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First, observe that for if $p_i \in \mathbb{P}^n$ is a fixed point of the standard torus action on the projective space, then the map on equivaraint cohomology is the map $\mathbb{Z}[t_0, \ldots, t_n][h]/\prod_i (h + t_i) \to \mathbb{Z}[t_0, \ldots, t_n]$ that sends $h \mapsto -t_i$. This follows from the fact that h corresponds to the equivariant first Chern class of the tautological bundle on $\mathbb{P}^n$, i.e. it corresponds to the first Chern class of the bundle $ET \times_T \mathscr{O}(1) \to ET \times_T \mathbb{P}^n$. The map on cohomology that comes form inclusion of a fixed point $\{p_i\} \hookrightarrow \mathbb{P}^n$ takes the first Chern class of our twisted tautological bundle to the stalk of this bundle at p_i, which is a torus representation of weight $-t_i$, hence the map $\mathbb{Z}[t_0, \ldots, t_n][h]/\prod_i (h + t_i) \to \mathbb{Z}[t_0, \ldots, t_n]$ sends $h \mapsto -t_i$.

Thus the map induced by $(\mathbb{P}^n)^T \hookrightarrow \mathbb{P}^n$ is the map of all evaluations

$$\mathbb{Z}[t_0, \ldots, t_n][h]/\prod_i (h + t_i) \to \bigoplus_i \mathbb{Z}[t_0, \ldots, t_n].$$

Clearly, the image of this map lies in the set of polynomials $f = (f_0, \ldots, f_n)$ such that $(t_i - t_j)|(f_i - f_j)$ for all i, j. On the other hand, fix a set of polynomials $(f_0, \ldots, f_n)$ such that $(t_i - t_j)|(f_i - f_j)$ for all i, j. By relation given on the cohomology of $\mathbb{P}^n$ we know that any polynomial that is mapped to f is of degree (with respect to h) at most n. Therefore we are looking for polynomials $g_0, \ldots, g_n$ such that

$$w(h) = g_0 + g_1 h + \cdots + g_n h^n \mapsto (f_0, \ldots, f_n).$$

This is a simple polynomial interpolation problem, which can be done with basic linear algebra. Our requirement is equivalent to a system of equations over $\mathbb{Z}[t_0, \ldots, t_n]$

$$\begin{bmatrix} 1 & -t_0 & \ldots & (-t_0)^n \\ \vdots & \vdots & & \vdots \\ 1 & -t_n & \ldots & (-t_n)^n \end{bmatrix} \begin{bmatrix} g_0 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix}$$

The matrix is a Vandermonde matrix, so its determinant is $\prod_{i \geq j}(t_i - t_j)$. By Cramer's rule we know that

$$g_k = \frac{1}{\prod_{i>j}(t_i - t_j)} \begin{vmatrix} 1 & \ldots & (-t_0)^{k-1} & f_0 & (-t_0)^{k+1} & \ldots & (-t_0)^n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \ldots & (-t_n)^{k-1} & f_n & (-t_n)^{k+1} & \ldots & (-t_n)^n \end{vmatrix}.$$

If these turn out to be polynomials, then we are done. This can be easily seen – fix $i > j$. After subtracting the j-th row from the i-th row we can take $t_i - t_j$ out of the determinant.