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We will start by stating the only one formulation of the Leray-Hirsch theorem, which we will use; is taken straight from Hatcher's "Algebraic Topology":

Theorem 1 (Leray-Hirsch)**.** *Let $F \to E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative ring coefficient R:*

1. *$H^n(F;R)$ is a finitely generated free R-module for each n.*
2. *There exists classes $c_j \in H^*(E;R)$ whose restrictions $i^*(c_j) \in H^*(F;R)$ form a basis of $H^*(F;R)$ for all inclusions of the fiber $i : F \to E$.*

Then the map $\Phi : H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$ given by $b \otimes i^*(c_i) \mapsto p^*(b) \cup c_i$ is an isomorphism.

In our case all cohomologies will be with coefficients in $\mathbb{Q}$ and we are going to abbreviate $H^*(-;\mathbb{Q})$ by $H^*(-)$. We denote by $\gamma_n : E_n \to Gr(k,n)$ the tautological vector bundle on the Grassmannian. We will not prove the fact that $B(U_n) \cong Gr(n,\infty)$ and $H^*(Gr(n,\infty)) \cong \mathbb{Q}[c_1,\dots,c_n]$, where c_i are Chern classes of γ_n; note that the Chern class c_i is of degree $2i$.

First observe that $B(U_k \times U_{n-k}) \cong B(U_k) \times B(U_{n-k}) \cong Gr(k,\infty) \times Gr(n-k,\infty)$, so its cohomology can be computed via the Künneth Formula

$$H^*\left(B(U_k \times U_{n-k})\right) \cong \mathbb{Q}\left[c'_1, \dots c'_k, c''_1, \dots, c''_{n-k}\right],$$

where c'_i are Chern classes of γ_k and c''_i are Chern classes of γ_{n-k}, the grading is given by $\deg(c'_i) = \deg(c''_i) = 2i$. Now that we understand almost all cohomologies that appear in the problem we can turn to Leray-Hirsch and our fibration.

To see the corollary that we have in the hint we can observe that $H^0(B) = \mathbb{Q}$ and so this is the $\mathbb{Q}$ that appears in the tensor product in the hint. The equation in the hint is in fact the same as

$$H^*(F) = H^*(E)/H^{>0}(B) \cdot H^*(E),$$

where the $H^*(B)$-module structure on $H^*(E)$ is of course given by multiplication with pullback $b \cdot e = p^*(b) \cup e$. Thus, tensoring $H^*(E)$ with $H^0(B)$ over $H^*(B)$ nullifies all higher gradations of $H^*(B)$ in $H^*(E)$. The Grassmannian being a complex, compact manifold has finitely dimensional vector spaces as its cohomology, thus the first part of requirements to use Leray-Hirsch is essentially for free. On the product of Grassmannians we can consider product bundles – three bundles worth mentioning here are:

1. Two bundles that arise as products of γ's and trivial bundles

$$\gamma_k \times 0 : E_k \times Gr(n-k,\infty) \to Gr(k,\infty) \times Gr(n-k,\infty),$$
$$0 \times \gamma_{n-k} : Gr(k,\infty) \times E_{n-k} \to Gr(k,\infty) \times Gr(n-k,\infty).$$

   Chern classes of these bundles are exactly the c'_i's and c''_i's that we have seen earlier.

2. The product of two tautological bundles $\gamma_k \times \gamma_{n-k} : E_k \times E_{n-k} \to Gr(k,\infty) \times Gr(n-k,\infty)$. This bundle is also the Whitney sum of the previously mentioned bundles

$$\gamma_k \times \gamma_{n-k} = (\gamma_k \times 0) \oplus (0 \times \gamma_{n-k}).$$

