}^0(\mathcal{U})) & = & IH_*(X)\otimes H^*(\mathcal{U}) \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
IH_*(N^n(V_\bullet)) & \to & \dots & \to & IH_*(N^1(V_\bullet)) & \to & IH_*(N^0(V_\bullet)) & = & IH_*(X)\,,
\end{array}$$

where the top row is written for the total polar variety of good flags and the bottom row is written for a fixed flag. The diagram commutes since $N^i(V_\bullet) \hookrightarrow \underline{\mathbf{N}}^i(\mathcal{U})$ is normally nonsingular. The class $(-1)^i[N^i(V_\bullet)] \in IH_{2n-2i}(\underline{\mathbf{N}}^i(\mathcal{U}))$ does not depend on the choice of the good flag $V_\bullet \in \mathcal{U}$ and is the class of a fiber in the bundle $\underline{\mathbf{N}}^i(\mathcal{U}) \to \mathcal{U}$. Applying the sequence of maps of intersection homology groups we obtain an element $(-1)^i[N^i(V_\bullet)] \in IH_{2n-2i}(X \times \mathcal{U})$ which is independent on $V_\bullet$. It can be written as $\tilde{c}^i(X) \otimes [pt] \in IH_{2n-2i}(X) \otimes H_0(\mathcal{U})$. $\square$

4. Chern class for quasi–projective variety

Let $X^n \subset \mathbb{P}^m$ be a smooth quasi–projective variety. Let T_{CX} be the tangent bundle of the affine cone $CX \subset \mathbb{C}^{n+1}$ over X away of origin. The bundle T_{CX} is induced from a bundle $\tau \to X$. The polar varieties of X are defined to be the projectivization of the ones of CX. They represent the Chern classes of the bundle τ. To recover the Chern classes of X we use the following formulas:

1. a formula for bundles (see [MS : 14.10] for the case $X^n = \mathbb{P}^n$):

$$T_X \oplus \Theta \simeq \tau \otimes \gamma^*\,;$$

   where Θ is the trivial bundle and $\gamma = \mathcal{O}(-1)$ is the tautological bundle.
2. a formula for Chern classes of a tensor product: let E be a k–dimensional bundle and let L be a line bundle, $c_i = c^i(E)$ and $a = c^1(L)$ then

$$
\begin{aligned}
c^*(E\otimes L) = 1 + c_1 + ka + \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & \\
c_2 + \binom{k-1}{1} ac_1 + \binom{k}{2} a^2 + & \\
c_3 + \binom{k-2}{1} ac_2 + \binom{k-1}{2} a^2c_1 + \binom{k}{3} a^3 + & \\
c_4 + \binom{k-3}{1} ac_3 + \binom{k-2}{2} a^2c_2 + \binom{k-1}{3} a^3c_1 + \binom{k}{4} a^4 + \dots &
\end{aligned}
\tag{4.1}
$$

If we put $k = n+1$, $E = \tau$, $L = \gamma^*$ then $a \in H^2(X)$ is the class of hyperplane section and we obtain a formula for the Chern class of X.

Suppose X is singular. There is no tangent bundle to CX nor a bundle τ. Instead we set $c_i = (-1)^i[N^i(V_\bullet)] \in H_{2n-2i}(X)$ where $N^i(V_\bullet) \subset X$ is the projectivization of the polar variety of CX. Then the formula (4.1) computes the Chern–Mather class of X, [Pi]. By the same formula we define a lift of the Chern–Mather class to intersection homology of X, but now c_i is the lift of $(-1)^i[N^i(V_\bullet)]$ to $IH_{2n-2i}(X)$ constructed in §3.

