wykład 13

Goal: Construct 2-fold connected covering of $SO(n)$. The simplyconnected group with lie algebra $so(n)$ is called $Spin(n)$.

$$\mathbb{Z}/2 \hookrightarrow Spin(n) \twoheadrightarrow SO(n)$$

In addition we get important representations of $Spin(n)$ which do not come from $SO(n)$ – "spinors".

Clifford algebra

V – vector space$/k$ $\quad Q: V \to k$ $\quad$ (k any field) char $k \neq 2$

$C(Q) :=$ Tensor algebra of $V / \underset{\sim}{I}$

Tensor algebra $:= k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \cdots = \bigoplus_{k=0}^{\infty} V^{\otimes k}$

(multiplication := concatention of tensors)

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) := x_1 \otimes \ldots \otimes x_n \otimes y_1 \ldots \otimes y_m$$

$\underset{\sim}{I}$ – two sided ideal generated by
$x \otimes x - Q(x)$ for $x \in V$.

(i.e we can simplyfy tensors writing $Q(x) \in k$ instead of x^2.)

Exercise: Formulate the universal property of $C(Q)$ in terms of the embedding $j: V \to C(Q)$.

The ideal is generated by homogeneous elements with respect to $\mathbb{Z}/2$ gradation $0 =$ even $1 =$ odd.

Therefore $C(Q)$ has $\mathbb{Z}/2$ gradation

$$C(Q) = C(Q)^0 \oplus C(Q)^1$$

2)

<u>Example 1</u> $Q \equiv 0$

then I generated by $x \otimes x$.

Hence in $C(Q)$ $x^2 = 0$ for $x \in V$

$$(x+y)^2 = x^2 + xy + yx + y^2$$

$$0 = xy + yx.$$

<u>Claim</u> $C(Q) = \Lambda^\cdot(V)$ - exterior algebra.

<u>Example 2</u> $K = \mathbb{R}$, $\dim V = 1$ $V = \mathbb{R}$ $Q(x) = -x^2$

Then in $C(Q)$ $j(e)^2 = -1$

<u>Claim</u> $C(Q) = \mathbb{C}$.

<u>Example 3</u> $K = \mathbb{R}$, $\dim V = 2$ $V = \mathbb{R}^2$ $Q = -$standard

$e_1 := j(e_1)$ $j(e_1)^2 = -1$

$e_2 := j(e_2)$ $j(e_2)^2 = -1$

$$(e_1 + e_2)^2 = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2$$

$$2 = 1 + e_1 e_2 + e_2 e_1 + 1$$

$$e_1 e_2 = -e_2 e_1$$

<u>Claim</u> $C(Q) = \mathbb{H}$.

The most interesting for us:

$C_n = C(Q)$ when $K = \mathbb{R}$, $V = \mathbb{R}^n$ $Q = -$standard

C_n generated by e_i $i = 1 \cdots n$ with relations $e_i^2 = -1$ $e_i e_j = -e_j e_i$ for $i \neq j$.

<u>Theorem</u> For <u>any</u> Q the dimension of $C(Q)$ is 2^n where $n = \dim V$. The additive basis of $C(Q)$ is indexed by increasing sequences $i_1 < i_2 \ldots < i_k$ $k \in n$ i.e. the subsets of $\{1 \ldots n\}$

3).

It is convenient to prove

Lemma If $V = V_1 \oplus V_2$ $Q = Q_1 + Q_2$ where Q_i a form on V_i,

then
$$C(Q) \simeq C(Q_1) \otimes C(Q_2)$$

and elements from $C(Q_1)$ super-commute with elements from $C(Q_2)$, i.e
$$(a \otimes 1)(1 \otimes b) = (-1)^{|a| \cdot |b|}(1 \otimes b)(a \otimes 1)$$
where $|a| = 0$ or 1 depending on the parity, i.e gradation of a.

