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Problem. *Suppose that $F \to X \to B$ is a fibration and F is a sphere. Write how $E_r^{p,q}$ table looks like for $r \leq \dim F + 1$ and deduce the Gysin long exact sequence.*

Solution:

Let $F = S^n$. Then the Serre spectral sequence starting from the second sheet is given by

$$E_2^{p,q} = H^p(B, H^q(S^n)) = \begin{cases} H^p(B) & , p = 0, n \\ 0 & , \text{otherwise} \end{cases}$$

Notice the only non-zero differential would appear when $r = n+1$, where $d_{n+1}^{p,n} : E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}$, i.e. $E_2 = E_r$ for $r \leq n+1$ and this differential is a map $d^p := d_{n+1}^{p,n} : H^p(B) \to H^{p+n+1}(B)$. On the $(n+2)$-th page we have

$$\begin{aligned} E_{n+2}^{p,n} &= \ker d^p, \\ E_{n+2}^{p,0} &= H^p(B) \text{ for } p \leq n+1, \\ E_{n+2}^{p+n+1,0} &= H^{p+n+1}(B)/\operatorname{im} d^p \text{ for } p \geq 0. \end{aligned}$$

We can put all of the above information into a not-so-long exact sequence for $p \geq 0$

$$0 \to E_{n+2}^{p,n} \to H^p(B) \xrightarrow{d^p} H^{p+n+1}(B) \to E_{n+2}^{p+n+1,0} \to 0$$

After $n+1$ all sheets have zero differentials, so $E_{n+1} = E_\infty$. The induced filtration on $H^*(X)$ thus consists of one nontrivial subgroup (as there are only two rows in the spectral sequence)

$$0 \subseteq E_{n+2}^{p,0} \subseteq H^p(X) \text{ and } H^p(X)/E_{n+2}^{p,0} \cong E_{n+2}^{p-n,n} \text{ if } p \geq n.$$

For $p < n$ the filtration becomes an equality $H^p(X) = H^p(B)$. For $p \geq n$ we get the Gysin long exact sequence:

$$H^n(X) \longrightarrow \dots \longrightarrow H^p(X) \dashrightarrow H^{p-n}(B) \xrightarrow{d^{p-n}} H^{p+1}(B) \dashrightarrow H^{p+1}(X) \longrightarrow \dots$$

$$H^p(X) \twoheadrightarrow E_{n+2}^{p-n,n} \hookrightarrow H^{p-n}(B), \qquad H^{p+1}(B) \twoheadrightarrow E_{n+2}^{p+1,0} \hookrightarrow H^{p+1}(X)$$

Exactness everywhere is trivial as we are using shorter exact sequences to construct this long exact sequence. For $p < n$ te inclusion $E_{n+2}^{p,0} \to H^p(X)$ is an isomorphism, so if we set $H^q(B) = 0$ for $q < 0$, then the same exact sequence holds. These sequences merge at n:

$$\dots \to 0 \to H^n(B) \dashrightarrow H^n(X) \dashrightarrow H^0(B) \xrightarrow{d^0} H^{n+1}(B) \dashrightarrow H^{n+1}(X) \to \dots$$

$$H^n(B) \xrightarrow{\sim} E_{n+2}^{n,0} \hookrightarrow H^n(X), \qquad H^n(X) \twoheadrightarrow E_{n+2}^{0,n} \hookrightarrow H^0(B), \qquad H^{n+1}(B) \twoheadrightarrow E_{n+2}^{n+1,0} \hookrightarrow H^{n+1}(X)$$

Exactness everywhere besides $H^n(B)$ is clear, but even there it is clear since $H^n(B) \to E_{n+2}^{n,0}$ is an isomorphism and not just an epimorphism. Thus we get the full Gysin long exact sequence.

Problem. *Consider the spectral sequence associated to the fibration $T \to ET \times X \to ET \times^T X$ for $T = S^1$. Show that $d_2^{p,1} : H_T^p(X) = E_2^{p,1} \to E_2^{p+2,0} = H_T^{p+2}(X)$ can be identified with the multiplication by the generator of $H_T^2(pt)$. What happens when X is equivariantly formal?*

Solution:

We will work with cohomology with coefficients in $\mathbb{Q}$. Let us first observe that cup product on $H_T^*(X)$ induces a product on E_2, namely $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the cup product in cohomology. We will only need to look at the product $E_2^{p,0} \times E_2^{q,1} \to E_2^{p+q,1}$ – this is the usual cup product on $H_T^*(X)$. All differentials d_2 are derivations under this product on E_2 – in particular $d_2^{p,1} : H_T^p(X) \to H_T^{p+2}(X)$ satisfies the usual Lebniz rule.

Now let $1 \in H_T^0(X)$, but we relabel it as $\iota \in E_2^{0,1}$. Note that for any $x \in E_2^{p,1}$ there exists a (unique) element $y \in E_2^{p,0}$ such that $\iota y = x$ – this is a matter of relabeling as $E_2^{p,0} = E_2^{p,1} = H_T^p(X)$. Thus we can utilize the Lebniz rule for the differentials

$$d_2^{p,1}(x) = d_2^{p,1}(\iota y) = d_2^{0,1}(\iota)y + \iota d_2^{p,0}(y),$$

but $d_2^{p,0} = 0$, so we are left with $d_2^{p,1}(x) = d_2^{0,1}(\iota)y$. Since on the level of $H_T^*(X)$ we have $x = y$, then $d_2^{p,1}$ considered as a map on $H_T^*(X)$ we have

$$d_2^{p,1}(x) = d_2^{0,1}(\iota)x,$$

where $d_2^{0,1}(\iota) \in H_T^2(X)$. We have thus shown that $d_2^{p,1}$ can be identified with multiplication with some degree 2 element of $H_T^*(X)$. There is an obvious commutative diagram of fibrations

$$\begin{array}{ccccc} T & \longrightarrow & ET \times X & \longrightarrow & ET \times^T X \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & ET & \longrightarrow & BT \end{array}$$

From this we get that the structural morhpism for equivariant cohomology $H_T^*(pt) \to H_T^*(X)$ commutes with derivations, i.e. we have

$$\begin{array}{ccc} H_T^*(pt) & \longrightarrow & H_T^*(X) \\ \downarrow{\scriptstyle d_2^{*,1}} & & \downarrow{\scriptstyle d_2^{*,1}} \\ H_T^{*+2}(pt) & \longrightarrow & H_T^{*+2}(X) \end{array}$$

Since the horizontal arrows are ring morphisms, then $d_2^{0,1}(\iota)$ can be regarded as a degree 2 element in $H_T^*(X)$, which comes from $H_T^*(pt)$. Note that to see that this element is in fact the generator of $H_T^*(pt)$ we only need to look at $d_2^{0,1}$ for $X = pt$. Since we are working with $\mathbb{Q}$-cohomology, then for $d_2^{0,1}(1) \in H_T^2(pt)$ to be a generator we only need that it is non-zero. Suppose that $d_2^{*,1} = 0$ then. This would imply that $E_2 = E_\infty$ as these are the only possibly non-zero differentials across all sheets of the Serre spectral sequence associated to $T \to ET \to BT = \mathbb{C}P^\infty$. Because of that we have

$$0 = H^n(ET) = H^n(BT) \oplus H^{n-1}(BT) \neq 0 \text{ for } n > 0$$

which is a contradiction.