5. A lift of Chern-Mather classes

The Proposition 3.2 and the formula 4.1 provides us with a method of defining a canonical lift of Chern-Mather classes:

Theorem 5.1. *The Chern-Mather classes of a quasi-projective complex variety can be lifted to intersection homology, in a canonical way, as soon as the embedding is fixed.*

The Chern-Schwartz-MacPherson class is a combination:

$$c_*(X) = \sum n_\alpha incl_* c_*^M(\overline{S}_\alpha)\,,$$

where $\{S_\alpha\}$ is the minimal stratification (see [Te]). Thus we obtain a canonical lift of Chern-Schwartz-MacPherson classes to

$$\bigoplus IH_*(\overline{S}_\alpha)\,.$$

Suppose X admits only isolated singularities $\{a_i\}$, then the total Chern-Schwartz-MacPherson class is equal to :

$$c_*(X) = c_*^M(X) + \sum (1 - Eu_{a_i})[a_i]\,.$$

It can be lifted to $IH_*(X)$ as soon as we lift the class $[a_i]$. Few canonical liftings can be defined but they coincide if X is irreducible.

Theorem 5.2. *The Chern-Schwartz-MacPherson classes of an irreducible quasi-projective complex variety which has only isolated singularities can be lifted to intersection homology, in a canonical way, as soon as the embedding is fixed.*

6. The class $\tilde{c}^1(X)$ and a small resolution

Let us recall that a *small resolution* $\varpi : \tilde{X} \to X$ is a resolution for which there exists a stratification S_α of X such that for any $x \in S_\alpha$, $\dim \varpi^{-1}(x) < 1/2\,\mathrm{codim}(S_\alpha)$. In this case, there is an identification of perverse sheaves $R\varpi_* \mathbb{Q}_{\tilde{X}} \cong IC_X$ and the intersection homology groups of X are identified with homology groups of $\tilde{X}$ ([GM], §6.2 and [MP2] §5).

We will show that:

Proposition 6.1. *For a small resolution we have:*
1. The lift $\tilde{N}^1(V_\bullet)$ of the cycle $N^1(V_\bullet)$ to $\tilde{X}$ (proper inverse image) represents the class $-\tilde{c}^1(X) \in H_{2n-2}(\tilde{X}) \simeq IH_{2n-2}(X)$;
2. If X does not have singularities in codimension one (e.g. if X is normal), then $\tilde{c}^1(X)$ coincides with $c^1(\tilde{X}) \in H_{2n-2}(\tilde{X})$.

Proof of 1. We have the following diagram of sheaves over X:

$$\begin{array}{ccc} R\varpi_* \mathbb{Q}_{\tilde{X}} & \xrightarrow{\simeq} & IC_X \\ \uparrow & & \uparrow{\scriptstyle \nu \text{ unique}} \\ R\varpi_* \mathbb{Q}_{\tilde{N}^1(V_\bullet)} & \underset{\text{direct summand}}{\hookleftarrow} & IC_{N^1(V_\bullet)} \end{array}$$

We may assume that it commutes away of singularities of $N^1(V_\bullet)$ and X. Then it commutes on the whole X since ν is unique. Thus the class $-[\tilde{N}^1(V_\bullet)] \in H_{2n-2}(\tilde{X})$ corresponds to $\tilde{c}^1(X) \in IH_{2n-2}(X)$. $\square$

The result is not true for higher classes (see Observation 7.2).

Proof of 2. Suppose that *codim* $\Sigma_X \geq 2$. Then $H^{2n-2}(X) \to IH_2(X)$ is surjective and thus $H_{2n-2}(\tilde{X}) = IH_{2n-2}(X) \to H_{2n-2}(X)$ is injective. In $H_{2n-2}(X)$ we have:

$$\varpi_* c^1(\tilde{X}) = c^1(X) + \sum n_\alpha [\overline{S}_\alpha^{n-1}] .$$

Since X has no singularities in codimension one, then all $S_\alpha^{n-1} = \emptyset$. Thus $\varpi_* c^1(\tilde{X}) = c^1(X)$ in $H_{2n-2}(X)$. The induced morphism ϖ_* is injective. The corresponding equality clearly holds in homology of $\tilde{X}$. $\square$

In general the first Chern class $\tilde{c}^1(X)$ differs from the Chern class of a small resolution. Let us consider two examples.

