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3. Let A be an algebra over a field $\mathbb{F}$ and $X = \operatorname{Spec} A$. Defining an action of $\mathbb{G}_m$ on X is equivalent to defining a $\mathbb{Z}$-gradation on A. Prove this correspondence and generalize it to an action of the algebraic torus $\mathbb{G}_m^r$.

Recall that given a scheme S, in the category of S-schemes an action of a group scheme G on a scheme X is a map

$$\sigma : G \times_S X \to X,$$

satisfying the following two axioms:

1. associativity, i.e.
$$\sigma \circ (\mathrm{Id}_G \times \sigma) = \sigma \circ (m \times \mathrm{Id}_X),$$
where $m : G \times_S G \to G$ is the group law on G;

2. unitality, i.e.
$$\sigma \circ (e \times \mathrm{Id}_X) = \mathrm{Id}_X,$$
where $e : S \to G$ is the identity section of G.

In our case $S = \operatorname{Spec} \mathbb{F}$, $G = \mathbb{G}_m$, $X = \operatorname{Spec} A$ for some $\mathbb{F}$-algebra A. The map σ corresponds to some map of rings

$$A \xrightarrow{\varphi} A \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}] \simeq A[t, t^{-1}].$$

Analyzing further the corresponding maps of rings, we conclude that the associativity axiom says that the maps

$$A \xrightarrow{\varphi} A \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}] \xrightarrow{\varphi \otimes \mathrm{Id}} A[t, t^{-1}, s, s^{-1}],$$
$$A \xrightarrow{\varphi} A \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}] \xrightarrow{t \mapsto ts} A[t, t^{-1}, s, s^{-1}]$$

have to coinside.
Analogically, the unitality axioms says that the map

$$A \xrightarrow{\varphi} A[t, t^{-1}] \xrightarrow{t \mapsto 1} A$$

must be the identity map of A.

Now suppose that we have a $\mathbb{Z}$-gradation $A = \bigoplus_{n \in \mathbb{Z}} A_n$. We can define an $\mathbb{F}$-linear map $\varphi : A \to A \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ by putting

$$\varphi(a) = a \otimes t^n \quad \text{for} \quad a \in A_n.$$

The definition of gradations ensures that it is a map of rings and one can easily check that it satiasfies the desired axioms.
Conversely, suppose that we have a map $\varphi : A \to A \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ satisfying the above axioms. The natural way to proceed is to take

$$A_n = \{a \in A : \varphi(a) = a \otimes t^n\}.$$

Clearly A_n's are additive groups, $A_n \cap A_m = \{0\}$ for $n \neq m$, and $A_n A_m \subseteq A_{n+m}$. It remains to show that $A = \bigoplus_{n \in \mathbb{Z}} A_n$.
For any $a \in A$ there are uniquely determined $a_n, n \in \mathbb{Z}$ such that

$$\varphi(a) = \sum_{n \in \mathbb{Z}} a_n \otimes t^n.$$

Observe that the associativity axioms implies the identity

$$\sum_{n\in\mathbb{Z}} \varphi(a_n)s^n = \sum_{n\in\mathbb{Z}} a_n t^n s^n,$$

so $\varphi(a_n) = a_n \otimes t^n$, i.e. $a_n \in A_n$. Now the unitality axiom implies that

$$a = \sum_{n\in\mathbb{Z}} a_n,$$

which completes the proof.

It is easy to see that in a more general situation $G = \mathbb{G}_m^r$ a group action is equivalent to a $\mathbb{Z}^r$-gradation on A. The above proof works without any change. For a given map $\varphi : A \to A[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ tha associated gradation is given by

$$A_{s_1,\dots,s_n} = \{a \in A : \varphi(a) = a t_1^{s_1} \cdots t_n^{s_n}\}.$$