Mathematical Models and Methods in Applied Sciences
Vol. 18, No. 7 (2008) 1073–1092

ON NON-NEWTONIAN FLUIDS WITH A PROPERTY OF RAPID THICKENING UNDER DIFFERENT STIMULUS

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Received 7 May 2007
Revised 14 October 2007
Communicated by N. Bellomo

The paper concerns the model of a flow of non-Newtonian fluid with nonstandard growth conditions of the Cauchy stress tensor. Contrary to standard power-law type rheology, we propose the formulation with the help of the spatially-dependent convex function. This framework includes e.g. rapidly shear thickening and magnetorheological fluids. We provide the existence of weak solutions. The nonstandard growth conditions yield the analytical formulation of the problem in generalized Orlicz spaces. Basing on the energy equality, we exploit the tools of Young measures.

Keywords: Non-Newtonian fluids; magnetorheological fluids; shear thickening fluids; Orlicz spaces; modular convergence; energy equality; Young measures.

AMS Subject Classification: 35K55, 35Q35, 46E30

1. Introduction

1.1. *Physical motivation and formulation of the model*

Our interest is directed to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under different stimulus, like the shear rate, magnetic or electric field. There is a wide range of possible applications of such fluids in numerous branches of industry, military and natural science, which entitled the intense research in this direction in the last years. We recall various potential applications and describe precisely the behavior of the fluids.

The recent development of advanced body armor is concerned with the so-called *liquid body armor* — a solution, which will provide the armor to be both flexible and lightweight. The improvement consists in soaking the existing armor materials with special fluids, cf. Ref. 10. The two types of fluids used for liquid body armor are of our interest: magnetorheological fluids and shear thickening fluids (STF). Although both solutions are on the level of laboratory research, the application of STF for combat was planned by the end of 2007 and magnetorheological fluids will require another few years until the technology is fully developed. Their common feature is that they are both colloids and consequently react strongly in response to a stimulus. Thus using them to impregnate e.g. kevlar armor provides that far few layers of kevlar are necessary which improves flexibility of the protection and reduces the weight significantly. Typically, to obtain the effect of bullet resistance in kevlar armor, one needs between 20 and 30 layers of kevlar, which resolves in vests of weight more than 4.5 kg, sometimes even increased by a ceramic insert for improved protection. Such a protection is obviously limited only to chest and head and limits the soldier's mobility. Contrary to this solution, the kevlar material soaked with the described fluids has an ability to transfer from flexible to completely rigid. After the stimulus causing the change in viscosity is removed, the substance retains its fluid state and the flexibility of the armor is regained. The first type of fluids — the magnetorheological fluids — consists of ferrous particles, usually spheres or ellipsoids, dispersed in oil. They form 20 to 40 percent of the fluid's volume and measure 3–10 microns. They are distributed randomly, see Fig. 1.

Nevertheless, their influence on the behavior of the fluid is significant. Once the fluid is exposed to magnetic field, the iron particles form the chains or column-like structures parallel to the applied magnetic field, which consequently hinder the movement of the fluid in the direction perpendicular to the magnetic field. The rheological properties of the fluid, like the viscosity or shape, change rapidly within ca. 0.02 second. One can easily observe the anisotropic character of the fluid when the magnetic field is applied. The schematic behavior of magnetorheological fluids exposed to the magnetic field is presented at Fig. 2.

Another type of fluids is shear thickening fluids, which increase their viscosity under the increase of the shear stress, like the impact of a bullet, knife or a needle. The fluid is capable of transferring from liquid to solid rapidly within a few miliseconds. It consists of tiny particles suspended in a liquid, e.g. small particles of silica in polyethylene glycol. They slightly repel each other, so they are able to float easily throughout the liquid. Once the high shear stress is applied, the repulsive

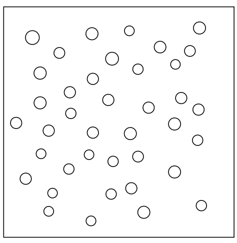

Fig. 1. The magnetorheological fluid without magnetic field.

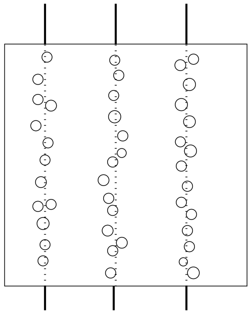

Fig. 2. The chains of particles aligned along the lines of magnetic field.

forces among the particles are overwhelmed and the particles aggregate forming the so-called hydroclusters.

The described liquids improve the resistance of materials to the bullet. However, the most remarkable improvement appears in the resistance to the needle or knife puncture. Thereby the studies on protective materials that prevent hypodermic needle puncture are of high interest for medical stuff and veterinarians. For the purpose of integrating theses materials into e.g. gloves, they must be thin and flexible, which is provided by application of shear thickening fluids, cf. Ref. 8.

The seismic protection is another branch of application of magnetorheological fluids, cf. Ref. 4. The devices called magnetorheological (MR) dampers play a role of shock absorbers for buildings. The MR dampers are filled with a fluid that includes suspended iron particles, hence they lessen the shaking by becoming solid. Once the vibrations are detected, the magnetic field is activated through the MR dampers and all of the iron particles align resulting in the increase of the viscosity of the fluid.

The magnetorheological fluids are used in production of automotive advances, like suspension system, clutches or crash-protection systems. Although these solutions cannot be found in casual cars yet, but the users of e.g. Cadillac Seville or Audi TT can recognize the dynamical shock absorbers in their suspension.

The behavior of the described fluids is non-Newtonian. Thus the flow is captured by the equations

$$\begin{aligned} v_t + v \cdot \nabla v - \operatorname{div} S(x, Dv) + \nabla p = f \quad &\text{in } Q, \\ \operatorname{div} v = 0 \quad &\text{in } Q, \end{aligned} \tag{1.1}$$

where $v : Q \to \mathbb{R}^d$ denotes the velocity field, $p : Q \to \mathbb{R}$ the pressure, $f : Q \to \mathbb{R}^d$ the given body forces, S the Cauchy stress–tensor,[a] $\Omega \subset \mathbb{R}^d$ is a bounded domain and we denote by $Q = (0, T) \times \Omega$ with some given $T > 0$ and $Dv = \frac{1}{2}(\nabla v + \nabla^{\mathrm{T}} v)$.

[a]We call S the Cauchy stress–tensor, however formally by the Cauchy stress–tensor one should mean the whole term $S(x, Dv) + pI$.