Example 6.2. Let $X \subset \mathbb{P}^2$ be given by an equation $f(x,y) = xy = 0$. It admits a small resolution which is its normalization

$$\tilde{X} \simeq \mathbb{P}^1_1 \sqcup \mathbb{P}^1_2 \xrightarrow{\varpi} X .$$

The Chern class of $\tilde{X}$, in $H_*(\tilde{X})$, is $[\tilde{X}]+2[\tilde{pt}_1]+2[\tilde{pt}_2]$, where $\tilde{pt}_1$ is a point in $\mathbb{P}^1_1$ and $\tilde{pt}_2$ is a point in $\mathbb{P}^1_2$. Let us compute $\tilde{c}^1(X)$. The affine cone CX consists of two planes. The projection of it along a general direction is nonsingular. Thus $N^1(V_\bullet)=\emptyset$. To compute $\tilde{c}^1(X)$ we use the formula 4.1:

$$\tilde{c}^*(X)=1+(-[N^1(V_\bullet)]+2a)+\dots,$$

where $a=[pt_1]+[pt_2]$ is the class of a hyperplane section. We obtain $\tilde{c}^1(X)=2[pt_1]+2[pt_2]$. In this case the class $\tilde{c}^1(X)$ coincides with $c^1(\tilde{X})$.

Example 6.3. Let $X\subset\mathbb{P}^2$ be given by the equation $f(x,y,z)=x^3+y^2z=0$. It has an singular point denoted by $\{x_o\}$. It admits a small resolution which is also its normalization:

$$\tilde{X}\simeq\mathbb{P}^1\xrightarrow{\varpi}X.$$

The Chern class of $\tilde{X}$, in $H_*(\tilde{X})$, is $[\tilde{X}]+2[pt]$. The class $\tilde{c}^1(X)$ is the homological Chern–Mather class since X is a topological manifold. We compute the Chern–Mather class $c^M_*(X)$ from the formula:

$$\varpi_*(c_*(\tilde{X}))=c^M_*(X)+k\,incl_*c^M_*(\{x_o\}),$$

where k is given by the following expression in terms of local Euler obstruction [MP1] :

$$\varpi_*(1_{\tilde{X}})=Eu_X+k\,Eu_{\{x_o\}}\quad\text{i.e.}\quad k=-1.$$

We obtain $\tilde{c}^*(X)=c^M_{1-*}(X)=[X]+3[pt]$. In this case the difference between $\tilde{c}^1(X)$ and $c^1(\tilde{X})$ is the class $[pt]$.

Let us give a general expression of the difference between $\tilde{c}^1(X)$ and the first Chern class of a small resolution.

We remind that $X\subset\mathbb{P}^m$. Let $U\subset\mathbb{P}^m$, $U\simeq\mathbb{C}^m$ be one of the standard affine charts. Let W be a irreducible component of $\varpi^{-1}\Sigma_X$ with $dim\,W=dim\,X-1=n-1$. Since $\varpi:\tilde{X}\to X$ is a small resolution thus $dim\,\varpi(W)=n-1$. To each such W we will assign a number.

Definition 6.4. Assume that $U\cap\varpi(W)\neq\emptyset$. We define the *Jacobian multiplicity* of W, denoted by n_W, as the order of zeros on W of the Jacobian of the composition:

$$\varpi^{-1}(U\cap X)\xrightarrow{\varpi}U\cap X\xrightarrow{p}\mathbb{C}^n,$$

where p is a general projection from $U\simeq\mathbb{C}^m$ to $\mathbb{C}^n$.

Remark 6.5. For a small resolution $\varpi : \tilde{X} \to X$ let us define by $C^\circ\tilde{X}$ the pull-back (fibred product):

$$\begin{array}{ccccccc} C^\circ\tilde{X} & \xrightarrow{C^\circ\varpi} & C^\circ X & = & CX\setminus\{0\} & \subset & \mathbb{C}^{m+1} \\ \downarrow & & \downarrow & & & & \\ \tilde{X} & \xrightarrow{\varpi} & X & \subset & \mathbb{P}^m . & & \end{array}$$

Then, in the Definition 6.4, instead of a local general projection p we can take a general global projection $p : \mathbb{C}^{m+1} \to \mathbb{C}^{n+1}$ and use the composition $\tilde{p} = p \circ C^\circ\varpi$:

$$C^\circ\tilde{X} \xrightarrow{C^\circ\varpi} C^\circ X \xrightarrow{p} \mathbb{C}^{n+1}$$

to compute the multiplicity of W.

