zadanie

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The quadric in question is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1],$$

The inverse is given by

$$[z_0 : z_1 : z_2 : z_3] \mapsto ([z_0 : z_2], [z_2 : z_3]).$$

Thus, we can view X as $\mathbb{P}^1 \times \mathbb{P}^1$ with the following torus action

$$(t_0, t_1) \cdot ([u_0 : u_1], [v_0 : v_1]) = ([t_0u_0 : t_1^{-1}u_1], [t_1^{-1}v_0 : t_0^{-1}v_1]) = ([t_0t_1u_0 : u_1], [t_0t_1^{-1}v_0 : v_1]).$$

There is a natural $T' = (\mathbb{C}^\times)^4$ action on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$(s_0, s_1, s_2, s_3) \cdot ([u_0 : u_1], [v_0 : v_1]) = ([s_0u_0 : s_1u_1], [s_2v_0 : s_3v_1]),$$

we will distinguish the two tori acting on the different $\mathbb{P}^1$'s by $T' = T_1 \times T_2$. We retrieve our desired torus action via the homomorphism $(t_0, t_1) \mapsto (t_0t_1, 1, t_0t_1^{-1}, 1)$. We can compute the equivariant cohomology ring $H^*_{T'}(\mathbb{P}^1 \times \mathbb{P}^1)$ easily, since this is a product of groups acting on a product of spaces. Hence

$$H^*_{T_1\times T_2}(\mathbb{P}^1 \times \mathbb{P}^1) \cong H_{T_1}(\mathbb{P}^1) \otimes H^*_{T_2}(\mathbb{P}^1).$$

Here we use the following easy fact

Proposition 1. *Let G, H be Lie groups and X and Y be G- and H-spaces respectively. Then we have an isomorphism*

$$H^*_{G\times H}(X \times Y) \cong H^*_G(X) \otimes H^*_H(Y),$$

where $G \times Y$ acts on $X \times Y$ in the natural fashion $(g, h) \cdot (x, y) = (g \cdot x, h \cdot y)$.

Proof. This follows from the fact that if we take universal bundles $EG \to BG$ and $EH \to BH$, then we can construct a universal bundle for $G\times H$ as the product bundle $E(G\times H) = EG\times EH \to BG \times BH = B(G \times H)$. We can therefore describe the twisted product of $X \times Y$ with $E(G \times H)$:

$$(EG \times EH) \times_{G\times H} (X \times Y) \cong (EG \times_G X) \times (EH \times_H Y),$$

where the isomorphism is given by $[(e, f), (x, y)] \mapsto ([e, x], [f, y])$. Thus we can use Künneth formula to compute the equivariant cohomology of the product

$$H^*_{G\times H}(X \times Y) \cong H^*_G(X) \otimes H^*_H(Y),$$

if our cohomology is not, say, over a field, then we would only get an exact sequence, as one usually gets with the Künneth formula. □

In our case equivariant cohomology of $\mathbb{P}^1$ with standard torus action is a free module over the polynomial ring, hence the Tor functors vanish, meaning that we can use the above Künneth formula even for cohomology over $\mathbb{Z}$. Explicitly, if $T = (\mathbb{C}^\times)^2$ acts on $\mathbb{P}^1$ in the standard fashion, then we have

$$H^*_T(\mathbb{P}^1) \cong \mathbb{Z}[t_0, t_1][h]/(h + t_0)(h + t_1).$$

Using this formula together with Proposition 1 we obtain

$$\begin{aligned} H_{T'}(\mathbb{P}^1 \times \mathbb{P}^1) &\cong \mathbb{Z}[t_0, t_1][\zeta]/(\zeta + t_0)(\zeta + t_1) \otimes \mathbb{Z}[t_2, t_3][\xi]/(\xi + t_2)(\xi + t_3) = \\ &= \mathbb{Z}[t_0, t_1, t_2, t_3, \xi, \zeta]/\left((\zeta + t_0)(\zeta + t_1), (\xi + t_2)(\xi + t_3)\right). \end{aligned}$$

Now, the homomorphism of tori $T \to T'$ induces a map on cohomology $H^*_{T'}(\mathbb{P}^1 \times \mathbb{P}^1) \to H^*_T(\mathbb{P}^1 \times \mathbb{P}^1)$; this map evaluates s_i at the weight that T acts on the i-th coordinate. In the end we get the following presentation of $H^*_T(\mathbb{P}^1 \times \mathbb{P}^1)$

$$H^*_T(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}[t_0, t_1, \xi, \zeta]/(\zeta(\zeta + t_0 + t_1), \xi(\xi + t_0 - t_1)).$$

As $\mathbb{P}^1$ has vanishing odd cohomology, then so does $\mathbb{P}^1 \times \mathbb{P}^1$, making it equivariantly formal. Thus it's equivariant cohomology $H^*_T(pt)$-module is free.