Proof In $C(Q)$ we have for $x \in V_1$, $y \in V_2$
$$(x+y)^2 = x^2 + xy + yx + y^2$$
$$Q(x+y) = Q(x) + xy + yx + Q(y)$$
$$Q_1(x) + Q_2(y) = Q_1(x) + xy + yx + Q_2(y)$$
$$xy = -yx$$
$$xy = (-1)^{|x||y|} yx$$

Therefore we get an algebra map
$C(Q_1) \otimes C(Q_2) \longrightarrow C(Q)$
(map defined on the generators which agree on the relations)

The inverse map also defined on the generators $V = V_1 \oplus V_2$. (Here we use the universal property)

Proof of the theorem: We assume that Q decomposes into direct sum of 1-dimensio-

4) and forms (we do not have to pass to a field extension). So it is enough to calculate $C(Q)$ for
$V = k \quad Q(x) = ax^2$
Then $C(Q) = \text{lin}(1, e) \quad e = j(1)$ ($\underline{\dim C(Q) = 2}$)
with multiplication $e^2 = a$. ⊠

<u>Important maps in $C(Q)$</u>

- $V \to V \quad x \to -x$ induces an algebra map $\alpha : C(Q) \to C(Q)$ ($C(-)$ is a functor)
$$\alpha \circ \alpha = id$$
this is an algebra homomorphism
- An antihomomorphism:
$$\varphi(x) \cdot \varphi(y) = \varphi(yx)$$
i.e. a homomorphism
$$t : C(Q) \to C(Q)^{op}$$
such that for $x \in V$
$$t(x) = x \qquad (t(x_1 x_2 \cdots x_k) = x_k x_{k-1} \cdots x_2 x_1)$$
well defined since defined on the generators and preserves the relation:
$$t(x \cdot x) \overset{?}{=} t(Q(x))$$
$$t(x) \cdot t(x) \overset{?}{=} Q(x)$$
$$x^2 = Q(x).$$
(Again one can use the universal property of $C(Q)$)
- "conjugation" $t\alpha = \alpha t$ denoted by $x \to \bar{x}$
this is an antihomomorphism:
for $C_1 = \mathbb{C} \quad i \mapsto -i$
$C_2 = \mathbb{H} \quad i \mapsto -i, \; j \mapsto -j \quad k = ij \mapsto (-j)(-i) = ji = -k$

5) α, t and conjugation are involutions.

Let $C(Q)^*$ - invertible elements in $C(Q)$

$$\Gamma(Q) = \{ x \in C(Q)^* : \alpha(x) v x^{-1} \in V \text{ for } v \in V \}$$

$\Gamma(Q)$ called "Clifford group" it a subgroup of $C(Q)^*$ (x preserves $V \Rightarrow x^{-1}$ preserves V)

$$N : C(Q) \to C(Q)$$

$$x \longmapsto x \cdot \bar{x} \quad \text{the norm}$$

$N(x) = -Q(x) \cdot \mathbb{1}$ for $x \in V$.

Later we will show that for $x \in \Gamma(Q)$ $N(x) \in \mathbb{R}^* \subset C(V)$ for $\mathbb{R}^n$ with -standard form.

<u>From now on</u> $Q(x) = -$standard

$$C_n = \underbrace{\mathbb{C} \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}}_{n} \supset \Gamma_n = \Gamma(Q)$$

Γ_n acts on $V = \mathbb{R}^n$ by definition

$$\Gamma_n \xrightarrow{\rho} Gl(\mathbb{R}^n)$$

<u>Lemma</u> $\ker(\rho) = \mathbb{R}^*$

<u>Proof</u> if $x \in \ker \rho$, i.e. $\forall v$ $\alpha(x) v x^{-1} = v$

$$\alpha(x) v = v x$$

We decompose $x = x^0 + x^1$ $x^i \in C_n^i$ $i = 0,1$

$$x^0 v = v x^0 \qquad -x^1 v = v x^1$$

$x^0 = a^0 + e_1 b^1$ a^0, b^1 contain no e_1 in the canonical basis.