The Jacobian multiplicity can be expressed by the local Euler obstruction. For X locally irreducible, the Jacobian multiplicity of W equals $Eu_X(x) - 1$, where x is a generic point of $\varpi(W)$. If X is not locally irreducible, we should take a suitable local component of X. For proving this, it is sufficient to look at the case where X is a curve.

Theorem 6.6. *Let $\varpi : \tilde{X} \to X$ be a small resolution. Then $c^1(\tilde{X}) = \tilde{c}^1(X) - \sum n_W\,[W]$, where the sum runs over the set of irreducible components of $\varpi^{-1}(\Sigma_X)$ such that $\dim W = n - 1$.*

Remark 6.7. The corresponding relation between Chern-Mather classes in homology can be found in [MP1].

Proof. Firstly notice that the polar variety $C^\circ\tilde{N}^1(V_\bullet)$ of $C^\circ\tilde{X}$ is the closure of zeros of the Jacobian of a generic projection $\tilde{p} : C^\circ\tilde{X} \to \mathbb{C}^{n+1}$. If the map $\tilde{p}$ is not generic, then we should take into account the multiplicity of the zeros of the Jacobian. The singularities of $\tilde{p}$ consist of the singularities of p and of the singularities of $C^\circ\varpi$. Thus the components of the polar variety in $C^\circ\tilde{X}$ come from $C^\circ N^1(V_\bullet) \subset C^\circ X$ or from the singularities of $C^\circ X$. If p is general then these two sets of components are disjoint. The components of $\varpi^{-1}\Sigma_X$ should be counted with multiplicities n_W. The Chern class of $\tilde{X}$ is

$$c^1(\tilde{X}) = -([\tilde{N}^1(V_\bullet)] + \sum n_W\,[W]) + (n+1)\tilde{a} = \tilde{c}^1(X) - \sum n_W\,[W]\,.$$

where $\tilde{a}$ is the class of a hyperplane section of $\tilde{X}$ which is the inverse image of the class of a hyperplane section of X. $\square$

Explanation of the examples. In the first case the components of $\varpi^{-1}\Sigma_X$ are two points, but with zero Jacobian multiplicity. In the second example consider a general local projection from $X \setminus \{z \neq 0\}$ to $\mathbb{C}$. A point t of normalization $\tilde{X} \simeq \mathbb{P}^1$ is sent to $[t^2 : t^3 : 1]$ and then projected to a point $at^2 + bt^3$. Thus the Jacobian multiplicity is one for $t = 0$.

7. The example of J. L. Verdier (see [V] and [BG])

Let $B = \mathbb{P}^1_x \times \mathbb{P}^1_y \hookrightarrow \mathbb{P}^3$ be the Segre embedding:

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1] .$$

The quadric B is described by the equation:

$$z_0z_3 - z_1z_2 = 0 .$$

Denote by X the projective cone over B:

$$X = cB \subset \mathbb{P}^4$$

defined by the same equation in $\mathbb{P}^4$. Topologically X is the Thom space of the bundle $\gamma_{|B}$; where γ is the tautological bundle over $\mathbb{P}^3$. The variety X admits two small resolutions. To see them consider the bundle $\gamma_{|B} \to B = \mathbb{P}^1_x \times \mathbb{P}^1_y$ as the family of bundles over $\mathbb{P}^1_y$ parameterized by $\mathbb{P}^1_x$. For each x the bundle $\gamma_{|\{x\}\times\mathbb{P}^1_y}$ is equivalent to the tautological bundle over $\mathbb{P}^1_y$. We apply the construction of Thom space for each x. We obtain a smooth space X_1 fibered over $\mathbb{P}^1_x$ with fiber $c\mathbb{P}^1_y \simeq \mathbb{P}^2$. The space X_1 is a small resolution of X; the inverse image of the singular point is the set of infinity points of the family of the Thom spaces, i.e. it is $\mathbb{P}^1$. The second small resolution X_2 is obtained by changing the role of x and y.

