Positivity of Thom polynomials II: the Lagrange singularities

Małgorzata Mikosz*
Warsaw University of Technology,
Pl. Politechniki 1, 00-661, Warszawa, Poland
emmikosz@mech.pw.edu.pl

Piotr Pragacz
Institute of Mathematics of Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
P.Pragacz@impan.gov.pl

Andrzej Weber†
Department of Mathematics of Warsaw University
Banacha 2, 02-097 Warszawa, Poland
aweber@mimuw.edu.pl

Abstract

We study Thom polynomials associated with Lagrange singularities. We expand them in the basis of $\widetilde{Q}$-functions. This basis plays a key role in the Schubert calculus of isotropic Grassmannians. We prove that the $\widetilde{Q}$-function expansions of Thom polynomials of Lagrange singularities have always nonnegative coefficients. This is an analog of a result on Thom polynomials of mapping singularities and Schur S-functions, established formerly by the two last authors.

1 Introduction

In the present paper, we study the *Thom polynomials* of Lagrange singularities. This paper concerns the structure of these polynomials and is a continuation of [16], where the case of singularities of maps was investigated. We look at the *positivity* properties of these Thom polynomials. Such positivity

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properties are nowadays widely investigated in algebraic geometry, see the monograph of Lazarsfeld [12].

Let L be a Lagrangian submanifold in the linear symplectic space

$$V = W \oplus W^*\,,$$

equipped with the standard symplectic form.

Classically, in real symplectic geometry, the *Maslov class* ([1]) is represented by the cycle

$$\Sigma = \{x \in L \ :\ \dim(T_x L \cap W^*) > 0\}.$$

This cycle is the locus of singularities of the projection $L \to W$. Its cohomology class is integral and modulo 2 it is equal to $w_1(T^*L)$, the first Stiefel-Whitney class. In complex symplectic geometry the same construction applied for a Lagrangian submanifold L contained in a symplectic manifold fibering over a base B with Lagrangian fibers, leads to the cohomology class which is equal to

$$c_1(T^*L - T^*B)\,.$$

The generalizations of the Maslov class are Thom polynomials associated with the higher order types of singularities. These types are defined by imposing conditions on the higher order jets of L, see Definition 3.

For real singularities, the associated cohomology classes were studied by Arnold and Fuks (see, e.g., [5]), Vassiliev [17], Audin [3], and others. The complex case was studied by Kazarian [11] who also computed a substantial number of examples. The Thom polynomials in this case can always be written as polynomials in the Chern classes of $T^*L \to T^*B$ (see Remark 13).

Given a Thom polynomial, one can expand it in different bases. The Thom polynomial of a singularity of map, in general, is not a positive combination of monomials in Chern classes. As shown in [16], such a Thom polynomial is always a positive $\mathbb{Z}$-linear combination of Schur S-functions.

The Thom polynomial of a Lagrange singularity is, in general, neither a positive combination of monomials in Chern classes nor a positive combination of Schur S-functions.

In the present paper, we use $\widetilde{Q}$-functions of [15] and show that the Thom polynomial of a Lagrange singularity, expanded in the basis of these $\widetilde{Q}$-functions, has nonnegative coefficients.

Here is a brief outline of the content of the paper.

In Section 2, we recall definitions and properties of Lagrange singularities. We introduce the space of jets of Lagrange submanifolds and define the notion of a "Lagrange singularity type".

In Section 3, we recall the algebraic properties of $\widetilde{Q}$-functions of [15]. We recall also their cohomological interpretation in terms of Schubert classes for the Lagrangian Grassmannian from [14].

In Section 4, we attach the "Thom polynomial" to a Lagrange singularity type. Then we state and prove our main result, Theorem 9, asserting that the Thom polynomial of a Lagrange singularity type is a nonnegative $\mathbb{Z}$-linear combination of $\widetilde{Q}$-functions.

The proof of the theorem is quite different from the one in [16]. There are two reasons for that. First, the $\widetilde{Q}$-functions do not admit – up to now – a characterization similar to that of Schur S-functions from [8]. Second, the Lagrangian case is "more rigid" than the one for singularities of maps, and does not admit a functorial interpretation like that in [16]. The proof in the present paper relies on the computation of some normal bundle in the space of jets of Lagrangian submanifolds, see Lemma 12. This lemma is based on a result about actions of linear transformations on jets of functions (see Proposition 11) which seems to be of independent interest. In the key step, the proof uses deformation to the normal cone.

We finish Section 4 with some discussion of Thom polynomials of Legendre singularities.

In the last section, we list the $\widetilde{Q}$-function expansions of Thom polynomials up to codimension 6.