$v := e_1$

$$(a_0 + e_1 b^1) e_1 = e_1 (a_0 + e_1 b^1)$$

6)

$$a_0 e_1 + e_1 b' e_1 = e_1 a^0 + e_1^2 b'$$

$$\cancel{e_1 a^0} - e_1^2 b' = \cancel{e_1 a^0} + e_1^2 b' \qquad e_1^2 = -1 \text{ , super-commutation}$$

$$b^1 = -b^1$$

$$b_1 = 0$$

Hence X_0 contains no factor e_1, ... etc.

$x_0 \in \mathbb{R}^*$.

The same for x': $x' = a' + e_1 b^0$, $v := e_1$

$$-(a' + e_1 b^0) e_1 = e_1 (a' + e_1 b^0)$$

$$-a' e_1 - e_1 b^0 e_1 = e_1 a' + e_1^2 b^0$$

$$\cancel{e_1 a'} - e_1^2 b^0 = \cancel{e_1 a'} + e_1^2 b^0$$

$$+b^0 = -b^0$$

$$b^0 = 0$$

So x_1 has no factor with e_1.

At the end: $x \in \mathbb{R} \cap \Gamma_n \subset \mathbb{R} \cap C_n^* = \mathbb{R}^*$

<u>Lemma</u> If $x \in \Gamma_n$ then $N(x) \in \mathbb{R}^*$

<u>Proof</u> we check that $N(x) \in \ker \rho$

(note $\alpha(x^{-1}) \in \Gamma_n$ so $t(x^{-1} \cdot v \cdot \alpha(x)) \in V$

hence

$$t\,\alpha(x) v t(x)^{-1} \in V$$

$$\alpha\, t(x) v t(x)^{-1} \in V \implies t(x) \in \Gamma_n$$

Therefore $N(x) = x \cdot \bar{x} = x \cdot t\,\alpha(x) \in \Gamma_n$.

Now: by the definition of Γ_n

(1) $\alpha(x) v x^{-1} = w \in \mathbb{R}^n \quad | \ t$

$$t(\alpha(x)\ v\ x^{-1}) = tw = w$$

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(2) $t(x^{-1})\, v\, t\alpha(x) = w$

(1) $\wedge$ (2) $\alpha(x)\, v\, x^{-1} = t(x^{-1})\, v\, t\alpha(x)$

$t(x)\alpha(x)\, v\, x^{-1}(t\alpha(x))^{-1} = v$

$\alpha(\alpha t(x)\cdot x)\, v\, (t\alpha(x)\cdot x)^{-1} = v$

$\alpha(\bar{x}\cdot x)\, v\, (\bar{x}x)^{-1} = v$

the same holds for $x := \bar{x}$

$\alpha(\underbrace{x\cdot\bar{x}}_{N(x)})\, v\, (\underbrace{x\bar{x}}_{N(x)})^{-1} = v$

Hence $N(x) \in \operatorname{Ker}\rho = \mathbb{R}^*$ ☒

Lemma $N : \Gamma_n \to \mathbb{R}^*$ is a homomorphism

Proof $N(xy) = xy\cdot\overline{xy} = xy\,\bar{y}\,\bar{x} = x\underbrace{N(y)}_{\in\,\mathbb{R}^*}\bar{x}$

$= x\bar{x}\,N(y) = N(x)N(y)$ ☒

Also $N(\alpha(x)) = N(x)$.

The elements of $V - \{0\}$ belong to Γ_n and they induce reflections of $V = \mathbb{R}^n$.