We have the canonical isomorphisms:

$$H_*(X_1) \simeq IH_*(X) \simeq H_*(X_2) .$$

We will calculate the intersection homology groups of X with rational coefficients. Since it is the Thom space, therefore

$$\tilde{H}_*(X) \simeq H_{*-2}(B) \simeq 0\,, 0\,, \mathbb{Q}\,, 0\,, \mathbb{Q}^2\,, 0\,, \mathbb{Q}\,,$$

and

$$IH_*(X) \simeq \begin{cases} H^{6-*}(X) & \text{for} \quad * < 3 \\ im\, PD & \text{for} \quad * = 3 \\ H_*(X) & \text{for} \quad * > 3 \end{cases} ,$$

where $PD : H^3(X) \to H_3(X)$ is the Poincaré homomorphism, cap-product by the fundamental class $[X]$, thus

$$IH_*(X) \simeq \mathbb{Q}\,, 0\,, \mathbb{Q}^2\,, 0\,, \mathbb{Q}^2\,, 0\,, \mathbb{Q}\,.$$

We will describe the generators (see the figures 1–6):

$IH_2(X)$ is generated by the projective lines: $[\mathbb{P}^1_x] = d_1$ and $[\mathbb{P}^1_y] = d_2$.

$IH_4(X)$ is generated by the cones: $c(d_1) = p_1$ and $c(d_2) = p_2$.

The corresponding generators in X_1 and X_2 are the proper inverse images of those in X and will be denoted by the same letter. The homological Chern class of X_1 and X_2 were calculated in [BG] and they are the following:

$$c^*(X_1) = [X_1] + (3p_1 + 3p_2) + (3d_1 + 5d_2) + 6\{pt\}\,,$$

$$c^*(X_2) = [X_2] + (3p_1 + 3p_2) + (5d_1 + 3d_2) + 6\{pt\}\,.$$

This shows, that the Chern class of X cannot be calculated using small resolution without correction terms in $H_2(X_i)$.

Now we will calculate $\tilde{c}^*(X)$ straightforward. Firstly we find suitable polar varieties. The cone over X in $\mathbb{C}^5$ is described by the equation:

$$f(z) = z_0 z_3 - z_1 z_2 = 0\,.$$

The gradient field of f is

$$grad\, f(z) = (z_3, -z_2, -z_1, z_0, 0)\,.$$

Fix the flag

$$V_\bullet = \{\{0\},\, lin\{e_0 - e_3\},\, lin\{e_0 - e_3, e_1 - e_2\},\, lin\{e_0, e_1 - e_2, e_3\},\, lin\{e_0, e_1, e_2, e_3\},\, \mathbb{C}^5\}\,.$$

Let $CX^o = C(X_{\text{reg}}) \setminus \{0\} = CX \setminus \{z_4 = 0\}$. Then

$$\begin{aligned}
CN^1(V_\bullet) &= \mathrm{cl}\{z \in CX^o : grad\, f(z)^\perp \oplus lin\{e_0 - e_3\} \neq \mathbb{C}^5\}\,,\\
&= \mathrm{cl}\{z \in CX^o : grad\, f(z) \perp (e_0 - e_3)\}\,,\\
&= \mathrm{cl}\{z \in CX^o : z_0 - z_3 = 0\}\,,\\
CN^2(V_\bullet) &= \mathrm{cl}\{z \in CX^o : grad\, f(z)^\perp \oplus lin\{e_0 - e_3, e_1 - e_2\} \neq \mathbb{C}^5\}\,,\\
&= \mathrm{cl}\{z \in CX^o : grad\, f(z) \perp (e_0 - e_3),\ grad\, f(z) \perp (e_1 - e_2)\}\,,\\
&= \mathrm{cl}\{z \in CX^o : z_0 = z_3,\ z_1 = z_2\}\,,\\
CN^3(V_\bullet) &= \mathrm{cl}\{z \in CX^o : grad\, f(z)^\perp \oplus lin\{e_0, e_1 - e_2, e_3\} \neq \mathbb{C}^5\}\,,\\
&= \mathrm{cl}\{z \in CX^o : grad\, f(z) \perp e_0,\ grad\, f(z) \perp (e_1 - e_2),\ grad\, f(z) \perp e_3\}\,,\\
&= \mathrm{cl}\{z \in CX^o : z_0 = z_3 = 0,\ z_1 = z_2\} = \emptyset\,,
\end{aligned}$$