   Now consider the pullback of $\gamma_n : E_n \to Gr(n,\infty)$ with respect to our fibration $\pi : Gr(k,\infty) \times Gr(n-k,\infty) \to Gr(n,\infty)$ – this fibration sends (V,W) to $V \oplus W$, so in fact we have

$$\pi^*\gamma_n = \gamma_k \times \gamma_{n-k}.$$

Over the Grassmannian $Gr(k,n)$ we also have the tautological bundle $\gamma : E \to Gr(k,n)$ and we also have the orthogonal bundle $\overline{\gamma} : \overline{E} \to Gr(k,n)$; their Whitney sum is trivial and so their Chern classes have to be inverse to each other:

$$c(\gamma) \cdot c(\overline{\gamma}) = 1.$$

We also know that $H^*(Gr(k,n))$ has a basis given by Schubert varieties and these correspond to Schur polynomials evaluated at the Chern classes of γ. Inclusion of any fiber $Gr(k,n) \to Gr(k,\infty) \times Gr(n-k,\infty)$ can be thought of fixing an n-dimensional subspace $V \subset \mathbb{C}^\infty$ and mapping $V \supseteq W \mapsto (W, W^\perp)$, where we take the orthogonal complement of W in V. For any subspace $V \subset \mathbb{C}^\infty$ we have the inlusion of the vector subspace and projection onto the subspace itself which give us two maps that factor the identity on V

$$V \xhookrightarrow{\ i\ } \mathbb{C}^\infty \xrightarrow{\ p\ } V, \qquad p \circ i = \mathrm{id}$$

The inclusion gives us a map $Gr(k,n) \to Gr(k,\infty)$ simply mapping $W \subseteq V$ to $i(W) \in Gr(k,\infty)$ and the projection induces a map $Gr(k,\infty) \to \bigsqcup_{i=0}^k Gr(i,n)$ given by $\mathbb{C}^\infty \supseteq W \mapsto p(W) \subseteq V$. Composition $Gr(k,n) \to Gr(k,\infty) \to \bigsqcup_{i=0}^k Gr(i,n)$ is the map given by inclusion $Gr(k,n) \subseteq \bigsqcup_{i=0}^k Gr(i,n)$. On cohomology we get maps

$$\bigoplus_{i=0}^{k} H^*(Gr(i,n)) \to H^*(Gr(k,\infty)) \to H^*(Gr(k,n))$$

Composing these maps results in the natural projection from the direct sum, which is an epimorphism. Hence the map $H^*(Gr(k,\infty)) \to H^*(Gr(k,n))$ must be an epimorpism as well. Same argument for $H^*(Gr(n-k,\infty))$ holds as well, but instead of inclusions of subspaces we would have to consider inclusions of orthogonal complements, but it is not important here.

Now note that the epimorphism $H^*(Gr(k,\infty)) \to H^*(Gr(k,n))$ sends the Chern classes of γ_k to Chern classes of γ (since γ_k pullbacks to γ) and similarly $H^*(Gr(n-k,\infty)) \to H^*(Gr(k,n))$ sends Chern classes of γ_{n-k} to Chern classes of $\overline{\gamma}$. By taking $S_\lambda(c_1,\ldots,c_n) \otimes 1$ we obtain elements of $H^*(Gr(k,\infty) \times Gr(n-k,\infty))$ that restrict to a basis on fibers, hence we can use Leray-Hirsch for our fibration.

In $H^*(Gr(k,n))$ we thus set to 0 the pullbacks $\pi^*(q)$ of any element $q \in H^*(BU_n) = \mathbb{Q}[c_1,\ldots,c_n]$ which is a sum of elements of positive degree. Since π^* is a graded morphism, then this is equivalent to saying that $\pi^*(c_1 + \cdots + c_n) = 0$ in $H^*(Gr(k,n))$. But we have just shown that

$$\pi^*(c_1 + \cdots + c_n) = \pi^*(c(\gamma_n) - 1) = \pi^*(c(\gamma_n)) - 1 = c(\gamma_k \times 0)c(0 \times \gamma_{n-k}) - 1.$$

In other words, in $H^*(Gr(k,n))$ we have only one new relation

$$1 = c(\gamma_k \times 0)c(0 \times \gamma_{n-k}) = (1 + c'_1 + \cdots + c'_k)(1 + c''_1 + \cdots + c''_{n-k}).$$

Thus we get the following presentation of the cohomology ring of the Grassmannian

$$H^*(Gr(k,n);\mathbb{Q}) \cong \mathbb{Q}[c'_1,\ldots,c'_k,c''_1,\ldots,c''_{n-k}]/(c'c''-1),$$

where $c' = 1 + c'_1 + \cdots + c'_k$ and $c'' = 1 + c''_1 + \cdots + c''_{n-k}$.