The standard growth conditions of the Cauchy stress–tensor, namely polynomial growth, see e.g. Refs. 11 and 12, i.e.

$$\begin{aligned} &|S(x,\xi)| \le c(1+|\xi|)^{q-1}, \\ &S(x,\xi)\cdot\xi \ge c|\xi|^q, \end{aligned} \tag{1.2}$$

cannot capture the described situation. Note that the fluids of power-law type rheology can be characterized by the constitutive relation $S(Dv) = |Dv|^{q-2}Dv$. The property of shear thickening is described by the case $q > 2$. However, we want to describe the processes where the growth is faster than polynomial and possibly different in various components of the shear rate. We are not aiming to provide explicit constitutive relation for the stress–tensor S. We formulate the system of conditions for S and provide examples of some functions satisfying these conditions. Therefore we will formulate analogue conditions to (1.2) with the help of a general convex function, the so-called *N-function*. A function $M(x,\xi)$, $M : \Omega \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}_+$ is called an N-function if it is a convex (w.r.t. ξ) Carathéodory function (i.e. measurable function of x for all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$ and continuous function of ξ for a.a. $x \in \Omega$) such that $M(x,\xi) = 0$ only if $\xi = 0$, and $M(x,\xi) = M(x,-\xi)$ a.e. in Ω. Moreover, $\lim_{|\xi|\to 0}\sup_{x\in\Omega}\frac{M(x,\xi)}{|\xi|} = 0$ and $\lim_{|\xi|\to\infty}\inf_{x\in\Omega}\frac{M(x,\xi)}{|\xi|} = \infty$. The complementary function M^* to a function M is defined by

$$M^*(x,\eta) = \sup_{\xi\in\mathbb{R}^{d\times d}_{\rm sym}} (\xi\cdot\eta - M(x,\xi))$$

for $\eta \in \mathbb{R}^{d\times d}_{\rm sym}$ and a.a. $x \in \Omega$. The complementary function M^* is again an N-function.

Now we are ready to formulate the assumptions on the Cauchy stress–tensor $S : \Omega \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$:

(S1) $S(x,\xi)$ is a Carathéodory function and $S(x,0) = 0$.

(S2) There exist positive constants c, α_1, α_2, an N-function M and its complementary function M^* and an integrable non-negative function $m(x)$ such that for all $\xi \in \mathbb{R}^{d\times d}_{\rm sym}$ and a.a. $x \in \Omega$ it holds

$$S(x,\xi)\cdot\xi \ge c\{M(x,\alpha_1\xi) + M^*(x,\alpha_2 S(x,\xi))\} - m(x). \tag{1.3}$$

(S3) S is strictly monotone, i.e. for all $\xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{\rm sym}$, $\xi_1 \ne \xi_2$ and a.a. $x \in \Omega$

$$[S(x,\xi_1) - S(x,\xi_2)]\cdot[\xi_1 - \xi_2] > 0.$$

In the above formulation, the condition (S2) captures the situation, when S grows fast w.r.t. the shear rate which allows to describe the effects of (rapidly) shear thickening fluids. The dependence on x provides the possibility of considering the influence of magnetic (or electric) field into the system. The other significant issue is that function M depends on the whole tensor Dv, not only on its absolute value. This generality allows to include some effects exhibited e.g. by magnetorheological fluids. As we mentioned before, the application of the magnetic field results in the formation of chains oriented in the direction of the magnetic field. Due to this orientation of the particles chains, the properties of the fluid (viscosity) changes differently in the direction perpendicular and parallel to the vector of magnetic field. The monotonicity condition (S3) is natural. It follows automatically in the case of stress–tensors having strictly convex potential (see e.g. Ref. 12).

Let us briefly mention some examples of constitutive relations which are captured by (S1)–(S3), but cannot be described by (1.2). The first example is

$$S_1(x,\eta) = (\eta\mathcal{E}(x)\eta)^{\alpha_1}\eta + (\eta\mathcal{G}(x)\eta)^{\alpha_2}\eta$$

with some constants $1 < \alpha_1 < \alpha_2$, whereas

$$\mathcal{E}(x) = (e_{ijkl}(x))_{i,j,k,l=1}^d, \quad \mathcal{E}\colon\Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

and

$$\mathcal{G}(x) = (g_{ijkl}(x))_{i,j,k,l=1}^d, \quad \mathcal{G}\colon\Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

satisfy for all $i,j,k,l = 1,\ldots,d$ and for all $\eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, some constant $c > 0$ and a.a. $x \in \Omega$ the conditions

$$\begin{aligned} \mathcal{E}(x) &> 0, & \mathcal{G}(x) &\geq 0,\\ e_{ijkl}(x) &= e_{klij}(x), & g_{ijkl}(x) &= g_{klij}(x),\\ e_{ijkl}(x) &= e_{jikl}(x), & g_{ijkl}(x) &= g_{jikl}(x). \end{aligned}$$

As other examples we mention

$$S_2(x,\eta) = \eta\mathcal{E}(x)e^{(\eta\mathcal{E}(x)\eta)}$$

or

$$S_3(x,\eta) = \alpha(x)\eta\mathcal{E}(x)(\eta\mathcal{E}(x)\eta)^{\alpha(x)-1}$$

with $\alpha\colon\Omega \to (1,\infty)$ and $\mathcal{E}(x)$ defined as for S_1.

Note that some authors, cf. Refs. 2 and 23, claim that one of the possible directions of describing the magnetorheological fluids is the Bingham-type response having variable yield strength (or Herschley–Bulkley fluids, see e.g. Ref. 13). This issue needs more general mathematical tools, which include implicit relations between the Cauchy stress and the symmetric part of the velocity gradient, cf. Ref. 17. The promising direction for such studies would be a generalization of the results presented in Ref. 7. However, we will not consider this approach here.

1.2. *Function spaces and main result*

The growth conditions (1.2) naturally impose the formulation of the problem in L^q spaces. The appropriate spaces to capture problem (1.1) with conditions (S1)–(S3) are the Orlicz spaces. In particular, since we allow the stress–tensor to depend on x, the generalized Orlicz spaces, often called *Orlicz–Musielak spaces*, cf. Ref. 16, are adequate. Let us introduce some notions. Our notation is already adjusted to further application to the considered situation.