see the figures 7–9. The Chern classes of the bundle τ over X_{reg} (see §4) are represented by the cycles $(-1)^i N^i(V_\bullet)$. The cycle $N^1(V_\bullet)$ is allowable in X. It is the projective cone over the hyperplane section of B. Thus $\tilde{c}^1(\tau) = -[N^1(V_\bullet)] = -(c(d_1) + c(d_2)) = -(p_1 + p_2)$. To calculate $\tilde{c}^2(\tau)$ we have firstly to find the class $[N^2(V_\bullet)] \in IH_2(N^1(V_\bullet))$. The cycle $[N^2(V_\bullet)]$ itself is not allowable in $N^1(V_\bullet)$. To see how it lifts to $IH_2(N^1(V_\bullet))$ let us examine the only singular point of $N^1(V_\bullet)$. In the affine chart $\{z_4 \neq 0\}$ the set $N^1(V_\bullet)$ is described by the equations

$$\{z_0 z_3 - z_1 z_2 = 0,\ z_0 = z_3\} = \{z_3^2 - z_1 z_2 = 0,\ z_0 = z_3\}\,.$$

It is a singularity of the type A_1.

Fact 7.1. *The surface in $\mathbb{C}^3$ with a singularity of type $A_1 : \{z_3^2 - z_1 z_2 = 0\}$ is a rational homology manifold.*

Proof. We compute the cohomology of the link L of the singular point. We use the Gysin sequence of the fibration $p : L \to L/S^1 = K$, where K is a quadric in $\mathbb{P}^2$:

$$\begin{array}{ccccccc} H^1(K) & \xrightarrow{p^*} & H^1(L) & \xrightarrow{\int} & H^0(K) & \xrightarrow{2\cdot} & H^2(K) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Q} & & \mathbb{Q} \end{array}$$

We see that $H^1(L) = H^2(L)^* = 0$. $\square$

The space $N^1(V_\bullet)$ is the projective cone over K. It is a rational homology manifold. The polar variety $N^2(V_\bullet)$ is the projective cone over two points in K. The group $IH_2(N^1(V_\bullet)) = H_2(N^1(V_\bullet))$ is generated by $[K]$. Thus $[N^2(V_\bullet)]$ is a multiple of $[K]$. To find the multiplier we intersect $[N^2(V_\bullet)]$ and $[K]$ with $[K]$:

$[N^2(V_\bullet)] \cdot [K] = 2,$

$[K] \cdot [K] = deg\, K = 2.$

We conclude that $[N^2(V_\bullet)] = [K]$ in $H_2(N^1(V_\bullet)) = IH_2(N^1(V_\bullet))$.

Now K is allowable in $N^1(V_\bullet)$ and in X; it is the hyperplane section of B. Thus $[K] = d_1 + d_2$. We find that

$$\tilde{c}^*(\tau) = [X] - (p_1 + p_2) + (d_1 + d_2)\,.$$

The class $c^1(\gamma^*)$ is represented by $b = [B] \in H^2(X)$ – the hyperplane section of X. To calculate the class $\tilde{c}^*(X)$ we use the formula 4.1:

$$\begin{aligned} \tilde{c}^*(X) = {} & 1 - (p_1 + p_2) + 4b + \\ & + (d_1 + d_2) - 3b(p_1 + p_2) + 6b^2 + \\ & + 0 + 2b(d_1 + d_2) - 3b^2(p_1 + p_2) + 4b^3. \end{aligned}$$

