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1. Find a CW-decomposition of $\mathbb{P}^n$ with the standard action of $(S^1)^{n+1}$.

Let us consider separately the cases $n = 1$ and $n = 2$.

For $n = 1$ we have

$$\mathbb{P}^1 = \{[1 : 0]\} \cup \{[0 : 1]\} \cup \{[a_0 : a_1] : a_0, a_1 \in \mathbb{C}^*\}.$$

We observe that the two distinguished points are fixed under the action of $(S^1)^2$. The remaining set is isomorphic to $\mathbb{C}^*$ via

$$[a_0 : a_1] = [1 : a_1/a_0] \mapsto a_1/a_0.$$

The action of $(S^1)^2$ is given by

$$(\alpha_0, \alpha_1)[1 : a] = [\alpha_0 : \alpha_1 a] = [1 : \frac{\alpha_1}{\alpha_0} a].$$

We see that the stabilizer of any of considered points is the diagonal of $(S^1)^2$ isomorphic to S^1. The orbits are of the form

$$\{[1 : a] : |a| = r\}$$

for any $r > 0$ and correspond to circles on the complex plane. We can now conclude that the 0-skeleton of our decomposition is

$$X_0 = \{D^0_0, D^0_1\} \times (S^1)^2/(S^1)^2$$

(where D^0_0 corresponds to $[1 : 0]$ and D^0_1 to $[0 : 1]$) and the 1-skeleton is

$$X_1 = D^1 \times (S^1)^2/S^1$$

Tu powinien być podany charakter, którego jądrem jest S^1, tzn t_1-t_0

with the obvious inclusion of $\{D^0_1, D^0_2\}$ from X_0 on the boundary of D_1 from X_1.

For $n = 2$ we have

$$\mathbb{P}^2 = \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\} \cup \{[0 : 0 : 1]\} \cup \{[a_0 : a_1 : 0] : a_0, a_1 \in \mathbb{C}^*\} \cup$$

$$\{[a_0 : 0 : a_2] : a_0, a_2 \in \mathbb{C}^*\}] \cup \{[0 : a_1 : a_2] : a_1, a_2 \in \mathbb{C}^*\} \cup \{[a_0 : a_1 : a_2] : a_0, a_1, a_2 \in \mathbb{C}^*\}.$$

We observe that the three distinguished points are fixed under the action of $(S^1)^3$. The action on the next set is given by

$$(\alpha_0, \alpha_1, \alpha_2)[0 : 1 : a_2] = [0 : \alpha_1 : \alpha_2 a_2] = [0 : 1 : \frac{\alpha_2}{\alpha_1} a_2]$$

and one can easily see that the stabilizer of any points is isomorphic to $(S^1)^2$. The same argument goes for the next two sets. Now for the remaining one, we have

character t_2-t_1 , other characters t_1-t_0 and t_2-t_0

$$(\alpha_0, \alpha_1, \alpha_2)[1 : a_1 : a_2] = [1 : \frac{\alpha_1}{\alpha_0} a_1 : \frac{\alpha_2}{\alpha_0} a_2]$$

and we see that, analogously as for $n = 1$, the stabilizer of any point is the diagonal of $(S^1)^3$ and the orbits are isomorphic to spheres in $(\mathbb{C}^*)^2$. So we see that the 0-skeleton is

$$X_0 = \{D^0_0, D^0_1, D^0_2\} \times (S^1)^3/(S^1)^3,$$

the 1-skeleton is

$$X_1 = D^1_{0,1} \times (S^1)^3/(S^1)^2 \cup D^1_{1,2} \times (S^1)^3 \cup D^1_{1,3} \times (S^1)^3/(S^1)^2$$

and the 2-skeleton is

$$X_2 = D^2 \times (S^1)^3/(S^1) \qquad \text{(S^1=diagonal torus)}$$

Now it should be clear how to proceed in the general case. For arbitrary n we have $n+1$ fixed points - these whose only one coordinate is non-zero. Hence we have the 0-skeleton

$$X_0 = \{D_0^0, D_1^0, \ldots, D_n^0\} \times (S^1)^{n+1}/(S^1)^{n+1}.$$

Then we have elements with exactly two coordinates non-zero which gives us $\binom{n}{2}$ components in 1-skeleton:

$$X_1 = \bigcup_{i,j} D_{i,j}^1 \times (S^1)^{n+1}/(S^1)^n$$

and so on until we obtain one component consisting of points with all coordinates non-zero which given us the n-skeleton

$$X_n = D^n \times (S^1)^{n+1}/S^1.$$