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Problem. *The torus $T = (\mathbb{C}^\times)^{n+1}$ acts in the standard way on $\mathbb{P}^n$. Let $h = c_1^T(\mathscr{O}(1)) \in H_T^*(\mathbb{P}^n)$. Using AB-BV formula compute*

$$p_*(h^{n+m}) \in H_T^*(pt) = \mathbb{Z}[t_0, \dots, t_n].$$

Solution:

By the AB-BV formula we have

$$\int_{\mathbb{P}^n} h^{n+m} = \sum_{k=0}^{n} \frac{i_k^*(h)^{n+m}}{e(T_{p_k}\mathbb{P}^n)},$$

where $i_k : \{p_k\} \hookrightarrow \mathbb{P}^n$ is the inclusion of the k-th fixed point. As was shown during the lecture or previous exercises, we have $i_k^*(h) = -t_k$ and $e(T_{p_k}\mathbb{P}^n) = \prod_{i\neq k}(t_i - t_k)$. Note that the k-th summand of the above sum can be rewritten as

$$\frac{i_k^*(h)^{n+m}}{e(T_{p_k}\mathbb{P}^n)} = \frac{(-t_k)^{n+m}}{\prod_{i\neq k}(t_i - t_k)} = (-1)^m \mathrm{Res}_{z=t_k} \frac{z^{n+m}}{\prod_{i=0}^{n}(z - t_i)}.$$

The rest of this solution will be devoted to finding a better closed form for this sum. Let $f(z) = z^{n+m}/\prod_{i=0}^{n}(z - t_i)$, where we consider t_i to be some points in $\mathbb{C}$. We can rewrite f as a series centered at 0

$$f(z) = z^{m-1}\prod_{i=0}^{n}\frac{1}{1-\frac{t_i}{z}} = z^{m-1}\prod_{i=0}^{n}\sum_{j\geq 0}\left(\frac{t_i}{z}\right)^j = \sum_{j\geq 0} S_j(t_0,\dots,t_n) z^{m-j-1}.$$

Thus

$$\int_{\mathbb{P}^n} h^{n+m} = (-1)^m \sum_{k=0}^{n} \mathrm{Res}_{z=t_k}\frac{z^{n+m}}{\prod_{i=0}^{n}(z-t_i)} = (-1)^m \frac{1}{2\pi i}\int_C f(z)dz = (-1)^m S_m(t_0,\dots,t_n).$$

Problem. *Let $L = \mathscr{O}(m), m \geq 0$. Using Riemann-Roch theorem and the localization theorem compute $\chi(\mathbb{P}^n, L)$. Check if it agrees with the formula well known for algebraic geometers:*

$$\chi(\mathbb{P}^n; L) = \dim H^0(\mathbb{P}^n; \mathscr{O}(m)) = \dim \mathbb{C}[t_0,\dots,t_n]_{\deg = m}.$$

Solution:

We adopt notation from the previous solution – $h = c_1^T(\mathscr{O}(1))$ and $i_k : \{p_k\} \hookrightarrow \mathbb{P}^n$ is the inclusion of the k-th fixed point of $\mathbb{P}^n$. We will compute the equivariant Euler characteristic, which first by Riemann-Roch and then by the localization theorem is given by

$$\chi^T(\mathbb{P}^n; L) = \int_{\mathbb{P}^n} \mathrm{td}(T\mathbb{P}^n)\mathrm{ch}(L) = \sum_{k=0}^{n} \frac{i_k^*\left(\mathrm{td}(T\mathbb{P}^n)\mathrm{ch}(L)\right)}{e(T_{p_k}\mathbb{P}^n)}, \tag{1}$$

the denominator is known – $e(T_{p_k}\mathbb{P}^n) = \prod_{i\neq k}(t_i - t_k)$. Since $\mathscr{O}(m) = \mathscr{O}(1)^{\otimes m}$, then the Chern character is given by

$$\mathrm{ch}(L) = (\mathrm{ch}(\mathscr{O}(1)))^m = e^{mh}.$$

Since $h \mapsto -t_k$ under i_k^*, then we get $i_k^*(\mathrm{ch}(L)) = e^{-mt_k}$. The restriction of the Todd class can be calculated in a similar fashion, since a torus representation decomposes into a sum of weight spaces

$$i_k^*(\mathrm{td}(T\mathbb{P}^n)) = \mathrm{td}(T_{p_k}\mathbb{P}^n) = \prod_{i \neq k} \frac{t_i - t_k}{1 - e^{t_k - t_i}}.$$

Plugging these expressions into (1) we get

$$\chi^T(\mathbb{P}^n; L) = \sum_{k=0}^{n} \frac{e^{-mt_k}}{\prod_{i \neq k}(1 - e^{t_k - t_i})}.$$

If we set $s_i = e^{-t_i}$, then we get back the same formula as for the previous exercise

$$\chi^T(\mathbb{P}^n; L) = \sum_{k=0}^{n} \frac{e^{-mt_k}}{\prod_{i \neq k}(1 - e^{t_k - t_i})} = \sum_{k=0}^{n} \frac{s_k^m}{\prod_{i \neq k}(1 - \frac{s_i}{s_k})} = S_m(s_0, \ldots, s_n).$$

Inclusion of the trivial subtorus $\{1\} \hookrightarrow T$ induces a map on cohomology $H_T^*(X) \to H^*(X)$, which sends each t_i to 0. Our characteristic is thus sent to

$$\chi^T(\mathbb{P}^n; L) \mapsto S_m(1, \ldots, 1) = \#\{t_0^{l_0} \ldots t_n^{l_n} : l_0 + \cdots + l_n = m\} = \dim \mathbb{C}[t_0, \ldots, t_n]_{\deg = m}.$$