2 Jets of Lagrangian submanifolds

Let us fix a positive integer n. Suppose that W be a complex linear space, where $\dim W = n$. Let

$$V = W \oplus W^*$$

be a linear symplectic space, equipped with the standard symplectic form $\langle -, - \rangle$, defined by

$$\langle (w_1, f_1), (w_2, f_2) \rangle = f_1(w_2) - f_2(w_1)$$

for $w_i \in W$ and $f_i \in W^*$, $i = 1, 2$. We shall view V as a symplectic manifold. Writing $q = (q_1, \ldots, q_n)$ for the coordinates of W and $p = (p_1, \ldots, p_n)$ for the dual coordinates of W^*, the symplectic form on V is $\sum_{i=1}^n dp_i \wedge dq_i$.

Denote by $\rho : V \to W$ the projection.

Any germ L of a Lagrangian submanifold of V with the nonsingular projection $\rho_{|L}$ is a graph of a 1-form $\alpha : W \to W^*$. The condition that L is Lagrangian is equivalent to $d\alpha = 0$. Since we deal with germs (in fact, with their jets), we can write $\alpha = dF$ for some function $F : W \to \mathbb{C}$.

In the present paper, we shall investigate (the germs of) *singular* Lagrangian submanifolds L, that is L itself is smooth but the projection $\rho_{|L}$

is singular.

Here is the simplest example. Suppose that $\dim W = 1$, and set

$$L = \{(q,p) \in W \oplus W^* \,:\, q = p^2\}\,. \tag{1}$$

(In the classification of Lagrange singularities, this corresponds to the singularity of type A_2.)

To classify the (germs of) Lagrangian submanifolds, one introduces a suitable notion of a *generating family of a Lagrangian germ* (for motivation, see [2, Example 5 in §18.3], and for a precise definition, see [2, §19.1]). (These generating families are usually versal deformations of suitable functions $f : \mathbb{C}^\bullet \to \mathbb{C}$ (*loc.cit.*)). Then one introduces, in [2, §19.4], the notions of R-equivalence and (stable) R^+-equivalence of two generating families of Lagrangian germs. The crucial result (see [2, Theorem in §19.4] p. 304) asserts that the germs of two Lagrangian submanifolds are *Lagrangian equivalent* if and only if the corresponding *generating families* are stably R^+-equivalent. We illustrate these issues by the following example.

Example 1 The Lagrangian submanifold corresponding to the singularity A_3, i.e. $f(x) = x^4$, is obtained in the following way. The generating family

$$F(x,q) = x^4 + q_1 x^2 + q_2 x$$

is a *universal deformation* of $f(x) = F(x,0) = x^4$ (with additional condition $F(0,q) = 0$). The generating family defines the Lagrangian submanifold L in the following way

$$L = \left\{ (q,p) \in W \oplus W^* \,:\, \exists\, x \in \mathbb{C}\,,\ \frac{\partial F(x,q)}{\partial x} = 0\,,\ \frac{\partial F(x,q)}{\partial q} = p \right\}\,,$$

that is,

$$4x^3 + 2q_1 x + q_2 = 0\,, \qquad x^2 = p_1\,, \qquad x = p_2\,.$$

We thus obtain

$$L = \left\{ (q,p) \in W \oplus W^* \,:\, p_1 = p_2^2\,,\ q_2 = -(4p_2^3 + 2q_1 p_2) \right\}\,.$$

We will study the space of germs of Lagrangian submanifolds $L \subset V$ passing through 0. This space has infinite dimension, which is inconvenient from the point of view of algebraic geometry. Therefore we fix once for all a nonnegative integer k and we identify two germs if they have the degree of tangency at 0 bigger than k. This way we obtain the space of k-jets of Lagrangian manifolds denoted by $\mathcal{L}(V)$. This space is homogeneous. Every germ of a Lagrangian submanifold can be obtained from the "distinguished" Lagrangian submanifold W by application of a germ symplectomorphism preserving 0. We have the following description

$$\mathcal{L}(V) = \mathrm{Aut}(V)/P\,, \tag{2}$$

where $\mathrm{Aut}(V)$ is the *group of k-jet symplectomorphisms preserving* $0 \in V$, and P is the stabilizer of W.

Denote now by $LG(V)$ the *Lagrangian Grassmannian* parametrizing all linear Lagrangian subspaces of V. This manifold is embedded in $\mathcal{L}(V)$ in a natural way.

On the other hand, we have the *Gauss map* $\pi : \mathcal{L}(V) \to LG(V)$, which is a retraction to $LG(V)$, defined for a Lagrangian submanifold L by

$$\pi(L) = T_0(L)\,,$$

the tangent space of L at $0 \in L$.

