Orbit-finite Linear Programming

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minimise $\mathbf{c}^T \cdot \mathbf{x}$
subject to $\mathbf{M} \cdot \mathbf{x} \geq \mathbf{b}$, $\mathbf{x}$ is finite

$\mathbf{M}$, $\mathbf{b}$, $\mathbf{c}$ are orbit-finite

Existence of orbit-finite solutions and finite solutions are equivalent as decision problems

minimise $\sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha)$

subject to $\sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \mathbf{x}(\alpha) \geq 1\,, \ \beta \in \mathbb{A}$

$\mathbf{x}$ is finite

minimise $[1 \ 1 \ 1 \ \cdot \ \cdot \ \cdot\,] \cdot \mathbf{x}$

subject to $\begin{bmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & & & \\ 1 & 1 & 0 & & & \\ & \vdots & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix} \cdot \mathbf{x} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$

$\mathbf{x}$ is finite

How do we solve it?

1. Consider a finite support $F \subseteq \mathbb{A}$
2. Merge variables related by permutations of F,

   equivalently

   add columns whose indices are related by permutations of F
3. **Fact :** Inequalities/rows with indices related by permutations of F are the same
4. **Fact :** The number of row and column indices up to permutation of F is independent of F when F is large enough
5. We get a finite system

   minimise $\vec{c}(n) \cdot \vec{x}$
   subject to $M(n) \cdot \vec{x} \geq \vec{b}(n)$

   with $n = |F|$ as a parameter
6. The optimum value is defined by

   $\text{optimum}_{M,b}(d)$ = for all $d' \geq d$ there exists n and $\vec{x}$ such that $M(n) \cdot \vec{x} \geq \vec{b}(n)$ and $\vec{c}(n) \cdot \vec{x} \geq d'$

   Fact : It doesn't matter whether n is real or integer!
7. Solve this "parametric" system
   - decision procedures for real arithmetic
   - Fourier-Mozkin elimination

minimise $[1 \ 1 \ \cdots \ 1] \cdot [\overbrace{x \ \cdots \ x}^{F}]$

subject to finite $F \subseteq \mathbb{A}$

$$\begin{matrix} F \\ \\ \mathbb{A} \setminus F \end{matrix} \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 \\ 1 & 1 & \cdots & 1 \\ & \vdots & & \end{bmatrix} \cdot \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

minimise $n \cdot x$

subject to $n \in \mathbb{N}$

$$\begin{bmatrix} (n-1) \\ n \end{bmatrix} \cdot [x] \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Lemma : There is a solution x which uses finitely set of atoms $F \subseteq \mathbb{A}$,

if and only if

there is a finite solution x which is invariant under permutations of F

Proof :
$(\Longleftarrow)$ Trivial
$(\Longrightarrow)$

$$x' = \frac{1}{|F|!} \cdot \left(\sum_{\text{permutations } \pi \text{ of } F} \pi(x) \right)$$

The optimum may not be achieved

$\frac{1}{|F|-1} \cdot \mathbf{1}_F$ is a solution for the constraints for any finite $F \subseteq \mathbb{A}$, with objective $\frac{|F|}{|F|-1}$

For any $\mathbf{x}$ satisfying the above

$$\sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha) \geq 1 + \max\{\mathbf{x}(\alpha) \mid \alpha \in \mathbb{A}\} > 1$$

computation with infinite alphabets
register automata
Petri nets with data
orbit-finite sets
sets with atoms
orbit-finite dimensional vector spaces
symbolic computation
equivariant linear algebra