The generalized Orlicz class $\mathcal{L}_M(Q)$ is the set of all measurable functions $\xi{:}\mathcal{Q} \to \mathbb{R}^{d\times d}_{\rm sym}$ such that

$$\int_Q M(x,\xi)dxdt < \infty.$$

By $L_M(Q)$ we denote the generalized Orlicz space which is the set of all measurable functions $\xi : Q \to \mathbb{R}^{d\times d}_{\rm sym}$ which satisfy

$$\int_Q M(x,\lambda\xi(x))dxdt \to 0 \quad \text{as } \lambda \to 0.$$

The generalized Orlicz space is a Banach space with respect to the Orlicz norm

$$\|\xi\| = \sup\left\{|\eta\xi| : \eta \in L_{M^*}(Q), \int_Q M^*(x,\eta)dxdt \le 1\right\}$$

or the equivalent Luxemburg norm

$$\|\xi\| = \inf\left\{\lambda > 0{:}\ \int_Q M\left(x,\frac{\xi}{\lambda}\right)dxdt \le 1\right\}.$$

By $E_M(Q)$ we denote the closure of all bounded functions in $L_M(Q)$. The space $L_{M^*}(Q)$ is the dual space of $E_M(Q)$.

The functional

$$\rho(\xi) = \int_Q M(x,\xi(x))dxdt$$

is a modular, see e.g. Ref. 16 for definition. We will say that a sequence $\{z^j\}$ converges modularly to z in $L_M(Q)$ if there exists $\lambda > 0$ such that

$$\int_Q M\left(x,\frac{z^j - z}{\lambda}\right)dxdt \to 0.$$

We will use the notation $z^j \xrightarrow{M} z$ for the modular convergence in $L_M(Q)$. The analytical properties of the spaces with M dependent on a vector-valued argument ξ, not only on the absolute value $|\xi|$, were extensively studied in Ref. 20 and also in Refs. 9 and 22.

We are interested in the case of rapidly growing N-functions where the so-called Δ_2-condition is not satisfied. We say that an N-function M satisfies Δ_2-condition if for some non-negative, integrable in Ω function h and a constant $k > 0$

$$M(x,2\xi) \le kM(x,\xi) + h(x) \quad \text{for all } \xi \in \mathbb{R}^{d\times d}_{\rm sym} \text{ and a.a. } x \in \Omega. \tag{1.4}$$

If this condition fails, we lose numerous properties of the space $L_M(Q)$ like separability, reflexivity, cf. Ref. 16 and many others.

An example of generalized Orlicz spaces are the spaces $L^{q(x)}$, namely the case $M(x,\xi) = |\xi|^{q(x)}$. The framework of $L^{q(x)}$ spaces is often exploit to capture the description of electrorheological fluids, which are, next to magnetorheological fluids, another type of smart fluids (i.e. fluids whose properties, for example the viscosity, can be changed by applying an electric field or a magnetic field). The usual assumptions on the variable exponent, cf. Ref. 19, namely $1 \le q_0 \le q(x) < q_\infty < \infty$, provide the function M satisfies Δ_2-condition.

An interesting obstacle in our analysis is the lack of the classical integration by parts formula, which is the most direct tool for the proof of the energy equality. To follow the lines of the proof of integration by parts formula, cf. Ref. 6 (Sec. 4.1), we would essentially need that $L_M(Q) = L_M(0,T;L_M(\Omega))$. Unfortunately this is not the case. We recall the proposition from Ref. 3 (although it is stated for Orlicz spaces with $M = M(|\xi|)$). One can conclude that (1.5) means that M must be equivalent to some power q, $1 \le q \le \infty$. Hence, if (1.5) should hold, very strong assumptions must be satisfied by M. Surely they would provide $L_M(\Omega)$ to be separable and reflexive.

Proposition 1.1. *Let I be the time interval and $\Omega \subset \mathbb{R}^d$, M an N-function, and $L_M(I\times\Omega), L_M(I;L_M(\Omega))$ the Orlicz spaces on $I\times\Omega$ and the vector valued Orlicz space on I respectively. Then*

$$L_M(I\times\Omega) = L_M(I;L_M(\Omega)),$$

if and only if there exist constants k_0, k_1 such that

$$k_0 M^{-1}(s)M^{-1}(x) \le M^{-1}(sx) \le k_1 M^{-1}(s)M^{-1}(x) \tag{1.5}$$

for every $s \ge 1/|I|$ and $x \ge 1/|\Omega|$.

Secondly, we have to face the problem, that C^∞-functions are not dense in $L_M(\Omega)$, unless M satisfies Δ_2-condition, whereas the density of C^∞ in the considered function space is necessary in the classical proof of the integration by parts formula. We can observe the efforts to generalize the formula for Orlicz spaces already in Ref. 3, and later in Ref. 5. The advantage that the authors of Ref. 5 possessed was independence of M on x, which enabled the regularization w.r.t. space and time variables. Then the result on the modular convergence of solution and time derivative, cf. Ref. 5 (Theorem 1), automatically resolved the case to classical integration. This is, however, impossible in our case. Moreover, it is not clear how to obtain that the solution and its time-derivative are in dual spaces. Besides the dependence on x in the N-function, we deal with the divergence-free projections of the functions. We overcome the problem without a direct characterization of the term $\int v_t \cdot v$. Only in very particular cases, the Helmholtz decomposition is well-defined in spaces more general than L^q spaces. Namely in the spaces with variable exponent $L^{q(x)}$ if we know that q is log-Hölder continuous, then the maximal operator is continuous from $L^{q(x)}(\mathbb{R}^d)$ to $L^{q(x)}(\mathbb{R}^d)$.

Our proof is completed with the tools of Young measures generated by the approximate sequence of the symmetric gradients $\{Dv^n\}$. We are using the strict monotonicity of the stress–tensor due to show that the Young measure is a Dirac measure, which yields the assertion.

Before we state our main result, let us present the notation. By $\mathcal{D}(\Omega)$ we mean the set of C^∞-functions with compact support in any set Ω. Let $\mathcal{V}(\Omega)$ be the set of all functions belonging to $\mathcal{D}(\Omega)$ and are divergence-free. Moreover by $L^q, W^{1,q}$ we mean the standard Lebesgue and Sobolev spaces, by L^2_{div} the closure of $\mathcal{V}$ w.r.t. the $\|\cdot\|_{L^2}$ norm. By q' we mean the conjugate exponent to q, namely $\frac{1}{q}+\frac{1}{q'}=1$.

We assume the initial and boundary conditions to Eq. (1.1)

$$\begin{aligned} v(0,x) &= v_0 \quad \text{in } \Omega,\\ v(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega. \end{aligned} \tag{1.6}$$

Now we can define weak solutions to (1.1), (1.6) and state the existence result.