Lemma 2 *The fiber of the projection π is the affine space isomorphic to $\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*)$.*[1]

Proof. The fiber $\mathcal{L}(V)_W = \pi^{-1}(W)$ consists of those (jets of) Lagrangian submanifolds which have the tangent space equal to W. Every Lagrangian submanifold L with nondegenerate projection onto W is the graph of the differential of a function $F : W \to \mathbb{C}$ (note that dF acts from W to W^*). The condition: $0 \in L$ corresponds to the condition: $dF(0) = 0$, and the condition: $T_0(L) = W$ corresponds to vanishing of the second derivatives of F at 0. This proves the lemma. □

We end this section with a definition of a *Lagrange singularity type*. Consider the following subgroup H of $\mathrm{Aut}(V)$ consisting of holomorphic symplectomorphisms preserving the fibration $\rho : V \to W$. Such symplectomorphisms are compositions of maps induced by biholomorphisms of the base and differentials of functions on the base (see [2, §18.5, Theorem, p. 284]).

We say, following [2, §18.6], that two jets of Lagrangian submanifolds are *Lagrangian equivalent*, if they belong to the same orbit of H.

We study not only individual orbits of the group H but also families of orbits. Also, together with an orbit we are forced to consider its closure. Therefore we introduce the following definition.

Definition 3 *A Lagrange singularity type Σ is any closed pure-dimensional algebraic subset of $\mathcal{L}(V)$ which is invariant with respect to the action of H.*

(In other words, a Lagrange singularity type is a closed algebraic set which is the union of some Lagrangian equivalence classes.)

For instance, the closure of the orbit of the singularity A_2 is described by the condition: $\mathrm{corank}(D\rho_{|L}(0)) \geq 1$. Similarly, the closure of the orbit

[1] We point out that $\pi : \mathcal{L}(V) \to LG(V)$ is *not* a vector bundle. This projection is of the form

$$\mathcal{L}(V) = Sp(V) \times_{P'} \mathcal{L}(V)_W \to Sp(V)/P' = LG(V)\,,$$

where $P' = P \cap Sp(V)$ is the group of linear symplectomorphisms stabilizing W. Calculation shows that P' *does not* act linearly on $\mathcal{L}(V)_W$ already for $k = 3$.

of the singularity D_4 is given by the condition: $\operatorname{corank}(D\rho_{|L}(0)) \geq 2$. The singularity class P_8 is not the closure of a single orbit. The family P_8 has one parameter (i.e. it has modality 1). An orbit of a germ belonging to P_8 has codimension 7, while the singularity type P_8 has codimension 6. It can be described by the condition: $\operatorname{corank}(D\rho_{|L}(0)) \geq 3$. To define the singularity type A_3 one has to consider the degeneracy locus $S \subset L$ of the differential $D\rho_{|L}$. The singularity type A_3 consists of the jets of L for which S is singular or $\rho_{|S}$ is not an immersion. These conditions can be translated into algebraic equations in $\mathcal{L}(V)$.

In the literature ([2, §21.3]), one can find a notion of *stable* Lagrangian equivalence classes. They are classified in small codimensions (*loc.cit.*).

3 Lagrangian Grassmannians and $\widetilde{Q}$-functions

We start with recollections on $\widetilde{Q}$-functions of [15].

Let $\mathbb{X}$ be an alphabet[2]. By $\mathbb{X}^2$ we shall denote the alphabet consisting of squares of elements of $\mathbb{X}$. Given an alphabet of variables $\mathbb{X}$, we shall denote by $\mathrm{Sym}(\mathbb{X})$ the ring of symmetric functions in $\mathbb{X}$. Given any alphabet $\mathbb{X} = \{x_1, x_2, \ldots\}$, we set

$$\widetilde{Q}_i(\mathbb{X}) = e_i(\mathbb{X}) = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i} \,, \tag{3}$$

the ith elementary symmetric function in $\mathbb{X}$.

Given two nonnegative integers $i \geq j$, we put

$$\widetilde{Q}_{i,j}(\mathbb{X}) = \widetilde{Q}_i(\mathbb{X})\widetilde{Q}_j(\mathbb{X}) + 2\sum_{p=1}^{j}(-1)^p \widetilde{Q}_{i+p}(\mathbb{X})\widetilde{Q}_{j-p}(\mathbb{X}) \,. \tag{4}$$

For example, we have $\widetilde{Q}_{i,i}(\mathbb{X}) = e_i(\mathbb{X}^2)$.

Given any partition $I = (i_1 \geq \cdots \geq i_h \geq 0)$, where we can assume h to be even, we set

$$\widetilde{Q}_I(\mathbb{X}) = \mathrm{Pfaffian}(M) \,, \tag{5}$$

where $M = (m_{p,q})$ is the $h \times h$ skew-symmetric matrix with

$$m_{p,q} = \widetilde{Q}_{i_p,i_q}(\mathbb{X})$$

for $1 \leq p < q \leq h$.