$x = r e_1$ $r > 0$ (we can change the coordinates)

Then $\rho(x) \in Gl(\mathbb{R}^n)$

$e_1 \mapsto \alpha(re_1)\, e_1\, (re_1)^{-1}$

what is $(re_1)^{-1} = r^{-1}e_1^{-1}$? $\quad \underset{e_1^{-1}}{(-e_1)}\cdot e_1 = -e_1^2 = 1$

$e_1 \longmapsto \underbrace{r(-e_1)}_{\alpha(e_1)}\cdot e_1\, \underbrace{r^{-1}(-e_1)}_{e_1^{-1}} = +e_1^3 = -e_1$

and for $v \in e_1^{\perp}$

$v \mapsto r(-e_1)\, v\, r^{-1}(-e_1) = -v\, r(-e_1)r^{-1}(-e_1)$

$= v$ ☒

Theorem Let $Pin(n) = \ker(N)$ then $\rho : Pin(n) \to O(n)$ is surjective with the kernel ± 1.

Proof We have to check that $\rho(x)$ preserves the norm in $\mathbb{R}^n$.

Let $v = \mathbb{R}^n$, then $v \in T_n$ and

$$N(\rho(x)(v)) = N(\alpha(x)\, v\, x^{-1}) = N(\alpha x) N(v) N(x^{-1}) = N(v).$$

But $N(v) = |v|^2$ for $v \in \mathbb{R}^n$.

ρ is surjective since $O(n)$ is generated by the reflections.

The kernel: $\ker \rho \cap \ker N = \{ r \in \mathbb{R}^* : r^2 = 1 \}$ ∎

Definition $Spin(n) \subset Pin(n)$ is defined as the inverse image of $SO(n)$.

We can also define $Pin(n)$ as a subgroup of $C^*(Q)$ generated by the vectors of the lenght 1 and $Spin(n) = Pin(n) \cap C_n^0$ (even elements). With this definition it is not clear that $Spin(n)$ is a closed subgroup.

Exceptional case $n = 1$ $Pin(1) = \{1, -1, i, -i\} \subset \mathbb{C}$

Theorem $Spin(n)$ for $n \geq 2$ is a double covering, nontrivial.

9) We have to find a path connecting 1 with $-1 \in C_n$.

$t \mapsto \cos(t) + \sin(t)\, e_1 \cdot e_2 \qquad t \in [0, \pi]$

check $(\omega(t))^{-1} = \bar{\omega}(t) = \cos(t) - \sin t\, e_1 e_2$

(we can think that $n = 2$ $C_2 = \mathbb{H}$ $e_1 \cdot e_2 = k$)
$e_1 = i, \; e_2 = j$

$\rho(\omega(t))\, e_1 = (\cos(t) + \sin(t) e_1 e_2)\, e_1\, (\cos t - \sin(t)\, e_1 e_2)$

i.e. $\mathbb{H}$: $(\cos t + \sin t \cdot k) \cdot i\, (\cos t - \sin t\, k)$

$$= (\underbrace{\cos^2 t - \sin^2 t}_{\cos 2t})\, i + (\underbrace{2 \sin t \cos t}_{\sin 2t})\, j$$

$\rho(\omega(t))\, e_2 = (\cos t + \sin t\, k)\, j\, (\cos t - \sin t\, k)$

$$= (\cos^2 t - \sin^2 t)\, j - 2 \sin t \cos t\, i$$

$\rightsquigarrow$ rotation by $2t$ in $\mathrm{lin}(e_1, e_2)$.*

* One can skip this calculation: everything happens in $\mathrm{Spin}(2) = S^1 \subset \mathrm{lin}(1, k) \subset \mathbb{H}$

(Spin(n) is the universal covering of SO(n))

$\mathrm{Spin}(n) \subset C_n^0$ comes with some complex representations which extend to representations of C_n^0, but are not induced by representations of $SO(n)$.

We have to distinguish 2 cases
$n =$ even or odd

We consider complexification of V, $C_n := C_n(V_{\mathbb{C}})$
If n is even we can assume that

$V_{\mathbb{C}} = W \oplus W^*$, the form $Q(w + \xi) = -2\xi(w)$, i.e. with matrix $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

(10) <u>Lemma</u> $C_{2n} \simeq \mathrm{End}(\underbrace{\Lambda^{\cdot} W}_{\substack{\text{exterior algebra} \\ \text{considered as a vector space}}})$ isomorphism of algebras.

