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We denote by $H^*(X)$ the cohomology of X with coefficients in $\mathbb{Q}$. Equivariant formality is equivalent to showing the following isomorphism of $H_T^*(*)$-modules

$$H_T^*(X) \cong H_T^*(*) \otimes H^*(X).$$

To prove this we will study the fibration $X \hookrightarrow ET \times^T X \to BT$. The second sheet of Serre's spectral sequence associated to this fibration has groups

$$E_2^{p,q} = H^p(BT; H^q(X)).$$

By assumption, $H^{odd}(X) = 0$ and from the CW decomposition of $BT = (\mathbb{C}P^\infty)^r$ we see that $H^{odd}(BT; H^q(X)) = 0$. Thus each differential in E_2 is zero and so is the case for all E_r – as $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ and $E_r^{p,q}$ is nonzero if and only if both p and q are even, then always either the domain or codomain of d_r is zero. Therefore $E_2 = E_\infty$ and thus $H_T^n(X)$ has a filtration

$$0 \subset F_n^n \subset \cdots \subset F_0^n = H_T^*(X),$$

where $F_p^n / F_{p+1}^n \cong E_\infty^{p,n-p} = E_2^{p,n-p}$. Since we are working with $\mathbb{Q}$-modules, then each of these inclusions split and therefore we can describe $H_T^*(X)$ neatly

$$H_T^*(X) = F_0^n \oplus E_2^{0,n} = F_1^n \oplus E_2^{1,n-1} \oplus E_2^{0,n} = \cdots = \bigoplus_{p+q=n} E_2^{p,q}. \tag{1}$$

Lastly, we need to get a handle on $E_2^{p,q} = H^p(BT; H^q(X))$. For this we use the Universal Coefficients Theorem

$$0 \to \operatorname{Ext}^1_{\mathbb{Q}}(H_{p-1}(BT), H^q(X)) \to H^p(BT; H^q(X)) \to \operatorname{Hom}_{\mathbb{Q}}(H_p(BT), H^q(X)) \to 0.$$

Again, we are working over $\mathbb{Q}$ and all finitely dimensional spaces are projective $\mathbb{Q}$-modules, i.e. the Ext vanish. From the CW decomposition of BT we get that $H_*(X)^\vee \cong H^*(X)$ and thus we obtain

$$H^p(BT; H^q(X)) \cong \operatorname{Hom}_{\mathbb{Q}}(H_p(BT), H^q(X)) \cong H_p(BT)^\vee \otimes H^q(X) \cong H^p(BT) \otimes H^q(X).$$

Plugging this into (1) we get the desired result

$$H_T^*(X) \cong \bigoplus_{n\geq 0} \bigoplus_{p+q=n} E_2^{p,q} \cong \bigoplus_{n\geq 0} \bigoplus_{p+q=n} H^p(BT) \otimes H^q(X) \cong H_T^*(*) \otimes H^*(X).$$