For an alphabet $\mathbb{X} = \{x_1, x_2, \ldots\}$ of degree 1 variables, the degree of $\widetilde{Q}_I(\mathbb{X})$ is equal to $|I| := i_1 + \cdots + i_h$.

For a fixed positive integer n, let X_n be an alphabet of n variables of degree 1. Then the set $\{\widetilde{Q}_I(\mathbb{X}_n)\}$ indexed by all partitions such that $i_1 \leq n$

[2]By an *alphabet* we understand a finite multiset of elements in a commutative ring.

forms an additive basis of the ring $\mathrm{Sym}(\mathbb{X}_n)$ (see [15]). We shall say that a partition is *strict* if its parts are distinct. Then the set $\{\widetilde{Q}_I(\mathbb{X}_n)\}$ indexed by all strict partitions such that $i_1 \le n$ forms a basis of the ring $\mathrm{Sym}(\mathbb{X}_n)$ as a free $\mathrm{Sym}(\mathbb{X}_n^2)$-module (*loc.cit.*). The same assertions hold for a countable alphabet of variables without restriction on i_1.

Let $c_1, c_2, \ldots$ be a sequence of commuting variables, where $\deg(c_i) = i$, and let $\mathbb{X} = \{x_1, x_2, \ldots\}$ be an alphabet of degree 1 variables. We get a ring isomorphism

$$\Phi : \mathrm{Sym}(\mathbb{X}) \to \mathbb{Z}[c_1, c_2, \ldots],$$

by setting $\Phi(e_i(\mathbb{X})) = c_i$ for $i = 1, 2, \ldots$.

Given a partition I, we put

$$\widetilde{Q}_I = \Phi(\widetilde{Q}_I(\mathbb{X})). \tag{6}$$

If E is a vector bundle, then we define

$$\widetilde{Q}_I(E) = \widetilde{Q}_I(\mathbb{X}), \tag{7}$$

where $\mathbb{X}$ is the alphabet of the *Chern roots* of E. In other words, $\widetilde{Q}_I(E)$ is equal to $\widetilde{Q}_I$, where c_i is specialized to $c_i(E)$, the ith Chern class of E, $i = 1, 2, \ldots$.

Remark 4 The family of $\widetilde{Q}$-functions was invented and investigated in [15] on the occasion of study of Lagrangian degeneracy loci. It is modelled on the classical *Schur Q-functions* (see, e.g., [14]). More precisely, for a strict partition I, the Schur Q-function of a vector bundle E is obtained from $\widetilde{Q}_I$ by the substitution

$$Q_I(E) = \widetilde{Q}_I(E - E^*).$$

See also Remark 5.3 in [13] for another link between these two families of functions. We refer the reader to [15] and [13] for detailed studies of $\widetilde{Q}$-functions.

We shall now use $\widetilde{Q}$-functions to describe some cohomological properties of the Lagrangian Grassmannian $LG(V)$. First, we recall presentation of the cohomology ring of $LG(V)$ by generators and relations, that goes back to Borel [4].

Proposition 5 *With the above notation, we have*

$$H^*(LG(V), \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]/(\widetilde{Q}_{i,i})_{i=1,2,\ldots,n}. \tag{8}$$

(Here the c_i's correspond to the Chern classes of the dual of the tautological subbundle on $LG(V)$.)

The Lagrangian Grassmannian $LG(V)$ has an algebraic cell decomposition which is a particular case of classical Schubert-Bruhat cell decomposition, but here admits the following concrete form. Suppose that a general flag

$$V_\bullet: \ V_1 \subset V_2 \subset \cdots \subset V_n \subset V$$

of isotropic subspaces with dim $V_i = i$, is given (i.e., equivalently, V_n is Lagrangian). Given a partition $I = (n \geq i_1 > \cdots > i_h > 0)$, we define

$$\Omega_I(V_\bullet) = \{L \in LG(V): \ \dim\bigl(L \cap V_{n+1-i_p}\bigr) \geq p, \ p = 1, \ldots, h\}. \qquad (9)$$

– a *Schubert variety* associated with I. Note that we have

$$\operatorname{codim}\bigl(\Omega_I(V_\bullet), LG(V)\bigr) = |I|\,, \qquad (10)$$

and that the cohomology class of $\Omega_I(V_\bullet)$, denoted by Ω_I, does not depend on the choice of the flag $V_\bullet$. Let us recall the following expression for Ω_I in terms of $\widetilde{Q}$-functions.

Theorem 6 *([14, Sect.6]) Let V be a $2n$-dimensional linear symplectic space. Then in $H^{2|I|}(LG(V), \mathbb{Z})$, we have*

$$\Omega_I = \widetilde{Q}_I(R^*)\,, \qquad (11)$$

where R is the tautological subbundle on $LG(V)$.