Definition 1.1. We call v a weak solution to (1.1), (1.6) if $v \in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)) \cap L^q(0,T;W^{1,q}_0(\Omega))$, $Dv \in L_M(Q)$ and the following is satisfied for all $\varphi \in \mathcal{D}(-\infty,T;\mathcal{V}(\Omega))$

$$\int_Q (-v\varphi_t + v\cdot\nabla v\cdot\varphi + S(x,Dv)\cdot D\varphi)dxdt + \int_\Omega v_0\varphi dx = \int_Q f\varphi dxdt. \tag{1.7}$$

Theorem 1.1. Let M be an N-function satisfying for some $c>0$ and

$$q \geq \frac{3d+2}{d+2} \tag{1.8}$$

the condition

$$M(x,\xi) \geq c|\xi|^q. \tag{1.9}$$

Given $f \in W^{-1,q'}(Q)$ and $v_0 \in L^2_{\mathrm{div}}(\Omega)$ there exists a weak solution to (1.1), (1.6).

Note that condition (1.8) has a technical character. However, since we direct our interest on shear thickening fluids, it does not restrict generality of our considerations.

The abstract parabolic equations with growth conditions formulated in Orlicz spaces defined by an N-function not satisfying the Δ_2-condition were considered in Refs. 3 and 5. However, to our best knowledge, the framework presented here is more general (space dependent and vector argument N-function) and new for the models of non-Newtonian fluids. Besides the analytical results on the existence of weak solutions, the proof of the main theorem provides information that the Young measure generated by the approximate sequences of gradients reduces to the Dirac measure. This excludes the oscillations of the sequence (stability of a weakly convergent sequence), which is of high interest for numerical analysis.

2. Preliminaries

2.1. *Generalized Orlicz spaces*

We start with an elementary estimate, see Ref. 16.

Proposition 2.1. (Fenchel-Young Inequality) *Let M be an N-function and M^* a complementary to M. Then the following inequality is satisfied*

$$|\xi \cdot \eta| \leq M(x,\xi) + M^*(x,\eta)$$

for all $\xi, \eta \in \mathbb{R}^{d\times d}_{\rm sym}$ and a.a. $x \in \Omega$.

Next, we recall an analogue to the Vitali's lemma, however for the modular convergence instead of the strong convergence in L^q.

Lemma 2.1. *Let $z^j : \Omega \to \mathbb{R}^d$ be a measurable sequence. Then $z^j \xrightarrow{M} z$ in $L_M(\Omega)$ modularly if and only if $z^j \to z$ in measure and there exist some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda z^j)\}$ is uniformly integrable*, i.e.

$$\lim_{R\to\infty} \left(\sup_{j\in\mathbb{N}} \int_{\{x:|M(x,\lambda z^j)|\geq R\}} M(x,\lambda z^j)dx \right) = 0.$$

Proof. Note that $z^j \to z$ in measure if and only if $M(\cdot, \frac{z^j - z}{\lambda}) \to 0$ in measure for all $\lambda > 0$. Moreover, the convergence $z^j \to z$ in measure implies that for all measurable sets $A \subset Q$ it holds

$$\liminf_{j\to\infty} \int_A M(x,z^j)dxdt \geq \int_A M(x,z)dxdt.$$

Note also that the convexity of M implies

$$\int_A M\left(x, \frac{z^j - z}{\lambda}\right) dxdt \leq \int_A M\left(x, \frac{z^j}{2\lambda}\right) dxdt + \int_A M\left(x, \frac{z}{2\lambda}\right) dxdt.$$

Hence by the classical Vitali's lemma for $f^j(x) = M(x, \frac{z^j - z}{\lambda})$ we obtain that $f^j \to 0$ strongly in $L^1(Q)$. □

The following technical facts will be used in a sequel.

Lemma 2.2. *Let M be an N-function and for all $j \in \mathbb{N}$ let $\int_\Omega M(x,z^j) \leq c$. Then the sequence $\{z^j\}$ is uniformly integrable.*

Proof. Let us define $\delta(R) = \min_{|\xi|=R} \frac{M(x,\xi)}{|\xi|}$. Then for all $j \in \mathbb{N}$ it holds

$$\int_{\{x:|z^j(x)|\geq R\}} M(x,z^j(x))dx \geq \delta(R) \int_{\{x:|z^j(x)|\geq R\}} |z^j(x)|dx.$$

Since the left-hand side is bounded, we then obtain

$$\sup_{j\in\mathbb{N}} \int_{\{x:|z^j(x)|\geq R\}} |z^j(x)|dx \leq \frac{c}{\delta(R)}.$$

□

Proposition 2.2. *Let M be an N-function and M^* its complementary function. Suppose that the sequences $\psi^j : Q \to \mathbb{R}^d$ and $\phi^j : Q \to \mathbb{R}^d$ are uniformly bounded in $L_M(Q)$ and $L_{M^*}(Q)$ respectively. Moreover, $\psi^j \xrightarrow{M} \psi$ modularly in $L_M(Q)$ and $\phi^j \xrightarrow{M^*} \phi$ modularly in $L_{M^*}(Q)$. Then $\psi^j \cdot \phi^j \to \psi \cdot \phi$ strongly in $L^1(Q)$.*

Proof. Due to Lemma 2.1 the modular convergence of $\{\psi^j\}$ and $\{\phi^j\}$ implies the convergence in measure of these sequences and consequently also the convergence in measure of the product. Hence it is sufficient to show the uniform integrability of $\{\psi^j \cdot \phi^j\}$. Notice that it is equivalent with the uniform integrability of the term $\{\frac{\psi^j}{\lambda_1} \cdot \frac{\phi^j}{\lambda_2}\}$ for any $\lambda_1, \lambda_2 > 0$. The assumptions of the proposition show that there exist some $\lambda_1, \lambda_2 > 0$ such that the sequences

$$\left\{M\left(x, \frac{\psi^j}{\lambda_1}\right)\right\} \quad \text{and} \quad \left\{M^*\left(x, \frac{\phi^j}{\lambda_2}\right)\right\}$$

are uniformly integrable. Hence let us use the same constants and estimate with the help of Fenchel–Young inequality

$$\left|\frac{\psi^j}{\lambda_1} \cdot \frac{\phi^j}{\lambda_2}\right| \le M\left(x, \frac{\psi^j}{\lambda_1}\right) + M^*\left(x, \frac{\phi^j}{\lambda_2}\right).$$

Obviously the uniform integrability of the right-hand side provides the uniform integrability of the left-hand side and this yields the assertion. □

Proposition 2.3. *Let ϱ^j be a standard mollifier*, i.e. $\varrho \in C^\infty(\mathbb{R})$, *$\varrho$ has a compact support and $\int_{\mathbb{R}} \varrho(\tau)d\tau = 1, \varrho(t) = \varrho(-t)$. We define $\varrho^j(t) = j\varrho(jt)$. Moreover, let $*$ denote a convolution in the variable t. Then for any function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in L^1(Q)$ it holds*

$$(\varrho^j * \psi)(t, x) \to \psi(t, x) \quad \textit{in measure.}$$

Proof. For a.a. $x \in \Omega$ the function $\psi(\cdot, x) \in L^1(0, T)$ and $\varrho^j * \psi(\cdot, x) \to \psi(\cdot, x)$ in $L^1(0, T)$ and hence $\varrho^j * \psi \to \psi$ in measure on the set $[0, T] \times \Omega$. □