<u>Proof</u> To define $C_n \to \mathrm{End}(\Lambda^* W)$ it is enough to define maps $V = W \oplus W^* \to \mathrm{End}(\Lambda^{\cdot} W)$ satisfying the relations defining $C(V)$:

for $w \in W$ (multiplication)

$$L_w : \Lambda^{\cdot} W \to \Lambda^{\cdot} W$$
$$a \mapsto w \wedge a$$

for $\xi \in W^*$ $D_\xi : \Lambda^{\cdot} W \to \Lambda^{\cdot} W$ a derivation

$\boxed{\xi \mapsto 2 D_\xi}$
$$\begin{cases} D_\xi(1) = 0 \\ D_\xi(w) = \xi(w) & w \in W \end{cases}$$

+ Leibniz formula to extend to ΛW.

We check $L_w^2 = 0$ $D_\xi^2 = 0$

$$L_w \circ 2D_\xi + 2D_\xi \circ L_w = 2\,\xi(w) \cdot \mathrm{Id}_{\Lambda W}$$

(Use explicit formula
$$D_\xi(w_1 \wedge \dots \wedge w_r) = \sum_{i=1}^{r} (-1)^{i-1} \xi(w_i)\, w_1 \wedge \dots \overset{i}{\vee} \wedge w_r \Big)$$

The map $C_{2n} \to \mathrm{End}(\Lambda^* W)$ is iso – – check on the basis that it is mono.

$\dim C_{2n} = 2^{2n}$ $\quad \dim \mathrm{End}(\Lambda^* W) = \dim(\Lambda^* W)^2 = (2^n)^2$

The vector space $\Lambda^{\cdot} W$ is denoted by S and called spinor space.

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Note

$$C^0_{2n} \simeq \mathrm{End}(\overbrace{\Lambda^{even} W}^{S^+}) \oplus \mathrm{End}(\overbrace{\Lambda^{odd} W}^{S^-})$$

The action of $Spin(2n)_{\mathbb{C}}$ preserves this decomposition $\dim S^{\pm} = 2^{n-1}$

$\dim V$ odd $\qquad V = W \oplus W^* + U$

The matrix of Q $\qquad \begin{bmatrix} 0 & I & \\ I & 0 & \\ & & 1 \end{bmatrix}$ $\qquad \dim W = n$, $\dim U = 1$

Theorem

$$C_{2n+1} = C(Q) \simeq \mathrm{End}(\Lambda^* W) \oplus \mathrm{End}(\Lambda^* W)$$

As before L_w, $2D_z$, aditionally we have the action of $u \in U$ $|u| = 1$.

We have two structures of $C(Q)$-module by letting $u \mapsto \pm \varepsilon$, where

$$\varepsilon = \begin{cases} id & \text{on } \Lambda^{even} W \\ -id & \text{on } \Lambda^{odd} W \end{cases}$$

The projections onto factors are isomorphisms $C(Q)^0 \to \mathrm{End}(\Lambda^* W)$.

Both representations of $Spin(2n+1)$ are isomorphic (we will compute the weights).

Again $\Lambda^* W$ is denoted by S and is called spinors. This time S does not split, it is irreducible.

Appendix

8 - Periodicity of Clifford algebras

n	$C_n = C(\mathbb{R}^n, -\text{standard})$
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}[2]$
5	$\mathbb{C}[4]$
6	$\mathbb{R}[8]$
7	$\mathbb{R}[8] \oplus \mathbb{R}[8]$

$$C_{n+8} = C_n \otimes \mathbb{R}[16]$$

$A[n]$ denotes the matrix algebra with coefficients in A.

This periodicity is related to Real Bott periodicity.