(In [14, Sect.6], this result was given in terms of the *special Schubert classes* Ω_i but $\Omega_i = c_i(R^*)$.)

Since the Schubert varieties are closures of the cells of a cellular decomposition of $LG(V)$, the Schubert classes $\{\Omega_I\}$, I strict and $i_1 \leq n$, form a $\mathbb{Z}$-basis of $H^*(LG(V), \mathbb{Z})$. Hence also the polynomials $\{\widetilde{Q}_I(R^*)\}$ indexed by the partitions from the same set, have this property.

In the proof of Theorem 9, we shall use the following result. For a strict partition $I = (n \geq i_1 > \cdots > i_h > 0)$, we denote by I' the strict partition whose set of parts complements $\{i_1, \ldots, i_h\}$ in $\{1, \ldots, n\}$.

Proposition 7 *For a strict partition I with $i_1 \leq n$, there exists only one strict partition J with $j_1 \leq n$ and $|J| = \dim LG(V) - |I|$, for which*

$$\widetilde{Q}_I(R^*) \cdot \Omega_J \neq 0\,.$$

In fact, this J is equal to I', and we have

$$\int_{LG(V)} \widetilde{Q}_I(R^*) \cdot \Omega_{I'} = 1. \qquad (12)$$

By virtue of Theorem 6, this proposition follows, e.g., from [15], Theorem 5.23. See also Example 4.2 (5) in [14].

4 Thom polynomials of Lagrange singularities and $\widetilde{Q}$-functions

A Lagrange singularity type $\Sigma \subset \mathcal{L}(V)$ defines the cohomology class

$$[\Sigma] \in H^*(\mathcal{L}(V), \mathbb{Z}) \cong H^*(LG(V), \mathbb{Z})\,.$$

Suppose that this class is equal to

$$\sum_I \alpha_I \widetilde{Q}_I(R^*)\,,$$

where the sum runs over strict partitions I with $i_1 \le n$, and $\alpha_I \in \mathbb{Z}$ (it is important here to use the bundle R^*). Then

$$\mathcal{T}^\Sigma := \sum_I \alpha_I \widetilde{Q}_I \tag{13}$$

is called the *Thom polynomial* associated with the Lagrange singularity type Σ.

Example 8 We list here the $\widetilde{Q}$-functions expansions of Thom polynomials of some Lagrange singularities. They were computed in [11] in the basis of monomials in Chern classes.

A_2: $\widetilde{Q}_1$

A_3: $3\widetilde{Q}_2$

D_4: $\widetilde{Q}_{21}$

D_6: $12\widetilde{Q}_{32} + 24\widetilde{Q}_{41}$

A_7: $135\widetilde{Q}_{321} + 1275\widetilde{Q}_{42} + 2004\widetilde{Q}_{51} + 2520\widetilde{Q}_6$

P_8: $\widetilde{Q}_{321}$.

Note that that all the coefficients in the formulas of Example 8 are nonnegative. For more extensive list of examples, see Section 5.

The Thom polynomials of the singularities A_2, D_4 and P_8 are equal to single $\widetilde{Q}$-functions because they are defined by conditions involving only the differential of $\rho_{|L}$, that is, by conditions defining single Schubert varieties in the Lagrangian Grassmannian.

For example, P_8 is defined by the condition: $\mathrm{corank}(D\rho_{|L}(0)) \ge 3$. This singularity type is the closure of one dimensional family of orbits. Therefore the germs of the type P_8 are *not* stable in the sense of [2, §21.3]. On the other hand, P_8 is worth mentioning since it is the first example of modality that appears in the classification. The Thom polynomial $\widetilde{Q}_{321}$ is an obstruction to avoid this singularity by deformation.

We state now our main result.

Theorem 9 *For any Lagrange singularity type Σ, all the coefficients α_I in (13) are nonnegative.*

For the proof of the theorem, we need several preliminary results.

First of all, we shall use the *nonnegativity property* of globally generated bundles [7] (see also [6], [12]). Let E be a vector bundle on a variety X. By a *cone* in E, we mean a subvariety of E which is stable under the natural $\mathbb{G}_m$-action on E. If $C \subset E$ is a cone, then one may intersect its cycle $[C]$ with the zero-section of the vector bundle:

$$z(C, E) := s_E^*([C]) , \tag{14}$$

where $s_E^* : H^*(E, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is the map of the cohomology groups determined by the zero-section $s_E : X \to E$.

We now record the following variant of a result from [7].