Proposition 2.4. *Let ϱ^j be defined as in Proposition* 2.3. *Given an N-function M and a function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in \mathcal{L}_M(Q)$ the sequence $\{M(x, \varrho^j * \psi)\}$ is uniformly integrable.*

Proof. We start with an abstract fact concerning the uniform integrability. Namely, the following two conditions are equivalent for any measurable sequence $\{z^j\}$

(a) $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$: $\displaystyle\sup_{j\in\mathbb{N}} \sup_{|A|\le\delta} \int_A |z^j(x)| dxdt \le \varepsilon,$

(b) $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$: $\displaystyle\sup_{j\in\mathbb{N}} \int_Q \left| |z^j(x)| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \le \varepsilon,$

where we use the notation

$$|\xi|_+ = \max\{0, \xi\}.$$

The implication (a) $\Rightarrow$ (b) is obvious. To show that (b) $\Rightarrow$ (a) also holds let us estimate

$$\begin{aligned}\sup_{j\in\mathbb{N}} \sup_{|A|\le\delta} \int_A |z^j| dxdt &\le \sup_{|A|\le\delta} |A| \cdot \frac{1}{\sqrt{\delta}} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \\ &\le \sqrt{\delta} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt.\end{aligned}$$

Notice that since M is a convex function, then the following inequality holds for all $\delta > 0$

$$\int_Q \left| M(x,\psi) - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \ge \int_Q \left| M(x, \varrho^j * \psi) - \frac{1}{\sqrt{\delta}} \right|_+ dxdt. \tag{2.1}$$

Finally, since $\psi \in \mathcal{L}_M(Q)$, then also $\int_Q |M(x,\psi) - \frac{1}{\sqrt{\delta}}|_+ dxdt$ is finite and hence taking supremum over $j \in \mathbb{N}$ in (2.1) we prove the assertion. □

2.2. *Preliminaries on Young measures*

We assume the basic facts on existence and properties of Young measures are known to the reader. The fundamental theorem on Young measures may be found in e.g. Refs. 1 and 15 and many others. We only recall the properties concerning the Carathéodory functions, which is the case of the stress–tensor S. Lemma 2.4 provides the relation between the support of the Young measure and convergence in measure, which is used to show the modular convergence of the approximate sequences of gradients. In the following by $\mathcal{M}(\mathbb{R}^d)$ we mean the space of bounded Radon measures. For the proofs of the next lemmas we refer to Ref. 15 (Corollaries 3.2 and 3.3).

Lemma 2.3. *Suppose that the sequence of measurable functions $z^j : \Omega \to \mathbb{R}^d$ generates the Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$. Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function. Let also assume that the negative part $F^-(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, z^j(x)) dx \ge \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda) d\nu_x(\lambda) dx.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$ then

$$F(\cdot, z^j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} F(x,\lambda) d\nu_x(\lambda) \quad \text{in } L^1(\Omega).$$

Remark 2.1. The second part of the above lemma can be easily extended to vector-valued functions F.

Lemma 2.4. *Suppose that a sequence of measurable functions $z^j : \Omega \to \mathbb{R}^d$ generates the Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$. Then*

$$z^j \to z \quad \text{in measure if and only if } \nu_x = \delta_{z(x)} \text{ a.e.}$$

3. Proof of Theorem 1.1

We construct Galerkin approximations to (1.1). For details on the Galerkin method we refer to Refs. 12 and 14. First, we describe the chosen basis $\{\omega_i\}$. Assume that

$$s > \frac{d}{2} + 1 \tag{3.1}$$

and denote

$$V_s \equiv \text{the closure of } \mathcal{V} \text{ w.r.t. the } W^{s,2}(\Omega)\text{-norm.}$$

Let then the scalar product in V_s be denoted by $((\cdot,\cdot))_s$ and $\{\omega_i\}$ be the set of eigenvectors to the problem

$$((\omega_i, \varphi))_s = \lambda_i(\omega_i, \varphi) \quad \text{for all } \varphi \in V_s.$$

Notice that condition (3.1) provides

$$W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \tag{3.2}$$

which we will use in the sequel. We define $v^n = \sum_{i=1}^n \alpha_i^n(t)\omega_i$, where $\alpha_i^n(t)$ solve the system

$$\begin{aligned} \int_\Omega \frac{d}{dt} v^n \cdot \omega_i + \int_\Omega v^n \cdot \nabla v^n \cdot \omega_i dx + \int_\Omega S(x, Dv^n) \cdot D\omega_i dx = \langle f, \omega_i \rangle, \\ v^n(0) = P^n v_0, \end{aligned} \tag{3.3}$$

where $i = 1, \ldots, n$ and by P^n we denote the orthogonal projection of $L^2_{\text{div}}(\Omega)$ on $\text{conv}\{\omega_1, \ldots, \omega_n\}$. Multiplying each equation of (3.3) by $\alpha_i^n(t)$, summing over $i = 1, \ldots, n$ and remembering that since $\operatorname{div} v^n = 0$, then $\int_\Omega v^n \cdot \nabla v^n \cdot v^n dx = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v^n\|^2_{L^2(\Omega)} + \int_\Omega S(x, Dv^n) \cdot Dv^n dx = \langle f, v^n \rangle. \tag{3.4}$$

To estimate the right-hand side of the above, observe first that due to the regularity theory of linear elasticity equations (i.e. Lame equation), see e.g. Ref. 21, there exists an $F \in L^{q'}(Q)$, such that F is a symmetric matrix and solves the equation

$$\operatorname{div} F = f. \tag{3.5}$$

Moreover if (1.9) holds, then one easily shows there exists some $c > 0$ such that

$$M^*(x, \xi) \le c|\xi|^{q'}. \tag{3.6}$$

Consequently we conclude that $F \in \mathcal{L}_{M^*}(Q)$ and estimate

$$\begin{aligned}
\langle \operatorname{div} F, v^n \rangle &\leq \int_\Omega \left| \frac{2}{c\alpha_1} F \cdot \frac{c\alpha_1}{2} Dv^n \right| dx \\
&\overset{F-Y}{\leq} \int_\Omega M^* \left(x, \frac{2}{c\alpha_1} F \right) dx + \int_\Omega M \left(x, \frac{c\alpha_1}{2} Dv^n \right) dx \\
&\leq \int_\Omega M^* \left(x, \frac{2}{c\alpha_1} F \right) dx + \frac{c}{2} \int_\Omega M(x, \alpha_1 Dv^n)\, dx. \qquad (3.7)
\end{aligned}$$