We have the following relations:

1. $b \cdot p_i = d_i$ in $IH_2(X)$;

2. $b \cdot d_i = [pt]$ in $IH_0(X)$;

3. $p_i \cdot b_i = 0$ and $p_i \cdot b_j = [pt]$ in $H_0(X)$ for $i \neq j$;

hence

4. the image of b in $IH_4(X)$ is $p_1 + p_2$ (from 2 and 3);

5. the image of b^2 in $IH_2(X)$ is $d_1 + d_2$ (from 1 and 4);

6. $b^2 \cdot p_i = [pt]$ in $IH_0(X)$ (from 1 and 2);

7. $b^3 = b \cdot (d_1 + d_2) = 2[pt]$ in $IH_0(X)$ (from 2 and 5).

We obtain:

$$\tilde{c}^*(X) = [X] + (3p_1 + 3p_2) + (4d_1 + 4d_2) + 6[pt] \in IH_*(X)\,.$$

The Chern-Schwartz-MacPherson class is

$$c^*_{MS}(X) = [X] + (3p_1 + 3p_2) + (4d_1 + 4d_2) + 6[pt] - [vertex] \in IH_*(X) \oplus IH_*(\{vertex\})\,.$$

If we compare it with $c^*(X_i)$, then we see that the difference is supported by $imH_*(\varpi^{-1}\Sigma_X) \subset H_*(\tilde{X})$. In homology, the difference is supported only by the image of $H_*(\Sigma_X)$ in $H_*(X)$.

Let us come back to the remark we made after Proposition 6.1.

Observation 7.2. *The proper inverse image of the cycle $N^2(V_\bullet)$ in the small resolution X_1 does not represent the same class as $\tilde{c}^2(\tau) \in IH_4(X)$, so it can not be used to compute the Chern class of X.*

Proof. We remind that X_1 is fibered over $\mathbb{P}^1_x$. Since $N^2(V_\bullet)$ is the projective cone over two points in B, thus $\tilde{N}^2(V_\bullet)$ is contained in the disjoint sum of two fibers in X_1. The element p_2 is represented by a fiber. Hence

$[\tilde{N}^2(V_\bullet)] \cdot p_2 = 0.$

We have also

$[\tilde{N}^2(V_\bullet)] \cdot p_1 = 2[pt],$

see the figures 10–12. This shows, that $[\tilde{N}^2(V_\bullet)] = 2d_2 \neq d_1 + d_2 = [N^2(V_\bullet)]$. $\square$

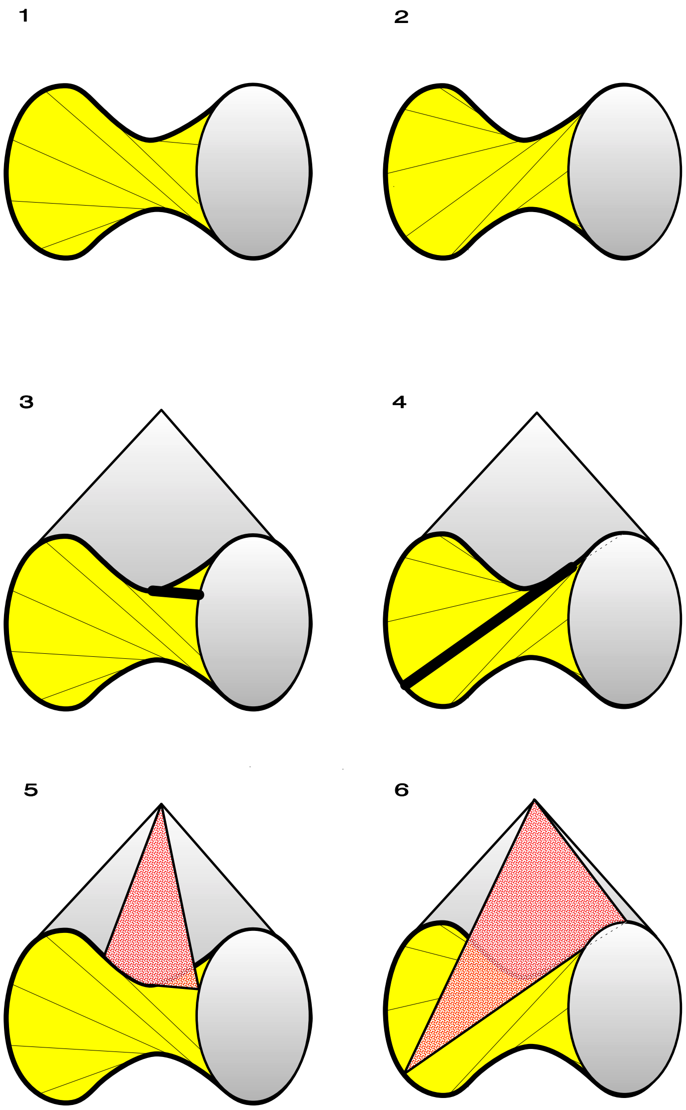
If one computes the Chern class of X using $[\tilde{N}^2(V_\bullet)] \in H^4(X_1)$ instead of $[N^2(V_\bullet)] \in IH_2(X_1)$ one obtains the Chern class of X_1.