Lemma 10 *Let $\pi : E \to X$ be a globally generated bundle on a proper homogeneous variety X. Let C be a cone in E, and let Z be any algebraic cycle in X of the complementary dimension. Then the intersection $[C] \cdot [Z]$ is nonnegative.*

Proof. We decompose C into Whitney strata C_i, each stratum fibered over $S_i \subset X$ (in C^∞-topology). Note that $\pi^{-1}Z$ is transverse to C_i if and only if Z is transverse to S_i. (One has to stratify the set Z into Whitney strata and check transversality of each pair of strata. For the notion of transversality of stratified cycles and their intersections, see [9].) Since X is homogeneous, we can move Z to make it transverse to each S_i. Therefore, we can assume that $\pi^{-1}Z$ is transverse to C. Then we have

$$[C] \cdot [Z] = [C] \cdot [\pi^{-1}Z] \cdot [X] = [C \cap \pi^{-1}Z] \cdot [X] . \tag{15}$$

The last number is equal to the degree of the cone class $z(C \cap \pi^{-1}Z, E)$ which is nonnegative by [7, Theorem 1 (A)]. $\square$

Our next aim, in the proof of Theorem 9, is to determine the normal bundle of $LG(V)$ in $\mathcal{L}(V)$. To this end, we need a general result about actions of linear transformations on jets. Let $\mathcal{J}$ be the space of jets (of order k fixed in the beginning) of functions $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, satisfying the condition $Df(0) = 0$.

Let $A : \mathbb{C}^n \to \mathbb{C}^m$ be a linear map. The map A acts on $\mathbb{C}^m \oplus \mathbb{C}^n$ by

$$(q, p) \mapsto (q + Ap, p)$$

and it acts on $\mathcal{J}$ by transforming the graphs of the functions. (Note that the action of A is well defined because $Df(0) = 0$.) More precisely, the image of the function f is the function f_A satisfying the following implicit equation:

$$f(q + Af_A(q)) = f_A(q)\,. \tag{16}$$

We state

Proposition 11 *The derivative at* 0 *of* A *acting on* $\mathcal{J}$ *is equal to the identity.*

(We shall use this proposition in the proof of Lemma 12, where the setting stems from Section 2 ; in particular, we have there: $m = n$ and q (resp. p) are the coordinates of W (resp. of W^*) used in that section.)
Proof. We compute the derivative of A in the direction of an arbitrary $f \in \mathcal{J}$. We set

$$g_t = A(tf)\,. \tag{17}$$

By virtue of Eq. (16), the function g_t is given by the implicit equation

$$tf(q + Ag_t(q)) = g_t(q)\,. \tag{18}$$

We want to show that $\frac{g_t}{t} \to f$ as $t \to 0$. Since we assume

$$g_t(0) = f(0) = 0\,,$$

it is enough to show that $\frac{Dg_t}{t} \to Df$ as $t \to 0$. Differentiating Eq. (18) gives

$$tDf \circ (Id + A \circ Dg_t) = Dg_t\,. \tag{19}$$

Hence, we get

$$Dg_t = (Id - tDf \circ A)^{-1} \circ tDf\,. \tag{20}$$

It follows from Eq. (20) that $\frac{Dg_t}{t} \to Df$ as $t \to 0$, and the proposition has been proved. □

Let us come back to the setting of Section 2, and put

$$\mathcal{L} = \mathcal{L}(V) \quad \text{and} \quad G = LG(V)\,.$$

Recalling that R denotes the tautological vector bundle on G, we give the following description of the normal bundle of G in $\mathcal{L}$.

Lemma 12 *We have a natural isomorphism*

$$N_G\mathcal{L} \cong \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(R^*)\,. \tag{21}$$

Proof. Let $L \in G$ be a Lagrangian linear subspace. Let us choose a splitting

$$V \cong L \oplus L^* \tag{22}$$

Using this splitting, by Lemma 2 we construct an isomorphism

$$\pi^{-1}(L) \cong \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(L^*) \tag{23}$$

Two splittings of the exact sequence

$$0 \to L \to V \to L^* \to 0\,,$$

differ by a linear map $A : L^* \to L$. By Proposition 11, the action of A on the tangent space to jets is the identity. Hence the isomorphism (23) does not depend of the choice of splitting (22), and we have (globally) a *natural* isomorphism

$$N_G\mathcal{L} \cong \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(R^*)\,. \tag{24}$$

The lemma has been proved. □

We are now ready to complete the proof of Theorem 9.