Integrating (3.4) over the time interval $(0,t)$, using estimate (3.7) and the coercivity conditions on S (1.3) we obtain

$$\begin{aligned}
&\frac{1}{2}\|v^n(t)\|^2_{L^2(\Omega)} + \frac{c}{2}\int_0^T \int_\Omega M(x, \alpha_1 Dv^n) dxdt + c\int_0^T \int_\Omega M^*(x, \alpha_2 S(x, Dv^n)) dxdt \\
&\quad \leq \int_0^T \int_\Omega M^* \left(x, \frac{2}{c\alpha_1} F \right) dxdt + \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \int_0^T \|m\|_{L^1(\Omega)} dt. \qquad (3.8)
\end{aligned}$$

Condition (1.9) provides that $\{\alpha_1 Dv^n\}$ is uniformly bounded in the space $L^q(Q)$ for $q \geq \frac{3d+2}{d+2}$ and hence there exists a subsequence such that

$$Dv^n \rightharpoonup Dv \quad \text{weakly in } L^q(Q).$$

Provided that M is an N-function, then M^* is also an N-function. This allows to apply Lemma 2.2 to M^* and conclude the uniform integrability, and hence the weak precompactness of the sequence $\{\alpha_2 S(x, Dv^n)\}$ in $L^1(Q)$ and consequently $\{S(x, Dv^n)\}$ in $L^1(Q)$. Namely there exists a $\chi \in L^1(Q)$ such that

$$S(\cdot, Dv^n) \rightharpoonup \chi \quad \text{weakly in } L^1(Q).$$

To establish the uniform bound for $\frac{dv^n}{dt}$, we take a test function $\varphi \in L^\infty(0,T;V_s)$ and using (3.2) to estimate the following integrals

$$\begin{aligned}
\left| \int_0^T \int_\Omega S(x, Dv^n) \cdot D(P^n\varphi) dxdt \right| &\leq \int_0^T \|S(\cdot, Dv^n)\|_{L^1(\Omega)} \|D(P^n\varphi)\|_{L^\infty(\Omega)} dt \\
&\leq \int_0^T \|S(\cdot, Dv^n)\|_{L^1(\Omega)} \|P^n\varphi\|_{V_s} dt \\
&\leq \int_0^T \|S(\cdot, Dv^n)\|_{L^1(\Omega)} \|\varphi\|_{V_s} dt \\
&\leq \|S(\cdot, Dv^n)\|_{L^1(Q)} \|\varphi\|_{L^\infty(0,T;V_s)}
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_0^T \int_\Omega v^n \cdot \nabla v^n \cdot P^n\varphi dxdt \right| &= \left| \int_0^T \int_\Omega (v^n \otimes v^n) \cdot \nabla P^n\varphi dxdt \right| dt \\
&\leq \int_0^T \|v^n \otimes v^n\|_{L^1(\Omega)} \|\nabla P^n\varphi\|_{L^\infty(\Omega)} dt
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^T \|v^n\|^2_{L^2(\Omega)} \|P^n\varphi\|_{V_s} dt \\
&\le \int_0^T \|v^n\|^2_{L^2(\Omega)} \|\varphi\|_{V_s} dt \\
&\le \|v^n\|^2_{L^2(Q)} \|\varphi\|_{L^\infty(0,T;V_s)}.
\end{aligned}$$

To handle the right-hand side term, recall (3.5)

$$\begin{aligned}
\left|\int_0^T \langle \operatorname{div} F, P^n\varphi\rangle dt\right| &= \left|\int_0^T \int_\Omega F \cdot D(P^n\varphi) dx dt\right| \\
&\le \int_0^T \|F\|_{L^1(\Omega)} \|D(P^n\varphi)\|_{L^\infty(\Omega)} dt \\
&\le \int_0^T \|F\|_{L^1(\Omega)} \|P^n\varphi\|_{V_s} dt \\
&\le \int_0^T \|F\|_{L^1(\Omega)} \|\varphi\|_{V_s} dt \\
&\le \|F\|_{L^1(Q)} \|\varphi\|_{L^\infty(0,T;V_s)}.
\end{aligned}$$

Hence we conclude that $\frac{dv^n}{dt}$ is bounded in $L^1(0,T;V_s^*)$. Because of the low regularity of the time derivative, we recall the following generalization of the classical Aubin–Lions lemma, cf. Ref. 18. We use the notation

$$W^{1,p,q}(I;X_1,X_2) := \left\{ u \in L^p(I;X_1); \frac{du}{dt} \in L^q(I;X_2) \right\}$$

for X_1 a Banach space and X_2 a locally convex space, $X_1 \subset X_2$. By $\frac{du}{dt}$ we denote the distributional derivative, $\hookrightarrow$ means the continuous embedding and $\hookrightarrow\hookrightarrow$ a compact embedding.

Lemma 3.1. (Aubin–Lions, generalized) *Let* X_1, X_2 *be Banach spaces, and* X_3 *be a metrizable Hausdorff locally convex space,* X_1 *be separable and reflexive,* $X_1 \hookrightarrow\hookrightarrow X_2, X_2 \hookrightarrow X_3, 1 < p < \infty, 1 \le q \le \infty$. *Then* $W^{1,p,q}(I;X_1,X_3) \hookrightarrow\hookrightarrow L^p(I;X_2)$.

Since the sequence v^n is bounded in $W^{1,q,1}(0,T;W^{1,q}_{\mathrm{div}}(\Omega),V_s^*)$ and $W^{1,q,1}(0,T;W^{1,q}_{\mathrm{div}}(\Omega),V_s^*) \hookrightarrow\hookrightarrow L^q(0,T;L^2_{\mathrm{div}}(\Omega))$, hence

$$v^n \to v \text{ strongly in } L^q(0,T;L^2_{\mathrm{div}}(\Omega)). \tag{3.9}$$

This allows to conclude that for a fixed $i \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \int_0^T \int_\Omega v^n \cdot \nabla v^n \cdot \omega_i dx dt = \int_0^T \int_\Omega v \cdot \nabla v \cdot \omega_i dx dt.$$

Letting $n \to \infty$ in $(3.3)_1$ provides

$$\int_0^T \langle v_t, \omega_i \rangle dt + \int_0^T \int_\Omega v \cdot \nabla v \cdot \omega_i dxdt + \int_0^T \int_\Omega \chi \cdot D\omega_i dxdt = \int_0^T \int_\Omega f \cdot \omega_i dxdt. \tag{3.10}$$