Cycles in X

The quadric B in $\mathbb{P}^3$ with two families of generatrices 1) and 2).

The projective cone $X = cB$ and the generators of $IH_2(X)$: 3) d_1, 4) d_2.

The generators of $IH_4(X)$: 5) p_1, 6) p_2.

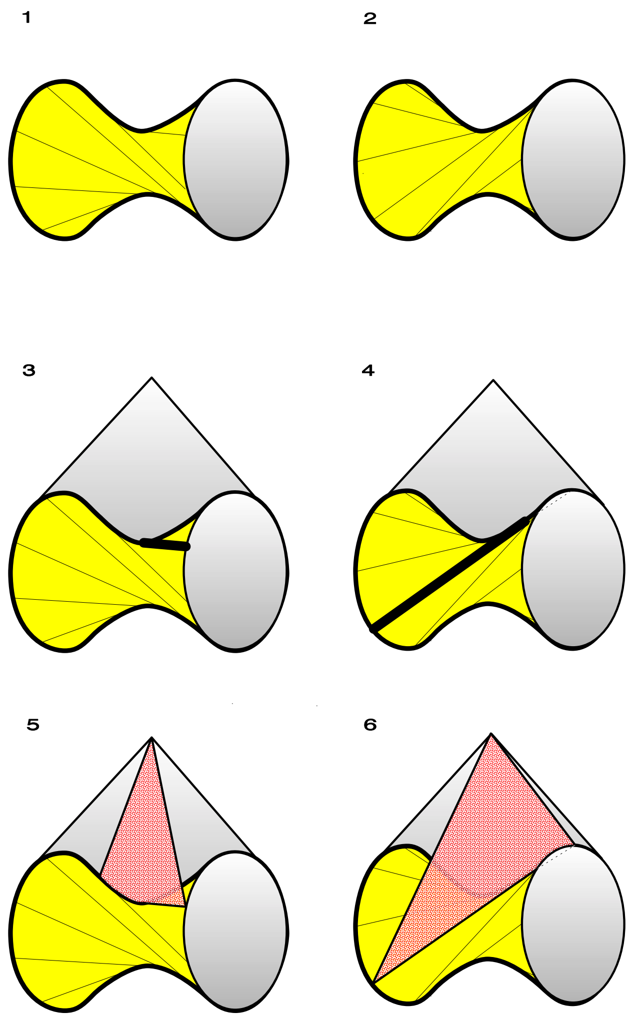


Cycles in X and X_1

The polar variety N^1 is the cone over K: 7) N^1, 8) K.

The polar variety N^2 and its proper inverse image in X_1: 9) N^2, 10) $\tilde{N}^2$.

Proper inverse image in X_1 of the generators of $IH_4(X)$: 11) p_1, 12) p_2.



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