Suppose that Σ is a Lagrange singularity class (in fact, it can be *any* algebraic pure-dimensional cycle in $\mathcal{L}$). Let $i : G \hookrightarrow \mathcal{L}$ be the inclusion, and denote by

$$i^* : H^*(\mathcal{L}, \mathbb{Z}) \to H^*(G, \mathbb{Z})$$

the induced map on cohomology rings. We have to examine the coefficients α_I of the expression

$$i^*[\Sigma] = \sum \alpha_I \ \widetilde{Q}_I(R^*)\,. \tag{25}$$

Let us fix now a strict partition I with $i_1 \le n$. Invoking Proposition 7, the coefficient α_I is equal to

$$i^*[\Sigma] \cdot \Omega_{I'} \tag{26}$$

(intersection in G). Since the cohomology ring and the Chow ring of G are equal, we can compute (26) using the Chow groups. Let

$$C = C_{G\cap\Sigma}\Sigma \subset N_G\mathcal{L}$$

be the *normal cone* of $G\cap\Sigma$ in Σ. Denote by $j : G \hookrightarrow N_G\mathcal{L}$ the zero-section inclusion. By deformation to the normal cone [6, §6.1, §6.2], we have in A_*G the equality

$$i^*[\Sigma] = j^*[C]\,, \tag{27}$$

where i^* and j^* are the pull-back maps of the corresponding Chow groups (see [6]). It follows that

$$\alpha_I = [C] \cdot \Omega_{I'}$$

(intersection in $N_G\mathcal{L}$). The bundle R^* is globally generated; therefore, by Lemma 12, the vector bundle $N_G\mathcal{L}$ is globally generated. The Lagrangian Grassmannian $G = LG(V)$ is a homogeneous space with respect to the action of the symplectic group $Sp(V)$. By Lemma 10, applied to the bundle $N_G\mathcal{L} \to G$, the intersection $[C] \cdot \Omega_{I'}$ is nonnegative.

This ends the proof of Theorem 9.

Remark 13 We may compare jets of Lagrangian manifolds enlarging the number of variables. For a Lagrangian submanifold $L \subset V = W \oplus W^*$, we consider

$$L \oplus \mathbb{C} \subset V \oplus (\mathbb{C} \oplus \mathbb{C}^*)\,.$$

This induces an embedding $\mathcal{L}(W \oplus W^*)$ into $\mathcal{L}(W' \oplus W'^*)$, where $W' = W \oplus \mathbb{C}$. In this way, we obtain a chain of inclusions:

$$\mathcal{L}(V) = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_r \subset \cdots$$

where $\mathcal{L}_r = \mathcal{L}((W \oplus \mathbb{C}^r) \oplus (W \oplus \mathbb{C}^r)^*)$. We say that the singularity type Σ is *closed with respect to suspension* if one can find a sequence of singularity classes $\Sigma_r \subset \mathcal{L}_r$ for $r \geq 0$, such that

$$\Sigma_r = \Sigma_{r+1} \cap \mathcal{L}_r \tag{28}$$

and $\Sigma_0 = \Sigma$. Moreover, we assume that the property (28) holds at the cohomological level, i.e. the restriction $H^*(\mathcal{L}_{r+1}, \mathbb{Z}) \to H^*(\mathcal{L}_r, \mathbb{Z})$ maps the cohomology class $[\Sigma_{r+1}]$ to $[\Sigma_r]$. (This holds when Σ_{r+1} and $\mathcal{L}_r$ intersect transversely along some smooth open and dense subset of Σ_r.)

Suppose now that E is a symplectic manifold, and $E \to B$ is a fibration with Lagrangian fibers. (Such objects have been recently widely investigated. We do not intend here to survey this activity, but refer the reader to, e.g., [10] and the references therein.) Let, in addition, $L \subset E$ be a Lagrangian submanifold. We denote by ρ the restriction of the projection to L. We study the singular points of the projection ρ. The definition of the singularity type (invariance with respect to change of coordinates) allows us to define the singular points of ρ of type Σ. Denote the set of these singular points by $\Sigma(\rho)$. Assume that the singularity type is closed with respect to suspension (see (28)). Then, if the map ρ satisfies suitable transversality conditions, the class $[\Sigma(\rho)] \in H^*(L, \mathbb{Z})$ is equal to $\mathcal{T}^\Sigma$ applied to the virtual bundle $T^*L - \rho^*T^*B$. See also [11, Theorem 2 and §3.1].