Let ϱ^j be a standard mollifier as described in Proposition 2.3. Let us then choose a test function

$$v^j = \varrho^j * ((\varrho^j * v)\mathbb{1}_{(s_0,s_1)}),$$

with $1/j < \min\{s_0, T - s_1\}$. We observe that such test functions are uniformly bounded in the space $C^\infty((0,T); W_0^{1,q}(\Omega))$ and Dv^j in $L_M(Q)$. With the standard estimates one can show that this is an admissible class of test functions. Hence for all $0 < s_0 < s_1 < T$ it follows that

$$\begin{aligned}
\int_0^T \langle v_t \cdot v^j \rangle dt &= \int_0^T \langle v_t, \varrho^j * ((\varrho^j) * v)\mathbb{1}_{(s_0,s_1)} \rangle dt \\
&= \int_{s_0}^{s_1} \int_\Omega (\varrho^j * v_t) \cdot (\varrho^j * v) dxdt \\
&= \int_{s_0}^{s_1} \int_\Omega (\varrho^j * v)_t \cdot (\varrho^j * v) dxdt \\
&= \frac{1}{2} \int_{s_0}^{s_1} \frac{d}{dt} \|\varrho^j * v\|^2_{L^2(\Omega)} dt \\
&= \frac{1}{2} \|\varrho^j * v(s_1)\|^2_{L^2(\Omega)} - \frac{1}{2} \|\varrho^j * v(s_0)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.11}$$

We pass to the limit with $j \to \infty$ and obtain for almost all s_0, s_1, namely for all Lebesgue points of the function $v(t)$

$$\lim_{j\to\infty} \int_0^T \langle v_t, v^j \rangle dt = \frac{1}{2} \|v(s_1)\|^2_{L^2(\Omega)} - \frac{1}{2} \|v(s_0)\|^2_{L^2(\Omega)}. \tag{3.12}$$

Next, we concentrate on the convergence of the term $\int_{s_0}^{s_1} \int_\Omega v \cdot \nabla v \cdot v^j dxdt$. Condition (1.9) provides that $Dv \in L^q(0,T; L^q(\Omega))$ and hence due to Korn's inequality $\nabla v \in L^q(0,T; L^q(\Omega))$. With the standard arguments we conclude that the sequence $\{\varrho^j * ((\varrho^j) * \nabla v)\mathbb{1}_{(s_0,s_1)}\}$ is also uniformly bounded in $L^q(0,T; L^q(\Omega))$. For q satisfying (1.8) the trilinear form $\int_0^T \int_\Omega v \cdot \nabla v \cdot v^j dxdt$ is continuous and hence letting $j \to \infty$ it converges to zero.

Observe now the term

$$\int_0^T \int_\Omega \chi \cdot (\varrho^j * ((\varrho^j) * Dv)\mathbb{1}_{(s_0,s_1)}) dxdt = \int_{s_0}^{s_1} \int_\Omega (\varrho^j * \chi) \cdot (\varrho^j * Dv) dxdt.$$

Both sequences $\{\varrho^j * \chi\}$ and $\{\varrho^j * Dv\}$ converge in measure in Q due to Lemma 2.3. Moreover, since M and M^* are convex functions, then the weak lower semicontinuity and estimate (3.8) provide that the integrals

$$\int_0^T \int_\Omega M(x, \alpha_1 Dv) dxdt \quad \text{and} \quad \int_0^T \int_\Omega M^*(x, \alpha_2 \chi) dxdt$$

are finite. Hence Lemma 2.4 implies that the sequences $\{\varrho^j * (\alpha_2\chi)\}$ and $\{\varrho^j * (\alpha_1 Dv)\}$ are uniformly bounded and hence according to Lemma 2.1 we have

$$\begin{aligned} \varrho^j * Dv &\xrightarrow{M} Dv \quad \text{in } L_M(Q), \\ \varrho^j * \chi &\xrightarrow{M^*} \chi \quad \text{in } L_{M^*}(Q). \end{aligned} \tag{3.13}$$

Applying Lemma 2.2 allows to conclude

$$\lim_{j\to\infty} \int_{s_0}^{s_1} \int_\Omega (\varrho^j * \chi) \cdot (\varrho^j * Dv) dxdt = \int_{s_0}^{s_1} \int_\Omega \chi \cdot Dv dxdt. \tag{3.14}$$

In the same manner we treat the source term. Since $f = \operatorname{div} F$, then

$$\begin{aligned} \int_0^T \langle f, \varrho^j * ((\varrho^j) * v) \mathbb{1}_{(s_0,s_1)} \rangle dt &= \int_0^T \langle \operatorname{div} F, \varrho^j * ((\varrho^j) * v) \mathbb{1}_{(s_0,s_1)} \rangle dt \\ &= - \int_{s_0}^{s_1} \int_\Omega (\varrho^j * F) \cdot D(\varrho^j * v) dxdt \\ &= - \int_{s_0}^{s_1} \int_\Omega (\varrho^j * F) \cdot (\varrho^j * Dv) dxdt. \end{aligned} \tag{3.15}$$

Hence

$$\lim_{j\to\infty} \int_{s_0}^{s_1} \int_\Omega (\varrho^j * F) \cdot (\varrho^j * Dv) dxdt = \int_{s_0}^{s_1} \int_\Omega F \cdot Dv dxdt = - \int_{s_0}^{s_1} \langle f, v \rangle dt. \tag{3.16}$$

Combining (3.12), (3.14) and (3.16) we may pass to the limit in (3.10) and obtain

$$\frac{1}{2} \|v(s_1)\|^2_{L^2(\Omega)} + \int_{s_0}^{s_1} \int_\Omega \chi \cdot Dv dxdt = \int_{s_0}^{s_1} \langle f, v \rangle dt + \frac{1}{2} \|v(s_0)\|^2_{L^2(\Omega)} \tag{3.17}$$

for almost all $0 < s_0 < s_1 < T$.

In the last step we concentrate on characterizing the limit χ. Since S is monotone and $S(x, 0) = 0$, then trivially the negative part is weakly relatively compact in $L^1(Q)$. Hence due to Lemma 2.3

$$\liminf_{n\to\infty} \int_{s_0}^{s_1} \int_\Omega S(x, Dv^n(x)) \cdot Dv^n dxdt \geq \int_{s_0}^{s_1} \int_\Omega \int_{\mathbb{R}^{d\times d}_{\text{sym}}} S(x, \xi) \cdot \xi d\nu_{t,x}(\xi) dxdt, \tag{3.18}$$

where $\nu_{t,x}$ is the Young measure generated by the sequence $\{Dv^n\}$. From (3.9) it follows that

$$\|v^n(s)\|_{L^2(\Omega)} \to \|v(s)\|_{L^2(\Omega)} \quad \text{for a.a. } s \in (0, T). \tag{3.19}$$

Hence integrating (3.4) over the interval (s_0, s_1) allows one to conclude that

$$\lim_{n\to\infty}\int_{s_0}^{s_1}\int_\Omega S(x, Dv^n)\cdot Dv^n dxdt = \int_{s_0}^{s_1}\langle f, v\rangle dt + \frac{1}{2}\|v(s_0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|v(s_1)\|^2_{L^2(\Omega)}. \tag{3.20}$$

Combining (3.17), (3.18) and (3.20) results

$$\int_{s_0}^{s_1}\int_\Omega\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} S(x,\xi)\cdot\xi d\nu_{t,x}(\xi)dxdt \le \int_{s_0}^{s_1}\int_\Omega \chi\cdot Dvdxdt. \tag{3.21}$$

Since the above inequality holds for a dense set $(s_0, s_1)\times\Omega$ in Q, we conclude that it is true in the whole Q. The monotonicity of S provides that

$$\int_Q\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(t,x,\xi)d\nu_{t,x}(\xi)dxdt \ge 0, \tag{3.22}$$

where h is defined by

$$h(t,x,\xi) := [S(x,\xi) - S(x,Dv)]\cdot[\xi - Dv]. \tag{3.23}$$

Since $\{Dv^n\}$ and $\{S(\cdot, Dv^n)\}$ are weakly relatively compact in $L^1(Q)$ and S is a Carathéodory function, then

$$Dv = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \xi d\nu_{t,x}(\xi) \quad \text{and} \quad \chi = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} S(x,\xi)d\nu_{t,x}(\xi). \tag{3.24}$$

Hence integrating $h(t,x,\xi)$ yields

$$\int_Q\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(t,x,\xi)d\nu_{t,x}(\xi)dxdt = \int_Q\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} S(x,\xi)\cdot\xi d\nu_{t,x}(\xi)dxdt - \int_Q \chi\cdot Dvdxdt, \tag{3.25}$$

which is nonpositive due to (3.21). Combining (3.21), (3.22) and (3.25) implies that $\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} h(x,\xi)d\nu_{t,x}(\xi) = 0$ for a.a. $(t,x)\in Q$. Moreover, since $\nu_{t,x}\ge 0$ is a probability measure and $S(x,\cdot)$ is strongly monotone, we conclude that

$$\mathrm{supp}\{\nu_{t,x}\} \overset{\text{a.e.}}{=} \{Dv(t,x)\}$$

and thus $\nu_{t,x} = \delta_{Dv(t,x)}$ a.e. Finally (3.24) yields $\chi \overset{\text{a.e.}}{=} S(x, Dv(t,x))$, which completes the proof of the existence of solutions. □

We additionally prove the modular convergence and continuity of solutions in $L^2_{\mathrm{div}}(\Omega)$, namely $v\in C([0,T]; L^2_{\mathrm{div}}(\Omega))$. These properties of solutions are formulated in the two lemmas below.

Lemma 3.2. *Let v be a weak solution to* (1.1), (1.6) *and $\{v^n\}$ be a solution to approximate problem* (3.3). *Then the following holds*

(1) $Dv^n \xrightarrow{M} Dv$ *in* $L_M(Q)$,
(2) $S(\cdot, Dv^n) \xrightarrow{M^*} S(\cdot, Dv)$ *in* $L_{M^*}(Q)$.

Proof. 1 and 2. A direct application of Lemma 2.4 implies that $Dv^n \to Dv$ in measure. To apply Lemma 2.1 we recall (1.3) and establish the uniform integrability of the term $\{S(x, Dv^n) \cdot Dv^n\}$. Observe that

$$S(x, Dv^n) \cdot Dv^n \geq -m(x), \quad S(x, Dv) \cdot Dv \in L^1(Q).$$

Moreover,

$$\lim_{n\to\infty} \int_Q S(x, Dv^n) \cdot Dv^n \, dxdt = \int_Q S(x, Dv) \cdot Dv \, dxdt$$

and

$$S(x, Dv^n) \cdot Dv^n \to S(x, Dv) \cdot Dv \quad \text{a.e. in } Q$$

hold. Noticing that

$$\begin{aligned}\int_Q &|S(x, Dv^n) \cdot Dv^n - S(x, Dv) \cdot Dv| \, dxdt \\ &= \int_Q (S(x, Dv^n) \cdot Dv^n - S(x, Dv) \cdot Dv) \, dxdt \\ &\quad + 2 \int_Q |S(x, Dv) \cdot Dv - S(x, Dv^n) \cdot Dv^n|_+ \, dxdt,\end{aligned}$$

we conclude by Lebesgue's Dominated Convergence Theorem that

$$S(x, Dv^n) \cdot Dv^n \to S(x, Dv) \cdot Dv \quad \text{in} \quad L^1(Q).$$

This implies the uniform integrability, which together with coercivity conditions (1.3) provides the uniform integrability of the sequences $\{M(x, \alpha_1 Dv^n)\}$ and $\{M^*(x, \alpha_2 S(x, Dv^n))\}$, which completes the proof. □

Lemma 3.3. *Let v be a weak solution to* (1.1). *Then $v \in C([0, T]; L^2_{\mathrm{div}}(\Omega))$.*

Proof. Since $v_t \in L^1(0, T; V_s^*)$, then we conclude that $v(t_k) \xrightarrow{t_k \to t} v(t)$ strongly, and also weakly, in V_s^*. Moreover, we know that $v \in L^\infty(0, T; L^2_{\mathrm{div}}(\Omega))$ thus $v(t_k) \rightharpoonup v(t)$ in $L^2(\Omega)$. In addition we observe from (3.12) and (3.17) that $\frac{d}{dt}\|v(\cdot)\|_{L^2} \in L^1(0, T)$, hence $\|v(\cdot)\|_{L^2}$ is continuous. Combining the facts that $v(t_k) \rightharpoonup v(t)$ in $L^2(\Omega)$ and $\|v(t_k)\|_{L^2} \to \|v(t)\|_{L^2}$, we conclude that $v(t_k) \to v(t)$ strongly in $L^2(\Omega)$, which yields the assertion. □

Acknowledgments

The paper was partially written during the stay of both authors at University of Heidelberg. The authors appreciate the support of Alexander von Humboldt Foundation (P.G.) and DAAD (A.Ś.-G.) and would like to thank Professor Willi Jäger for hospitality. They are also grateful to Professor Henryk Hudzik for fruitful discussions on Orlicz spaces. Both authors were supported by the Grant of Ministry of Science and Higher Education, Nr N201 033 32/2269.

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