Remark 14 In the case of Legendre singularities, using some refinement of our methods, we can get the following positivity result (not as transparent as the one in the Lagrangian case). We adopt the definition of the Thom polynomial of a Legendre singularity class from [11][3], p.730. This definition

[3] Our reference for Legendre singularities and characteristic classes is [11, §3.2].

makes use of the *classifing space of Legendre singularities* (*loc.cit.*). Let ξ be the canonical line bundle on $BU(1)$, appearing in this definition. Let $\mathbf{1}^n$ denote the rank n trivial bundle. The Thom polynomial of a Legendre singularity class can be presented uniquely as a $\mathbb{Z}$-linear combination

$$\sum_{j\geq 0}\sum_{I} \alpha_{Ij}\ \widetilde{Q}_I((T^*L-\mathbf{1}^n)\otimes \xi^{\frac{1}{2}})\cdot t^j\,, \qquad (29)$$

where

$$t := \tfrac{1}{2}c_1(\xi^*) \in H^2(BU(1),\mathbb{Z}[\tfrac{1}{2}])\,,$$

the second sum is over strict partitions I and $\alpha_{Ij}\in\mathbb{Z}$. We then have

$$\alpha_{Ij}\geq 0$$

for any strict partition I and $j\geq 0$. In other words, we get a positivity result after "perturbing" the argument "T^*L" by subtracting a bundle and twisting by some line bundle. If we formally assume that ξ is trivial, then we get the Lagrangian case.

5 Examples

We list now the Thom polynomials of Lagrange and Legendre singularities up to codimension 6. These polynomials were computed in [11] in the basis of square free monomials in the Chern classes of the involved bundles. We present them as $\mathbb{Z}$-linear combinations of the products of the "twisted" $\widetilde{Q}$-functions from (29):

$$\widetilde{Q}_I := \widetilde{Q}_I((T^*L-\mathbf{1}^n)\otimes \xi^{\frac{1}{2}})$$

(note the difference in notation with respect to Eq. (6)), and powers of $t=\frac{1}{2}c_1(\xi^*)$. The terms marked with bold give the Thom polynomials of the corresponding Lagrange singularities.

A_2: $\widetilde{\mathbf{Q}}_{\mathbf{1}}$

A_3: $\mathbf{3\widetilde{Q}_2} + t\widetilde{Q}_1$

A_4: $\mathbf{3\widetilde{Q}_{21}} + \mathbf{12\widetilde{Q}_3} + t\,10\widetilde{Q}_2 + t^2 2\widetilde{Q}_1$

D_4: $\widetilde{\mathbf{Q}}_{\mathbf{21}}$

A_5: $\mathbf{27\widetilde{Q}_{31}} + \mathbf{60\widetilde{Q}_4} + t(22\widetilde{Q}_{21}+86\widetilde{Q}_3) + t^2 40\widetilde{Q}_2 + t^3 6\widetilde{Q}_1$

D_5: $\mathbf{6\widetilde{Q}_{31}} + t\,4\widetilde{Q}_{21}$

A_6: $\mathbf{87\widetilde{Q}_{32}} + \mathbf{228\widetilde{Q}_{41}} + \mathbf{360\widetilde{Q}_5} + t(343\widetilde{Q}_{31}+756\widetilde{Q}_4) + t^2(151\widetilde{Q}_{21}+584\widetilde{Q}_3) + t^3 196\widetilde{Q}_2 + t^4 24\widetilde{Q}_1$

D_6: $\mathbf{12\widetilde{Q}_{32}} + \mathbf{24\widetilde{Q}_{41}} + t\,32\widetilde{Q}_{31} + t^2 12\widetilde{Q}_{21}$

E_6: $\mathbf{9\widetilde{Q}_{32}} + \mathbf{6\widetilde{Q}_{41}} + t\,9\widetilde{Q}_{31} + t^2 3\widetilde{Q}_{21}$

A_7: $\mathbf{135\widetilde{Q}_{321}} + \mathbf{1275\widetilde{Q}_{42}} + \mathbf{2004\widetilde{Q}_{51}} + \mathbf{2520\widetilde{Q}_{6}} +$
$t(7092\widetilde{Q}_5 + 4439\widetilde{Q}_{41} + 1713\widetilde{Q}_{32}) + t^2(3545\widetilde{Q}_{31} + 7868\widetilde{Q}_4) +$
$t^3(1106\widetilde{Q}_{21} + 4292\widetilde{Q}_3) + t^4 1148\widetilde{Q}_2 + t^5 120\widetilde{Q}_1$

D_7: $\mathbf{24\widetilde{Q}_{321}} + \mathbf{120\widetilde{Q}_{42}} + \mathbf{144\widetilde{Q}_{51}} + t(152\widetilde{Q}_{32} + 288\widetilde{Q}_{41}) + t^2 208\widetilde{Q}_{31} + t^3 56\widetilde{Q}_{21}$

E_7: $\mathbf{9\widetilde{Q}_{321}} + \mathbf{60\widetilde{Q}_{42}} + \mathbf{24\widetilde{Q}_{51}} + t(56\widetilde{Q}_{41} + 66\widetilde{Q}_{32}) + t^2 42\widetilde{Q}_{31} + t^3 10\widetilde{Q}_{21}$

P_8: $\mathbf{\widetilde{Q}_{321}}$.

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